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Proper Magic Sigma Coloring of Specific Graphs

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Abstract

A coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ is called a *magic sigma coloring* of G if the sum of colors of all the vertices in the neighborhood of each vertex of G is the same. A graph that admits such a coloring is said to be *magic sigma colorable*. The minimum number k required in a magic sigma coloring of a graph G is called the *magic sigma chromatic number*, denoted by $\sigma_m(G)$. These concepts have been extensively studied and motivate us to define a new type of coloring as follows.

A coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ is called a *proper magic sigma coloring* of G if it is both a magic sigma coloring and a proper vertex coloring. The minimum number k required for a proper magic sigma coloring of G is called the *proper magic sigma chromatic number*, denoted by $\sigma_{p,m}(G)$.

In this work, we introduce the concept of the proper magic sigma chromatic number and study its properties. Additionally, we determine $\sigma_{p,m}(G)$ for several specific graphs.

Keywords: Graph coloring, proper magic sigma coloring, proper magic sigma chromatic number.

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1 Introduction

This section presents definitions and previous results that are essential in this work.

Note that throughout the paper, every graph G with the vertex $V(G)$ is a nontrivial, simple, and undirected graph. Graph coloring is a widely studied and researched topic in graph theory. Among various

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types of vertex colorings, one notable concept is **sigma coloring**, introduced by Chartrand et al. in 2010 [2]. The **neighborhood** of a vertex v, denoted by $N(v)$, is the set of vertices that are adjacent to v. A vertex coloring $\varphi : V(G) \to \mathbb{N}$ of a nontrivial connected graph G is called a **sigma coloring** if $\sigma_\varphi(u) \neq \sigma_\varphi(v)$ for every pair of adjacent vertices $u, v \in V(G)$, where $\sigma_\varphi(v) = \sum_{w \in N(v)} \varphi(w)$ is the **neighborhood sum** of v, i.e., the sum of the colors of all vertices in the neighborhood $N(v)$ of v. The minimum number of colors required for a sigma coloring of G is called the **sigma chromatic number** of G, denoted by $\sigma(G)$.

In [2], it was proven that the sigma chromatic number of a graph G is bounded by its chromatic number $\chi(G)$. Furthermore, sigma chromatic numbers were determined for complete graphs, cycles, and complete r-partite graphs for $r \geq 2$. Many exact values of sigma chromatic numbers have been obtained, and extensive studies have been conducted on the sigma coloring of specific families of graphs (see, for instance, [3]–[6], [8], [9]).

In contrast, some researchers have explored whether there exists a coloring φ for a graph such that all vertices share the same neighborhood sum. In 2021, N. Narasimha Swamy et al. [7] introduced the concept of **magic sigma coloring** to investigate this idea. A coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ is called a **magic sigma coloring** of G if the sum of the colors of all vertices in the neighborhood of each vertex in G is the same. If a graph G admits such a coloring, it is said to be **magic sigma colorable**, and we define $\sigma_\varphi(G) = \sigma_\varphi(v)$ for any $v \in V(G)$. The least positive integer k required in a magic sigma coloring of G is called the **magic sigma chromatic number**, denoted by $\sigma_m(G)$. Note that a magic sigma coloring need not be surjective, that is, some colors in $\{1, 2, \ldots, k\}$ may not be used.

Properties of magic sigma colorings were determined for various graphs, including non-trivial trees, regular graphs, wheels, complete multipartite graphs, and powers of paths [7]. Since a magic sigma coloring is not necessarily proper, this inspired us to define the **proper magic sigma coloring** and study its analogous properties. This paper characterizes proper magic sigma colorable graphs such as paths, cycles, wheels, non-trivial trees, and complete multipartite graphs. Furthermore, we determine the proper magic sigma chromatic numbers for these graphs.

For definitions of graph-related terminology not mentioned in this paper, we refer the reader to [1].

Definition 1. *Let v be a vertex of a graph G. Consider a graph coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$. A* ***neighborhood sum*** *of a vertex v is the sum of $\varphi(u)$ for all $u \in N(v)$. If all vertices of G have the same neighborhood sum, then we call φ a* ***magic sigma coloring*** *of G. Furthermore, a common neighborhood sum of vertices in G is called* ***neighborhood sum*** *of G, denoted by $\sigma_\varphi(G)$.*

Definition 2. *A coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ is a* ***proper magic sigma coloring*** *of G if it is a magic sigma coloring and a proper vertex coloring. If a graph G admits a proper magic sigma coloring, then G is* ***proper magic sigma colorable****.*

Definition 3. *Let G be a proper magic sigma colorable graph. The least positive integer k such that a proper magic sigma coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ exists for G is called the* ***proper magic sigma chromatic number*** *of G, denoted by $\sigma_{p,m}(G)$.*

Definition 4. *([5]) Vertices u and v with $N(u) \cup \{u\} = N(v) \cup \{v\}$ are called* ***strong twins****.*

From the definition, any strong twins u and v are adjacent.

Definition 5. *([7]) Let G be a graph with a vertex coloring φ. The* ***φ-color sum*** *of G, denoted by $S_\varphi(G)$, is the sum of the colors of all the vertices in G, i.e., $S_\varphi(G) = \sum_{v \in V(G)} \varphi(v)$.*

2 Proper Magic Sigma Coloring of Some Graphs

We begin this section with some fundamental results relating to the magic sigma colorability of a graph and the proper magic sigma colorability of a graph. Further, we explore some graphs that are not proper magic sigma colorable.

It is straightforward to observe that a connected graph with exactly two vertices is not proper magic sigma colorable, as the two vertices must have distinct colors, resulting in different neighborhood sums. Additionally, a graph with one vertex is trivial. Therefore, we only consider graphs with at least three vertices from this point onward.

Lemma 1. *([7]) If a graph G contains two vertices such that the neighborhood of one vertex is a proper subset of the neighborhood of the other, then G is not magic sigma colorable.*

Lemma 2. *If a graph G contains vertices u and v that are strong twins, then G is not proper magic sigma colorable.*

Proof. We suppose to the contrary that G has a proper magic sigma coloring φ. By the definition of magic sigma coloring, we have $\sigma_\varphi(u) = \left(\sum_{x\in N(u)\setminus\{v\}} \varphi(x)\right) + \varphi(v)$ and $\sigma_\varphi(v) = \left(\sum_{x\in N(v)\setminus\{u\}} \varphi(x)\right) + \varphi(u)$. Since u and v are strong twins, it follows that

$$\sum_{x\in N(u)\setminus\{v\}} \varphi(x) = \sum_{x\in N(v)\setminus\{u\}} \varphi(x).$$

This implies $\varphi(u) = \varphi(v)$, contrary to φ is a proper coloring. Hence, G is not proper magic sigma colorable. $\square$

Lemma 3. *Let G be a connected graph with $|V(G)| \geq 3$ and a pendant vertex. Then G contains a path of length 3 if and only if it is not proper magic sigma colorable.*

Proof. Necessity. Let v be a pendant vertex of G, and assume that G has a path of length 3. Since G is connected, G contains a path of length 3 with an endpoint v, say $vwxu$. Since v is a pendant vertex, $N(v) \subsetneq N(x)$. It follows from Lemma 1 that G is not magic sigma colorable, and consequently, it is not proper magic sigma colorable.

Sufficiency. We prove the contrapositive. Assume that G has no paths of length 3. It follows that $\mathrm{diam}(G) \leq 2$ and G is acyclic. Consequently, G is a star graph $K_{1,n}$, where $n \geq 2$. A proper magic sigma coloring of G can be easily obtained. $\square$

Corollary 1. *A connected graph G with diam$(G) \geq 3$ and having a pendant vertex is not proper magic sigma colorable.*

We have the following result as a direct consequence of Lemma 3.

Corollary 2. *A non-trivial tree T with $|V(T)| \geq 3$ is proper magic sigma colorable if and only if T is $K_{1,n}$, where $n \geq 2$.*

Corollary 3. *Let $n \geq 3$. The path P_n is proper magic sigma colorable if and only if $n = 3$.*

Proof. Necessity. Using Lemma 3 to complete the case $n \geq 4$.

Sufficiency. Give a coloring $1, 2, 1$ to vertices of P_3 to obtain a proper magic sigma coloring. □

Observation 1. *K_n is not proper magic sigma colorable when $n \geq 3$.*

Proof. Since any two vertices of K_n for $n \geq 3$ are strong twins, we use Lemma 2 to complete the proof. □

Observation 2. *The graph P_n^k, the k-th power of a path with $n \geq 3$ and $k \geq 2$, is not proper magic sigma colorable.*

Proof. Consider the graph $G := P_n^k$ with vertices $v_1, v_2, \ldots, v_n$ in the usual arrangement.

We prove that P_n^k is not proper magic sigma colorable by considering the following three cases.

Case 1: $n \leq k + 1$.

In this case, we have that $P_n^k \cong K_n$. By Observation 1, the result follows.

Case 2: $n = k + 2$. Here, adjacent vertices v_2 and v_3 are strong twins. By Lemma 2, the result follows.

Case 3: $n \geq k + 3$. In this case, $N(v_1) = \{v_2, v_3, \ldots, v_{k+1}\}$ is a proper subset of $N(v_{k+2})$. By Lemma 1, the result follows. □

Observation 3. *Let a fan graph F_n be the join of K_1 and P_n. Then F_n is not proper magic sigma colorable for all n.*

Proof. Let $v_0 \in V(K_1)$ and $v_1, v_2, \ldots, v_n$ be the vertices of P_n in the usual order.

Case 1: If $n = 1$, then $F_n \cong P_2$ which is not proper magic sigma colorable.

Case 2: If $n = 2$, then $F_n \cong K_3$. By Observation 1, the result follows.

Case 3: If $n = 3$, then vertices v_0 and v_2 are strong twins. By Lemma 2, the result follows.

Case 4: For $n \geq 4$, we observe that $N(v_1)$ is a proper subset of $N(v_3)$. By Lemma 1, the result follows. □

Lemma 4. *The cycle C_n is proper magic sigma colorable if and only if n is divisible by 4.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n in the usual arrangement.

Necessity. Assume that C_n has a proper magic sigma coloring φ, and let $n = 4t + r$, where t, r are integers, with $1 \leqslant r \leqslant 3$. Define $\varphi(v_1) = a$, $\varphi(v_2) = b$, and $\varphi(v_3) = c$, where $a \neq b$ and $b \neq c$.

Since $\sigma_\varphi(v_3) = \sigma_\varphi(v_2) = a+c$, we have $\varphi(v_4) = a+c-b$. Continuing this reasoning, the vertices $v_1, v_2, \ldots, v_{4t}$ are colored periodically as $a, b, c, a+c-b, \ldots, a, b, c, a+c-b$.

Case 1: $r = 1$. Then $\varphi(v_n) = a$. Since v_1 and v_n are adjacent, this leads to a contradiction.

Case 2: $r = 2$. Here, $\varphi(v_{n-1}) = a$ and $\varphi(v_n) = b$. It follows that $\sigma_\varphi(v_n) = 2a$ and $\sigma_\varphi(v_1) = 2b$ which leads to $a = b$, a contradiction.

Case 3: $r = 3$. We have $\varphi(v_{n-2}) = a$ and $\varphi(v_{n-1}) = b$. Thus, $\sigma_\varphi(v_n) = a + b$, and since $\sigma_\varphi(v_2) = a + c$, it follows that $b = c$, a contradiction.

Sufficiency. Assume $n = 4t$, where t is an integer. Assign the coloring φ to vertices $v_1, v_2, \ldots, v_{4t}$ as $1, 2, 3, 2, \ldots, 1, 2, 3, 2$. This coloring is a proper magic sigma coloring since any two adjacent vertices have distinct colors and $\sigma_\varphi(C_n) = 4$. □

Lemma 5. *For $n \geq 5$, the wheel graph W_n is proper magic sigma colorable if and only if $n - 1$ is divisible by 4.*

Proof. Let W_n be the wheel graph on n vertices, where v_n is the central vertex, and the vertices $v_1, v_2, \ldots, v_{n-1}$ form the cycle.

Necessity. Assume that W_n has a proper magic sigma coloring φ, and let $n-1 = 4t+r$, where t and r are integers with $1 \leqslant r \leqslant 3$. Using an argument similar to the proof of Lemma 4, we color the vertices $v_1, v_2, \ldots, v_{4t}$ as $a, b, c, a+c-b, \ldots, a, b, c, a+c-b$.

Case 1: $r = 1$. Then $\varphi(v_{n-1}) = a$, but $\varphi(v_1) = a$, leading to a contradiction.

Case 2: $r = 2$. We have $\varphi(v_{n-2}) = a$ and $\varphi(v_{n-1}) = b$. Since $\sigma_\varphi(v_{n-1}) = \sigma_\varphi(v_1)$, it follows that $2a = 2b$, a contradiction.

Case 3: $r = 3$. Here, $\varphi(v_{n-3}) = a$ and $\varphi(v_{n-2}) = b$. Then $\sigma_\varphi(v_{n-1}) = a + b + \varphi(v_n)$, but $\sigma_\varphi(v_2) = a + c + \varphi(v_n)$, which leads to $b = c$, a contradiction.

Sufficiency. Now assume $n = 4t+1$, where t is a positive integer. We assign the coloring to $v_1, v_2, \ldots, v_{n-1}$ as $= 1, 2, 3, 2, \ldots, 1, 2, 3, 2$ and set $\varphi(v_n) = 4(2t-1)$. This coloring is a proper magic sigma coloring with $\sigma(W_n) = 8t$.

□

Observation 4. *Let φ be a vertex coloring (not necessarily proper) of the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with the partite sets $X_1, X_2, \ldots$, X_k, where $k \geq 2$. The graph $K_{n_1,n_2,\ldots,n_k}$ has a magic sigma coloring φ if and only if*

$$\sum_{u \in X_i} \varphi(u) = \sum_{v \in X_j} \varphi(v)$$

for any two distinct partite sets X_i and X_j.

Proof. Necessity. Suppose $G := K_{n_1,n_2,\ldots,n_k}$ has a magic sigma coloring φ. Let X_i and X_j be two distinct partite sets with $x \in X_i$ and $y \in X_j$. Since x and y have the same neighborhood sum, it follows that $\sum_{v \notin X_i} \varphi(v) = \sum_{v \notin X_j} \varphi(v)$, so that $S_\varphi(G) - \sum_{v \in X_i} \varphi(v) = S_\varphi(G) - \sum_{v \in X_j} \varphi(v)$. Thus, $\sum_{v \in X_i} \varphi(v) = \sum_{v \in X_j} \varphi(v)$.

Sufficiency. Suppose $\sum_{u \in X_i} \varphi(u) = \sum_{v \in X_j} \varphi(v) := \alpha$ for any two distinct partite sets. Thus, $\sigma_\varphi(v) = \alpha(k-1)$ for each vertex $v \in V(K_{n_1,n_2,\ldots,n_k})$. □

Lemma 6. *For positive integers m and n that are not simultaneously 1, the graph $K_{m,n}$ is proper magic sigma colorable.*

Proof. Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be the partite sets of $K_{m,n}$.

Case 1: $m = n$. We define the coloring φ_1 as:

$$\varphi_1(x_i) = 1 \text{ for } 1 \le i \le m-1, \varphi_1(x_m) = m+1, \varphi_1(y_j) = 2 \text{ for } 1 \le j \le n.$$

Since $m + 1 \ge 3$ and $\sigma_{\varphi_1}(x_i) = \sigma_{\varphi_1}(y_j) = 2m$ for $1 \le i \le m$ and $1 \le j \le n$, this is a proper magic sigma coloring.

Case 2: $m < n$. Let $n = mt + r$, where $t \ge 1$ and $0 \le r \le m - 1$.

Case 2.1: $r = 0$. We define the coloring φ_2 as:

$$\varphi_2(x_i) = t \text{ for } 1 \le i \le m, \quad \varphi_2(y_j) = 1 \text{ for } 1 \le j \le n.$$

Since $m < n$, it follows that $t > 1$, and thus, φ is a proper magic sigma coloring with $\sigma_{\varphi_2}(K_{m,n}) = n$.

Case 2.2: $1 \le r \le m - 1$. We define the coloring φ_3 as: $\varphi_3(x_i) = 1$ for $1 \le i \le m-1$, $\varphi_3(x_m) = m(2t-1)+2r+1$, and $\varphi_3(y_j) = 2$ for $1 \le j \le n$. Since $m(2t - 1) + 2r + 1 \ge 4$ and $\sigma_{\varphi_3}(x_i) = \sigma_{\varphi_3}(y_j) = 2n$ for $1 \le i \le m$ and $1 \le j \le n$, this is a proper magic sigma coloring.

□

Theorem 1. *Let $n_1 \le n_2 \le \cdots \le n_k$. The complete k-partite graph $K_{n_1,n_2,\dots,n_k}$ is proper magic sigma colorable if and only if $n_2 \ge 2$.*

Proof. Consider the complete k-partite graph $K_{n_1,n_2,\dots,n_k}$, where the partite sets are $X_1 = \{v_{1,1}, v_{1,2}, \dots, v_{1,n_1}\}$, $X_2 = \{v_{2,1}, v_{2,2}, \dots, v_{2,n_2}\}$, $\dots, X_k = \{v_{k,1}, v_{k,2}, \dots, v_{k,n_k}\}$.

Suppose $K_{n_1,n_2,\dots,n_k}$ is proper magic sigma colorable. By Lemma 2, there can be at most one partite set of size 1, otherwise, the graph would contain strong twins. Thus, $n_2 \ge 2$.

Let $\varphi(v_{1,j}) = (2k+1)(n_k - 1)+1$, where $1 \le j \le n_1$. For $2 \le i \le k$, define a coloring as:

$$\varphi(v_{i,j}) = \begin{cases} i, & \text{if } 1 \le j \le n_i - 1, \\ n_1 \left[(2k+1)(n_k - 1) + 1\right] - i(n_i - 1), & \text{if } j = n_i. \end{cases}$$

We verify the proper colorability of φ as follows:

1. If $i_1 = 1$, $1 \leq j_1 \leq n_{i_1}$ and $2 \leq i_2 \leq k$, $1 \leq j_2 \leq n_{i_2} - 1$, then $\varphi(v_{i_1,j_1}) = (2k+1)(n_k - 1) + 1 \geq 2k + 2$ and $\varphi(v_{i_2,j_2}) = i_2$. Thus, $\varphi(v_{i_1,j_1}) > \varphi(v_{i_2,j_2})$.

2. If $i_1 = 1$, $1 \leq j_1 \leq n_{i_1}$ and $2 \leq i_2 \leq k$, $j_2 = n_{i_2}$, then $\varphi(v_{i_1,j_1}) = (2k+1)(n_k - 1) + 1$ and $\varphi(v_{i_2,j_2}) = n_1(2k+1)(n_k - 1) + 1 - i_2(n_{i_2} - 1)$. Thus, if $n_1 = 1$, then $\varphi(v_{i_1,j_1}) > \varphi(v_{i_2,j_2})$, otherwise $\varphi(v_{i_1,j_1}) < \varphi(v_{i_2,j_2})$.

3. If $2 \leq i_1 < i_2$, $1 \leq j_1 \leq n_{i_1} - 1$, and $1 \leq j_2 \leq n_{i_2} - 1$, then $\varphi(v_{i_1,j_1}) = i_1 < i_2 = \varphi(v_{i_2,j_2})$.

4. If $i_1 \geq 2$, $j_1 = n_{i_1}$, and $i_2 \geq 2$, $1 \leq j_2 \leq n_{i_2} - 1$, then $\varphi(v_{i_1,j_1}) = n_1\left[(2k+1)(n_k - 1) + 1\right] - i_1(n_{i_1} - 1) > k + 2$ and $\varphi(v_{i_2,j_2}) = i_2$. Thus, $\varphi(v_{i_1,j_1}) > \varphi(v_{i_2,j_2})$.

5. If $2 \leq i_1 < i_2$, $j_1 = n_{i_1}$, and $j_2 = n_{i_2}$, then $\varphi(v_{i_1,j_1}) = n_1\left[(2k+1)(n_k - 1) + 1\right] - i_1(n_{i_1} - 1)$ and $\varphi(v_{i_2,j_2}) = n_1[(2k+1)(n_k - 1) + 1] - i_2(n_{i_2} - 1)$. Thus, $\varphi(v_{i_1,j_1}) > \varphi(v_{i_2,j_2})$.

Altogether, φ is a proper coloring with

$$\sum_{u \in X_i} \varphi(u) = n_1\left[(2k+1)(n_k - 1) + 1\right]$$

for each partite set X_i, ensuring magic sigma colorability by Observation 4. $\square$

Lemma 7. *For any graphs G and H, the proper magic sigma colorability of the join $G+H$ implies the proper magic sigma colorability of both G and H.*

Proof. Let φ be a proper magic sigma coloring of $G + H$. We show that the restriction of φ to G is a proper magic sigma coloring of G.

Consider adjacent vertices x and y in G. Since $\sigma_\varphi(x) = \sigma_\varphi(y)$ in $G + H$, we have

$$\left(\sum_{u \in V(G) \cap N(x)} \varphi(u)\right) + S_\varphi(H) = \left(\sum_{v \in V(G) \cap N(y)} \varphi(v)\right) + S_\varphi(H),$$

which simplifies to: $\sum_{u \in V(G) \cap N(x)} \varphi(u) = \sum_{v \in V(G) \cap N(y)} \varphi(v)$, implying that $\sigma_\varphi(x)$ under the restriction of G and $\sigma_\varphi(y)$ under the restriction of G are equal.

Since $\varphi(x) \neq \varphi(y)$ for any two adjacent vertices x and y in G (otherwise, it would contradict the proper colorability of $G + H$), it follows that φ under the restriction of G is proper magic sigma coloring of G.

The proper magic sigma colorability of H follows similarly by applying the same reasoning. $\square$

Remark 1. *It is always true that a graph G is magic sigma colorable if and only if its complement $\overline{G}$ is magic sigma colorable [7].*

However, it is not necessarily true that if a graph G is proper magic sigma colorable, then its complement $\overline{G}$ is also proper magic sigma colorable. For example, the graph $K_{2,3}$ is proper magic sigma colorable, but its complement, which is the disjoint union of P_2 and K_3, is not proper magic sigma colorable.

3 Proper Magic Sigma Chromatic Numbers of Some Graphs

In this section, we determine the proper magic sigma chromatic numbers for some fundamental graphs.

Remark 2. *If a graph G is proper magic sigma colorable, then $\sigma_{p,m}(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G.*

Theorem 2. *For the cycle graph C_n, if n is divisible by 4, then $\sigma_{p,m}(C_n) = 3$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n in the usual arrangement.

Assume $n = 4t$, where t is an integer. By the proof of Lemma 4, we have $\sigma_{p,m}(C_n) \leq 3$.

To establish that $\sigma_{p,m}(C_n) \geq 3$, we suppose to the contrary that there exists a proper magic sigma coloring $\varphi : V(C_n) \to \{1, 2\}$. Without loss of generality, assume $\varphi(v_1) = 1$. Then we must have $\varphi(v_2) = 2$, $\varphi(v_3) = 1$, and $\varphi(v_4) = 2$. However, this leads to a contradiction, as

$\sigma_\varphi(v_2) = 2$ and $\sigma_\varphi(v_3) = 4$, which are not equal. Thus, $\sigma_{p,m}(C_n) \geq 3$, and combining both bounds, we conclude that $\sigma_{p,m}(C_n) = 3$. $\square$

Theorem 3. *For the wheel graph W_n, if $n-1$ is divisible by 4, then $\sigma_{p,m}(W_n) = 4(2t-1)$.*

Proof. Let W_n be the wheel graph on n vertices $v_1, v_2, \ldots, v_n$, with v_n as the central vertex and the vertices $v_1, v_2, \ldots, v_{n-1}$ forming the cycle.

Let $n-1 = 4t$, where t is a positive integer. By the proof of Lemma 5, we have $\sigma_{p,m}(W_n) \leq 4(2t-1)$.

To establish that $\sigma_{p,m}(W_n) \geq 4(2t-1)$, following similar arguments as in the proof of Lemma 5, we assign the coloring $a, b, c, a+c-b, \ldots, a, b, c, a+c-b$ to the vertices $v_1, v_2, \ldots, v_{4t}$, and assign $(2t-1)(a+c)$ to v_n.

To show that $a+c \geq 4$, suppose to the contrary that $a+c < 4$. If $a+c = 2$, then $\varphi(v_1)+\varphi(v_3) = \varphi(v_2)+\varphi(v_4) = 2$, which implies $\varphi(v_1) = \varphi(v_2) = \varphi(v_3) = \varphi(v_4) = 1$, a contradiction. If $a+c = 3$, then $\varphi(v_1)+\varphi(v_3) = \varphi(v_2)+\varphi(v_4) = 3$, which implies $\{\varphi(v_1), \varphi(v_3)\} = \{1, 2\}$ and $\{\varphi(v_2), \varphi(v_4)\} = \{1, 2\}$, also a contradiction.

Thus, $\varphi(v_n) = (2t-1)(a+c) \geq 4(2t-1)$, and therefore, $\sigma_{p,m}(W_n) \geq 4(2t-1)$. Since we know $\sigma_{p,m}(W_n) \leq 4(2t-1)$ from the proof of Lemma 5, we conclude that $\sigma_{p,m}(W_n) = 4(2t-1)$. $\square$

Theorems 4-6 determine the proper magic sigma chromatic numbers for all complete bipartite graphs that are proper magic sigma colorable.

Theorem 4. *Let $n \geq 2$. Then*

$$\sigma_{p,m}(K_{n,n}) = \begin{cases} 3, & \text{if } n \text{ is even,} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $V(K_{n,n})$ be the vertex set of $K_{n,n}$ with partite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Case 1: n is even. To show $\sigma_{p,m}(K_{n,n}) \geq 3$, suppose to the contrary that there exists a proper magic sigma coloring $\varphi : V(K_{n,n}) \to \{1, 2\}$. We may assume that $\varphi(x_i) = 1$ and $\varphi(y_i) = 2$ for each $i = 1, 2, \ldots, n$. This leads to $\sum_{i=1}^{n} \varphi(x_i) < \sum_{j=1}^{n} \varphi(y_j)$, a contradiction. Thus, $\sigma_{p,m}(K_{n,n}) \geq 3$.

To show $\sigma_{p,m}(K_{n,n}) \leq 3$, we define a coloring $\tilde{\varphi} : V(K_{n,n}) \to \{1,2,3\}$ as follows:

$$\tilde{\varphi}(x_i) = \begin{cases} 1, & \text{for } i = 1,2,\ldots,\frac{n}{2}, \\ 3, & \text{for } i = \frac{n}{2}+1,\ldots,n, \end{cases}$$

and $\tilde{\varphi}(y_j) = 2$ for each $j = 1,2,\ldots,n$. It is straightforward to verify this is a proper magic sigma coloring with $\sigma_{\tilde{\varphi}}(K_{n,n}) = 2n$. Thus, $\sigma_{p,m}(K_{n,n}) = 3$ when n is even.

Case 2: n is odd. To show $\sigma_{p,m}(K_{n,n}) \geq 4$, suppose to the contrary that there exists a proper magic sigma coloring $\varphi : V(K_{n,n}) \to \{1,2,3\}$. If $\varphi(x_i) = 1$ for each $i = 1,2,\ldots,n$, then $\sum_{i=1}^{n} \varphi(x_i) = n < \sum_{j=1}^{n} \varphi(y_j)$, a contradiction. If $\varphi(x_i) = 2$ for each $i = 1,2,\ldots,n$, then $\sum_{i=1}^{n} \varphi(x_i) = 2n$, but $\sum_{j=1}^{n} \varphi(y_j)$ is odd, a contradiction. If $\varphi(x_i) = 3$ for each $i = 1,2,\ldots,n$, then $\sum_{i=1}^{n} \varphi(x_i) = 3n > \sum_{j=1}^{n} \varphi(y_j)$, a contradiction.

To show $\sigma_{p,m}(K_{n,n}) \leq 4$, we define a coloring $\tilde{\varphi} : V(K_{n,n}) \to \{1,2,3,4\}$ as follows:

$$\tilde{\varphi}(x_i) = \begin{cases} 1, & \text{for } i = 1,2,\ldots,\frac{n+1}{2}, \\ 4, & \text{for } i = \frac{n+1}{2}+1,\ldots,n, \end{cases}$$

and

$$\tilde{\varphi}(y_j) = \begin{cases} 2, & \text{for } j = 1,2,\ldots,\frac{n+1}{2}+1, \\ 3, & \text{for } j = \frac{n+1}{2}+2,\ldots,n. \end{cases}$$

It is straightforward to verify that this is a proper magic sigma coloring with $\sigma_{\tilde{\varphi}}(K_{n,n}) = \frac{5n-3}{2}$. Thus, $\sigma_{p,m}(K_{n,n}) = 4$ when n is odd. $\square$

Theorem 5. *Let $n = mt + r$, where $t > 1$ and $0 \leq r < m$ are integers. Then*

$$\sigma_{p,m}(K_{m,n}) = \begin{cases} t, & \text{if } r = 0, \\ t+1, & \text{if } 1 \leq r < m. \end{cases}$$

Proof. Let $V(K_{m,n})$ be the vertex set of $K_{m,n}$, with partite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Case 1: $r = 0$. In this case, $n = mt$. By the proof of Lemma 6, we know that $\sigma_{p,m}(K_{m,n}) \leq t$.

Next, we prove the upper bound. By Observation 4,

$$\sum_{i=1}^{m} \varphi(x_i) = \sum_{j=1}^{n} \varphi(y_j).$$

Since $\varphi(y_j) \geq 1$ for $j = 1, 2, \ldots, n$, it follows that $\sum_{j=1}^{n} \varphi(y_j) \geq n$. By the Pigeonhole principle, there must exist a vertex x_i such that $\varphi(x_i) \geq \lceil \frac{n}{m} \rceil = t$. Therefore, $\sigma_{p,m}(K_{m,n}) \geq t$. We conclude that $\sigma_{p,m}(K_{m,n}) = t$.

Case 2: $1 \leq r < m$. To show that $\sigma_{p,m}(K_{m,n}) \leq t + 1$, we define a coloring $\varphi : V(K_{m,n}) \to \{1, 2, \ldots, t, t+1\}$ as: $\varphi(x_i) = t + 1$ for $i = 1, 2, \ldots, r$, $\varphi(x_i) = t$ for $i = r + 1, r + 2, \ldots, m$, and $\varphi(y_j) = 1$ for $j = 1, 2, \ldots, n$. This is a proper magic sigma coloring with $\sigma_\varphi(K_{m,n}) = n$.

Next, we prove the upper bound. Similar to the previous case, there must exist a vertex x_i such that $\varphi(x_i) \geq \lceil \frac{n}{m} \rceil = t + 1$. Therefore, $\sigma_{p,m}(K_{m,n}) \geq t + 1$. We conclude that $\sigma_{p,m}(K_{m,n}) = t + 1$.

□

Theorem 6. *Let $n = m + r$, where r is an integer and $1 \leq r < m$. Then*

$$\sigma_{p,m}(K_{m,n}) = \begin{cases} 4, & \text{if } m, n \text{ are odd and } m > 2r, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Let $V(K_{m,n})$ be the vertex set of $K_{m,n}$, with partite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. We obtain an upper bound of each $\sigma_{p,m}(K_{m,n})$ in the following four cases.

Case 1: n is even.

Define the coloring $\varphi : V(K_{m,n}) \to \{1, 2, 3\}$ as: $\varphi(x_i) = 2$ for $1 \leq i \leq m$, $\varphi(y_j) = 1$ for $1 \leq j \leq r + \frac{n}{2}$, and $\varphi(y_j) = 3$ for $r + \frac{n}{2} + 1 \leq j \leq n$.

Note that $0 \leq r + \frac{n}{2} \leq n$. Thus, φ is well-defined, and it is straightforward to verify that this is a proper magic sigma coloring with $\sigma_\varphi(K_{m,n}) = 2m$.

Case 2: n is odd and m is even.

Case 2.1: $m \geq 2r$. Define the coloring $\varphi : V(K_{m,n}) \to \{1, 2, 3\}$ as:

$\varphi(x_i) = 1$ for $1 \leq i \leq \frac{m}{2} - r$, $\varphi(x_i) = 3$ for $\frac{m}{2} - r + 1 \leq i \leq m$, and $\varphi(y_j) = 2$ for $1 \leq j \leq n$.

Note that $\varphi(x_i) = 3$ for each i if $m = 2r$. This coloring is a proper magic sigma coloring with $\sigma_\varphi(K_{m,n}) = 2n$.

Case 2.2: $m < 2r$. Define the coloring $\varphi : V(K_{m,n}) \to \{1, 2, 3\}$ as: $\varphi(x_i) = 3$ for $1 \leq i \leq m$, $\varphi(y_j) = 1$ for $1 \leq j \leq 2r - m$, and $\varphi(y_j) = 2$ for $2r - m + 1 \leq j \leq n$. This coloring is a proper magic sigma coloring with $\sigma_\varphi(K_{m,n}) = 3m$.

Case 3: m, n are odd, and $m < 2r$.

A proper magic sigma coloring with $\sigma_\varphi(K_{m,n}) = 3m$ can be defined as in Case 2.2.

Case 4: m, n are odd, and $m > 2r$.

Define the coloring $\varphi : V(K_{m,n}) \to \{1, 2, 3, 4\}$ as: $\varphi(x_i) = 3$ for $1 \leq i \leq m$, $\varphi(y_j) = 1$ for $1 \leq j \leq 3$, $\varphi(y_j) = 4$ for $4 \leq j \leq \frac{m-2r+9}{2}$, and $\varphi(y_j) = 2$ for $\frac{m-2r+11}{2} \leq j \leq n$.

It is straightforward to verify this is a proper magic sigma coloring with $\sigma_\varphi(K_{m,n}) = 3m$.

Next, we show a lower bound of $\sigma_{p,m}(K_{m,n})$ to complete the proof.

To show that $\sigma_{p,m}(K_{m,n}) \geq 3$, suppose $\sigma_{p,m}(K_{m,n}) = 2$. If $\varphi(x_i) = 2$ for $1 \leq i \leq m$ and $\varphi(y_j) = 1$ for $1 \leq j \leq n$, then $n = 2m$, leading to $m = r$, which is a contradiction. If $\varphi(x_i) = 1$ and $\varphi(y_j) = 2$, then $m = 2n$, which is also a contradiction.

To show $\sigma_{p,m}(K_{m,n}) \geq 4$ for $K_{m,n}$ in Case 4, suppose to the contrary that there exists a proper magic sigma coloring $\varphi : V(K_{m,n}) \to \{1, 2, 3\}$.

- If $\varphi(x_i) = 1$ for $1 \leq i \leq m$, then $\sum_{i=1}^{m} \varphi(x_i) = m < \sum_{j=1}^{n} \varphi(y_j)$, a contradiction.

- If $\varphi(x_i) = 2$ for $1 \leq i \leq m$, then $\sum_{i=1}^{m} \varphi(x_i) = 2m$, but $\sum_{j=1}^{n} \varphi(y_j)$ is odd, a contradiction.

- If $\varphi(x_i) = 3$ for $1 \leq i \leq m$, then $\sum_{i=1}^{m} \varphi(x_i) = 3m$. Additionally, $\sum_{j=1}^{n} \varphi(y_j) \leq 2n$. Since $m > 2r$, it follows that $\sum_{i=1}^{m} \varphi(x_i) > \sum_{j=1}^{n} \varphi(y_j)$, a contradiction.

- If $\varphi(y_j) = 1$ for $1 \leq j \leq n$, then $\sum_{j=1}^{n} \varphi(y_j) = n$ and $\sum_{i=1}^{m} \varphi(x_i) \geq 2m$. But $2m > n$, a contradiction.

- If $\varphi(y_j) = 2$ for $1 \leq j \leq n$, then $\sum_{j=1}^{n} \varphi(y_j) = 2n$, but $\sum_{i=1}^{m} \varphi(x_i)$ is odd, a contradiction.

- If $\varphi(y_j) = 3$ for $1 \leq j \leq n$, then $\sum_{j=1}^{n} \varphi(y_j) = 3n > \sum_{i=1}^{m} \varphi(x_i)$, a contradiction.

This completes the proof. □

4 Conclusion

Properties of magic sigma coloring for various graphs, including non-trivial trees, regular graphs, wheels, complete multipartite graphs, and powers of paths, have been studied extensively. Since this type of coloring is not necessarily proper, we were motivated to introduce the concept of proper magic sigma coloring and investigate analogous properties for this variation. We provide characterizations of paths, cycles, wheels, non-trivial trees, and complete multipartite graphs that are proper magic sigma colorable. Additionally, we determine the proper magic sigma chromatic numbers for several of these proper magic sigma colorable graphs.

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