Regularities in ordered ternary semigroups

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Abstract. We present various types of regularities in ordered ternary semigroups and describe connections between these regularities.

1. Preliminaries

A nonempty set $S$ is called a ternary semigroup if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, such that

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$. For any $x, y, z$ in a ternary semigroup $S$, we will write $xyz$ instead of $[xyz]$.

Siemons [8] introduced the concept of regularities in $n$-ary semigroups. Dudek and Groździńska [2] gave characterizations of a regular $n$-ary semigroup using its $j$-ideals. As a special case of a regular $n$-ary semigroup, a regular ternary semigroup was studied by Santiago and Sri Bala [7]. Connections between ternary and binary semigroups were firstly studied in [3].

An ordered ternary semigroup $(S, [], \leq)$ is a ternary semigroup $(S, [])$ together with a partial order relation $\leq$ on $S$ which is compatible with the ternary operation, i.e.,

$$x \leq y \Rightarrow xuv \leq yuv, \quad wuv \leq uvy, \quad wux \leq uyv$$

for all $x, y, u, v \in S$.

Ordered ternary semigroups have been studied by many authors (see, e.g., [4], [5], [6]). Daddi and Pawar [1] introduced the concepts of ordered quasi-ideals and ordered bi-ideals in ordered ternary semigroups and characterized a regular ordered ternary semigroup using its ordered ideals.

Throughout this paper, we write $S$ for an ordered ternary semigroup, unless specify otherwise.

Let $A, B, C$ be nonempty subsets of $S$. We denote

$$(A) = \{x \in S \mid x \leq a \text{ for some } a \in A\},$$

and note that $A \subseteq (A)$, $(A) = (A)[]$, $(A)[B](C) \subseteq (ABC)$, $(A)BC \subseteq (ABC)$, $A(B)C \subseteq (ABC)$, $AB(C) \subseteq (ABC)$, $(A \cup B) = (A) \cup (B)$ and $A \subseteq B$ implies $(A) \subseteq (B)$.

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A nonempty subset \( I \) of \( S \) is called an ordered left (resp. right, lateral) ideal of \( S \) if \( SSI \subseteq I \) (resp. \( ISS \subseteq I \), \( SIS \subseteq I \)) and \( (I) = I \).

If \( I \) is an ordered left, right and lateral ideal of \( S \), then it is called an ordered ideal of \( S \).

A nonempty subset \( Q \) of \( S \) is called an ordered quasi-ideal of \( S \) if \( Q \) is an ordered left, right and lateral ideal of \( S \), then it is called an ordered ideal of \( S \).

A nonempty subset \( B \) of \( S \) is called an ordered quasi-ideal (bi-ideal) of \( S \) if \( B \) is an ordered left, right and lateral ideal of \( S \), then it is called an ordered ideal of \( S \).

Let \( A \) be a nonempty subset of \( S \). Then

(i) \( L(A) = (A \cup SSA] \)

(ii) \( R(A) = (A \cup ASS] \)

(iii) \( M(A) = (A \cup SAS \cup SSASS] \)

(iv) \( I(A) = (A \cup SSA \cup ASS \cup SAS \cup SSASS] \)

(v) \( B(A) = (A \cup AAA \cup ASASA] \)

(vi) \( Q(A) = (A \cup SSA] \cap (A \cup SAS \cup SSASS] \cap (A \cup ASS] \)

In particular case, for \( a \in S \), we write \( L(a), R(a), M(a), I(a), Q(a) \) and \( B(a) \) instead of \( L(\{a\}), R(\{a\}), M(\{a\}), I(\{a\}), Q(\{a\}) \) and \( B(\{a\}) \), respectively.

2. Regularities in ordered ternary semigroups

An ordered ternary semigroup \( S \) is called regular, if each its element is regular, i.e., for each \( a \in S \) there exists \( x \in S \) such that \( a \leq axa \).

We note that \( S \) is called regular if and only if for each \( a \in S \) there exist \( x, y \in S \) such that \( a \leq axa \).

**Lemma 2.2.** (cf. [1]) The following statements are equivalent:

(i) \( S \) is regular,

(ii) \( A \subseteq (ASA] \) for any \( A \subseteq S \),

(iii) \( a \in (aSa] \) for any \( a \in S \),

(iv) \( A \subseteq (ASASA] \) for any \( A \subseteq S \),

(v) \( a \in (aSaSa] \) for any \( a \in S \).
Lemma 2.7. The following statements are equivalent:

(a) \( S \) is left (right) regular ordered quasi-ideal of \( S \).

(b) For each \( a \in S \), there exists \( x \in S \) such that \( a \leq xxax \) (\( a \leq aax \)).

Note that \( S \) is left (right) regular if and only if for each \( a \in S \) there exist \( x, y \in S \) such that \( a \leq xyaa \) (\( a \leq aaxy \)).

Lemma 2.4. The following statements are equivalent:

(i) \( S \) is left (resp. right) regular,

(ii) \( A \subseteq (SAA) \) (resp. \( A \subseteq (AAS) \)) for any \( A \subseteq S \),

(iii) \( a \in (Saa) \) (resp. \( a \in (aaS) \)) for any \( a \in S \),

(iv) \( A \subseteq (SSAAA) \) (resp. \( A \subseteq (AAASS) \)) for any \( A \subseteq S \),

(v) \( a \in (SSaaa) \) (resp. \( a \in (aaaSS) \)) for any \( a \in S \).

Theorem 2.5. \( S \) is both left regular and right regular ordered ternary semigroup if and only if every ordered quasi-ideal of \( S \) is semiprime.

Proof. Let \( S \) be both left regular and right regular and \( \emptyset \neq A \subseteq S \). Let \( Q \) be an ordered quasi-ideal of \( S \) such that \( A^3 \subseteq Q \). By Lemma 2.4,

\[
A \subseteq (AAS) \subseteq (A(AAS)(S)) \subseteq (AAASS) \subseteq (QSS),
\]

\[
A \subseteq (SAA) \subseteq (S[SAA](A)) \subseteq (SSAAA) \subseteq (SQS),
\]

\[
A \subseteq (AAS) \subseteq ((SAA)(A)[S]) \subseteq (SAAAS) \subseteq (SQS).
\]

Hence, \( A \subseteq (QSS) \cap (SQS) \cap (SSQ) \subseteq Q \).

Conversely, assume that every ordered quasi-ideal of \( S \) is semiprime and \( \emptyset \neq A \subseteq S \). We have \( A^3 \subseteq Q(A^3) = (A^3 \cup SSA^3) \cap (A^3 \cup SA^3S) \cap (A^3 \cup A^3S) \cap (A^3 \cup SSA^3). \)

By assumption, \( A \subseteq (A^3 \cup SSA^3) \cap (A^3 \cup SA^3S) \cap (A^3 \cup A^3S) \subseteq (A^3 \cup SSA^3). \) Thus,

\[
A^3 \subseteq (AA(A^3 \cup SSA^3) \cup SSA^3) \subseteq ((A^3 \cup AAASSA^3) \cup (SSA^3)) \subseteq (SSA^3)
\]

and then \( A \subseteq (A^3 \cup SSA^3) \subseteq ((SSA^3) \cup SSA^3) \subseteq (SSA^3) \subseteq (SAA) \). Similarly, we have \( A \subseteq (AAS) \). By Lemma 2.4, \( S \) is both left regular and right regular ordered ternary semigroup.

Definition 2.6. An ordered ternary semigroup \( S \) is called intra-regular, i.e., for each \( a \in S \) there exist \( x, y \in S \) such that \( a \leq xaxy \).

Note that \( S \) is intra-regular if and only if for each \( a \in S \) there exist \( w, x, y, z \in S \) such that \( a \leq wxyz \).

Lemma 2.7. The following statements are equivalent:
Theorem 2.8. The following statements are equivalent:

(i) \( S \) is intra-regular,

(ii) \( L \cap X \cap R \subseteq (LXR) \) for any ordered left ideal \( L \), ordered right ideal \( R \) and \( \emptyset \neq X \subseteq S \).

Proof. \((\Rightarrow)\) : Let \( a \in L \cap X \cap R \). Since \( S \) is intra-regular, there exist \( w, x, y, z \in S \) such that \( a \leq wxaaayz \in LaR \subseteq LXR \subseteq (LXR) \). Hence, \( L \cap X \cap R \subseteq (LXR) \).

\((\Leftarrow)\) : Let \( a \in S \). By assumption and Lemma 1.1,

\[
\begin{align*}
a &\in L(a) \cap \{a\} \cap R(a) \subseteq (L(a)\{a\}R(a)) \subseteq ((a \cup SSA\{a\}(a \cup aSS)) \\
&\subseteq (a^3) \cup (a^3SS) \cup (SSa^3) \cup (SSS^3SS).
\end{align*}
\]

Case 1: \( a \in \{a^3\} \); \( a \leq aaaxxx \leq aaaaaaa \leq SSa^3SS \).

Case 2: \( a \in \{aaaSS\} \); there exist \( x, y \in S \), \( a \leq aaaxyy \leq (aaaxyy)xy \in SSa^3SS \).

Case 3: \( a \in \{SSaaa\} \); there exist \( x, y \in S \), \( a \leq xzxyyz \leq xzxyz \in SSa^3SS \).

Case 4: \( a \in \{SSaaaSS\} \); it is obvious. By Lemma 2.7, \( S \) is intra-regular.

Definition 2.9. An ordered ternary semigroup \( S \) is called completely regular, if it is regular, left regular and right regular.

Lemma 2.10. The following statements are equivalent:

(i) \( S \) is completely regular,

(ii) \( A \subseteq (A^3SASA^3) \) for any \( A \subseteq S \),

(iii) \( a \in (a^3SaSa^3) \) for any \( a \in S \).

Theorem 2.11. \( S \) is completely regular if and only if every ordered quasi-ideal of \( S \) is completely regular.

Proof. Assume that \( S \) is completely regular. Let \( Q \) be an ordered quasi-ideal of \( S \) and \( \emptyset \neq A \subseteq Q \). By Lemma 2.10,

\[
\begin{align*}
A \subseteq (A^3SASA^3) \subseteq ([A](A^3SASA^3)(ASASA)(A^3SASA^3)(A]) \\
\subseteq ([A](A^3SASA^3)(ASA)(A^3SASA^3)(A]) \\
\subseteq (A^3(ASASA)A(A(ASASAAS)ASA)A^3) \\
\subseteq (A^3(QSASQ)A(QSASQ)A^3) \\
\subseteq (A^3QAQA^3).
\end{align*}
\]

By Lemma 2.10, \( Q \) is completely regular.

The converse is clear because \( S \) itself is an ordered quasi-ideal.
Theorem 2.12. $S$ is completely regular if and only if every ordered bi-ideal of $S$ is semiprime.

Proof. Assume that $S$ is completely regular and $\emptyset \neq A \subseteq S$. Let $B$ be an ordered bi-ideal of $S$ and $A^3 \subseteq B$. By Lemma 2.10 and Lemma 2.4,

$$A \subseteq (A^3SASA) \subseteq (BSASB) \subseteq (BS(SSAAA)SB) \subseteq (BSBSB) \subseteq (B) = B.$$

Hence, every ordered bi-ideal of $S$ is semiprime.

Conversely, assume that every ordered bi-ideal of $S$ is semiprime. Let $\emptyset \neq A \subseteq S$. First we show that $(A^3SASA)$ is an ordered bi-ideal of $S$. Thus,

$$(A^3SASA)(A^3SASA)S(A^3SASA) \subseteq (A^3SASA)(SSASA)(SASA)(SASA)$$


Clearly, $(A^3SASA) = (A^3SASA)$. So, $(A^3SASA)$ is ordered bi-ideals of $S$. Since $A^3 \subseteq (A^3SASA)$, by assumption, $A^3 \subseteq (A^3SASA)$, and $A \subseteq (A^3SASA)$. By Lemma 2.10, $S$ is completely regular.

Now, we define the notions of a left lightly regularity and a right lightly regularity of an ordered ternary semigroups as follows.

Definition 2.13. An ordered ternary semigroup $S$ is called left (right) lightly regular, if each its element is left (light) lightly regular, i.e., for each $a \in S$ there exist $x, y, z \in S$ such that $a \leq xyza$ ($a \leq axyz$).

Lemma 2.14. The following statements are equivalent:

(i) $S$ is left (resp. right) lightly regular,

(ii) $A \subseteq (SSASA)$ (resp. $A \subseteq (ASASS)$) for any $A \subseteq S$,

(iii) $a \in (SSaSa)$ (resp. $a \in (aSaSS)$) for any $a \in S$.

Theorem 2.15. The following statements are equivalent:

(i) $S$ is left lightly regular,

(ii) $R \cap M \cap L \subseteq (SSRML)$ for any ordered left ideal $L$, ordered right ideal $R$ and ordered lateral ideal $M$ of $S$,

(iii) $L \subseteq (LSL)$ for any ordered left ideal $L$ of $S$,

(iv) $L \cap M \subseteq (LML)$ for any ordered left ideal $L$ and ordered lateral ideal $M$ of $S$. 

Proof. (i) $\Leftrightarrow$ (ii): Let $L$, $R$ and $M$ be an ordered left ideal, an ordered right ideal and an ordered lateral ideal of $S$, respectively and $a \in R \cap M \cap L$. Since $S$ is left lightly regular, there exist $x, y, z \in S$ such that $a \leq xyaza \leq xz(axyza) = xy(axa)z(a) \in SSRML$. Hence, $R \cap M \cap L \subseteq (SSRML)$.

Conversely, let $\emptyset \neq A \subseteq S$. Then $A \subseteq R(A) \cap M(A) \cap L(A)$. By assumption and Lemma 1.1,

$$A \subseteq R(A) \cap M(A) \cap L(A) \subseteq (SSR(A)M(A)L(A)) = ((S)[S][A \cup AS][A \cup SAS \cup SSASS][A \cup SSA]) \subseteq (S^2A^3 \cup S^2A^2S^2A \cup S^2ASASA \cup S^2ASAS^3A \cup S^2AS^2AS^2A$$

$$\cup S^2AS^2AS^4A \cup S^2AS^2A \cup S^2AS^2AS^3A \cup S^2AS^3AS^3A \cup S^2AS^4AS^2A \cup S^2AS^2A) \subseteq (SSASA).$$

By Lemma 2.14, $S$ is left lightly regular.

(i) $\Rightarrow$ (iv): Let $L$ and $M$ be an ordered left ideal and an ordered lateral ideal of $S$ and $a \in L \cap M$. Since $S$ is left lightly regular, there exist $x, y, z \in S$ such that $a \leq xyaza \leq xz(axyza) = xy(axa)z(a) \in LML$. Hence, $L \cap M \subseteq (LML)$.

(iv) $\Rightarrow$ (iii): It is clear because $S$ itself is an ordered lateral ideal of $S$.

(iii) $\Rightarrow$ (i): Let $a \in S$. Then $a \in L(a)$. By assumption and Lemma 1.1,

$$a \in L(a) \subseteq (L(a)L(a)) = ((a \cup S[a][a \cup SSA]) \subseteq (aS[a] \cup aSSa \cup SSaSSa) \subseteq (aS[a] \cup aSSa \cup SSaSSa).$$

Case 1: $a \in (aS[a])$: there exists $x \in S$, $a \leq axa \leq ax(axa) \in SSaSa$.

Case 2: $a \in (aSSa)$: there exist $x, y, z \in S$, $a \leq axya \leq axz(axya) \in SSaSa$.

Case 3: $a \in (SSaS[a])$: it is obvious.

Case 4: $a \in (SSaSSa)$: it is obvious, since $(SSaSSa) \subseteq (SSaSa)$.

Thus, $S$ is left lightly regular.

The next theorem can be similarly proved as Theorem 2.15.

**Theorem 2.16.** The following statements are equivalent:

(i) $S$ is right lightly regular,

(ii) $R \cap M \cap L \subseteq (RMLSS)$ for any ordered left ideal $L$, ordered right ideal $R$ and ordered lateral ideal $M$ of $S$,

(iii) $R \subseteq (RSR)$ for any ordered right ideal $L$ of $S$,

(iv) $R \cap M \subseteq (RMR)$ for any ordered right ideal $R$, ordered lateral ideal $M$ of $S$.  

[End of natural text]
Definition 2.17. An ordered ternary semigroup $S$ is called generalized regular, if each its element is generalized regular, i.e., for each $a \in S$ there exist $w, x, y, z$ such that $a \leq wxayz$.

Lemma 2.18. The following statements are equivalent:

(i) $S$ is generalized regular,

(ii) $A \subseteq (SSASS)$ for any $A \subseteq S$,

(iii) $a \in (SSaSS)$ for any $a \in S$.

Theorem 2.19. The following statements are equivalent:

(i) $S$ is generalized regular,

(ii) $L \subseteq (SSLSS)$ for any ordered left ideal $L$ of $S$,

(iii) $R \subseteq (SSRSS)$ for any ordered right ideal $R$ of $S$,

(iv) $M \subseteq (SSMSS)$ for any ordered lateral ideal $M$ of $S$,

(v) $I \subseteq (SSISS)$ for any ordered ideal $I$ of $S$.

Proof. $(i) \iff (v)$: Let $I$ be an ordered ideal of $S$. By Lemma 2.18, $I \subseteq (SSISS)$. Conversely, let $a \in S$. Then $a \in I(a)$. By assumption and Lemma 1.1,

$$a \in I(a) \subseteq (SSI(a)SS) \subseteq ((S)(S)[a \cup SSa \cup aSS \cup SaS \cup SSaSS](S)(S)$$

$$\subseteq (S^2aS^2 \cup S^4aS^2 \cup S^2aS^4 \cup S^2aS^3 \cup S^4aS^4)$$

$$= (S^2aS^2) \cup (S^4aS^2) \cup (S^2aS^4) \cup (S^2aS^3) \cup (S^4aS^4).$$

Case 1: $a \in (S^2aS^2)$; it is obvious.

Case 2: $a \in (S^4aS^2)$; it is obvious, since $(S^4aS^2) \subseteq (S^2aS^2)$.

Case 3: $a \in (S^2aS^4)$; it is obvious, since $(S^2aS^4) \subseteq (S^2aS^2)$.

Case 4: $a \in (S^3aS^3)$; there exist $u, v, w, x, y, z \in S$, $a \leq uwaxyz \leq uw(uwaxyz)xyz = (uw)(uw)a(xyz)(xyz) \in SSaSS$.

Case 5: $a \in (S^3aS^4)$; it is obvious, since $(S^3aS^4) \subseteq (S^2aS^2)$.

Thus, $S$ is generalized regular.

$(i) \iff (ii) \iff (iii) \iff (iv)$ Can be proved similarly.

3. Connections between regularities

The proof of following proposition is not difficult.

Proposition 3.1. Let $S$ be an ordered ternary semigroup.

(i) If $S$ is completely regular, then it is regular, left regular and right regular.
(ii) If $S$ is left or right regular, then it is intr-regular.

(iii) If $S$ is left (resp. right) regular, then it is left (resp. right) lightly regular.

(iv) If $S$ is regular, then it is left and right lightly regular.

(v) If $S$ is intr-regular or left lightly regular or right lightly regular, then it is generalized regular.

Now, we give examples to show that the converses statements are not true.

**Example 3.2.** Let $S = \{a, b, c, d\}$. A ternary operation $[\ ]$ on $S$ and the figure of a partial order relation $\leq$ on $S$ are as follows:

$$
\begin{array}{cccc|cccc|cccc}
 & a & b & c & d & a & b & c & d & a & b & c & d \\
\hline
aa & a & a & a & a & ba & b & b & b & cb & a & a & a \\
ab & a & a & a & d & bb & b & b & b & cc & a & a & a \\
ac & a & a & a & d & bc & b & b & b & cd & d & d & d \\
ad & d & d & d & d & bd & d & d & d & cd & d & d & d \\
\end{array}
$$

It is clear that $a, b, d$ are left lightly regular. Since $c \notin (SScS) = S$, $S$ is left lightly regular. However, $S$ is neither regular nor right lightly regular because $c \notin (cSc) = \{a, d\} = (cScS)$.

**Example 3.3.** Let $S = \{a, b, c, d\}$. A ternary operation $[\ ]$ on $S$ and the figure of a partial order relation $\leq$ on $S$ are as follows:

$$
\begin{array}{cccc|cccc|cccc}
 & a & b & c & d & a & b & c & d & a & b & c & d \\
\hline
aa & a & b & a & a & ba & a & b & a & cb & a & b & a \\
ab & a & b & a & d & bb & a & b & a & cc & a & b & a \\
ac & a & b & a & d & bc & a & b & a & cd & a & b & a \\
ad & d & d & d & d & bd & d & d & d & cd & d & d & d \\
\end{array}
$$

It is clear that $a, b, d$ are right lightly regular. Since $c \notin (cScS) = S$, $S$ is right lightly regular. However, $S$ is neither regular nor left lightly regular because $c \notin (cSc) = \{a, d\} = (SScS)$. 
Example 3.4. Let $S = \{a, b, c, d, e, f\}$. A ternary operation $[\ ]$ on $S$ is as follows:

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Define a partial order relation $\leq$ on $S$ by $\leq := \{(x, x) \mid x \in S\}$. It is clear $a, b, c, d, e$ are regular. Since $f \in (fSf) = \{a, e, f\}$, $S$ is regular. So, $S$ is left lightly regular. However, $S$ is neither right regular nor intra-regular because $f \notin (Sff) = \{a, e\} = (SSf)^3SS$.

Example 3.5. Let $S = \{a, b, c, d, e, f\}$. A ternary operation $[\ ]$ on $S$ is as follows:

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Example 3.6. Let $S = \{a, b, c, d, e\}$. A ternary operation $[\ ]$ on $S$ and the figure of a partial order relation $\leq$ on $S$ are as follows:
It is clear \(b, c, d, e\) are left regular. Since \(a \in (Saa) = \{a, b, c, e\}\), \(S\) is left regular. So, \(S\) is intra-regular and generalized regular. However, \(S\) is neither right lightly regular nor regular because \(a \notin (aSaSa) = \{b, e\} = (aSa)\).

Example 3.7. Let \(S = \{a, b, c, d, e\}\). A ternary operation \([\ ]\) on \(S\) and the figure of a partial order relation \(\leq\) on \(S\) are as follows:

\[\begin{array}{cc}
aa & ab \\
ba & bb \\
ca & ac \\
db & ad \\
ae & ae \\
\end{array}\]

\[\begin{array}{cc}
aa & ba \\
bb & bb \\
cc & ac \\
dd & ad \\
ea & ae \\
\end{array}\]

\[\begin{array}{cc}
aa & ca \\
ba & bb \\
cc & ab \\
dd & ae \\
ea & ee \\
\end{array}\]

\[\begin{array}{cc}
ba & bb \\
bb & bb \\
cc & ac \\
dx & ae \\
ea & ee \\
\end{array}\]

It is clear \(b, c, d, e\) are right regular. Since \(a \in (SaS) = \{a, b, c, e\}\), \(S\) is right regular. So, \(S\) is intra-regular and generalized regular. However, \(S\) is neither left lightly regular nor regular because \(a \notin (SSaSa) = \{b, e\} = (SSa)\).

Example 3.8. Let \(S = \{a, b, c, d\}\). A ternary operation \([\ ]\) on \(S\) and the figure of a partial order relation \(\leq\) on \(S\) are as follows:

\[\begin{array}{cc}
aa & ba \\
ab & aa \\
ac & ab \\
ad & ad \\
ae & ae \\
\end{array}\]

\[\begin{array}{cc}
ba & bb \\
bb & bb \\
cc & ac \\
dd & ad \\
ea & ae \\
\end{array}\]

\[\begin{array}{cc}
aa & ca \\
bb & bb \\
cc & ab \\
dx & ae \\
ea & ee \\
\end{array}\]

\[\begin{array}{cc}
aa & ba \\
bb & bb \\
cc & ac \\
dx & ae \\
ea & ee \\
\end{array}\]

It is clear \(b, c, d, e\) are right regular. Since \(a \in (aaS) = \{a, b, c, e\}\), \(S\) is left regular. So, \(S\) is intra-regular and generalized regular. However, \(S\) is neither left lightly regular nor regular because \(a \notin (SSaSa) = \{b, e\} = (SSa)\).
It is clear $a, b, d$ are generalized regular. Since $c \in (SScSS) = \{a, b, c\}$, $S$ is generalized regular. $S$ is not intra-regular because $c \notin (SSc^3SS) = \{a, b\}$.

**Example 3.9.** Let $S = \{a, b, c, d, e, f, g\}$. A ternary operation $[\cdot]$ on $S$ and the figure of a partial order relation $\leq$ on $S$ are as follows:
It is clear $a, b, d, e, f, g$ are left regular. Since $c \in (Scc) = \{a, b, c, e, f\}$, $S$ is left regular. Similarly, $a, b, d, e, f, g$ are right regular and $c \in (ccS) = \{a, b, c, d, e\}$, $S$ is right regular. However, $S$ is not regular because $c \notin (cSc) = \{a, b, e\}$.

Now, we conclude the connections of the eight regularities as the figure.

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References


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