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On the trees with maximum Cardinality-Redundance number

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Abstract

A vertex v is said to be over-dominated by a set S if $|N[u] \cap S| \geq 2$. The cardinality–redundance of S, $CR(S)$, is the number of vertices of G that are over-dominated by S. The cardinality–redundance of G, $CR(G)$, is the minimum of $CR(S)$ taken over all dominating sets S. A dominating set S with $CR(S) = CR(G)$ is called a $CR(G)$-set. In this paper, we prove an upper bound for the cardinality–redundance in trees in terms of the order and the number of leaves, and characterize all trees achieving equality for the proposed bound.

Keywords: Dominating set, Cardinality-Redundance, trees.
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1 Introduction

We consider here undirected and simple graphs $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is given by $n = n(G) = |V(G)|$. The *open neighborhood* $N(v)$ of a vertex v is the set of vertices that are adjacent to v, and the *close neighborhood* $N[v]$ is $N(v) \cup v$. For any subset $S \subseteq V(G)$, denote $N(A) = \cup_{v \in A} N(v)$ and $N[A] = \cup_{v \in A} N[v]$. The *degree* of v is the cardinality of $N(v)$, denoted by $\deg(v)$. A vertex v is said to be a *leaf* if $\deg(v) = 1$. A vertex is a *support* vertex if it is adjacent to a leaf. We denote by $L(G)$ and $S(G)$ the collections of all leaves and support vertices of G, respectively. We also denote by $L(v)$ the leaves adjacent to v. A *star* is the graph $K_{1,k}$, where $k \geq 1$. Further if $k > 1$, the vertex of degree k is called the *center* vertex of the star, while if $k = 1$, arbitrarily designate either

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vertex of P_2 as the center. A *double star* is a tree with precisely two vertices of degree at least two, namely the centers of the double star. We denote by $S(a,b)$ a double star in which the centers have degrees a and b. We call a double star *strong* if at least one of its centers has degree at least three. A *bipartite graph* is a graph G that the vertex set can be partitioned into two sets X and Y such that any edge of G has one end-point in X and the other end-point in Y. The *diameter*, $diam(G)$, of a graph G is the maximum distance among all pairs of vertices in G. A *diametrical path* in G is a shortest path whose length is equal to the diameter of G. A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the parent of v is the neighbor of v on the unique (r,v)-path, while a child of v is any other neighbor of v. If T is a rooted tree, then for any vertex v, we denote by T_v the sub-rooted tree rooted at v.

A set $S \subseteq V$ of vertices in a graph is called a *dominating set*, if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set among all dominating sets of G. For fundamentals of domination theory in graphs, we refer the reader to the so-called domination books by Haynes, Hedetniemi, and Slater [2], [3].

Johnson and Slater [4] introduced the concept of cardinality–redundance in graphs. A vertex v is said to be *over-dominated* by a set S if $|N[u] \cap S| \geq 2$. The *cardinality–redundance* of S, $CR(S)$, is the number of vertices of G that are over-dominated by S. The cardinality–redundance of G, $CR(G)$, is the minimum of $CR(S)$ taken over all dominating sets S. A dominating set S with $CR(S) = CR(G)$ is called a $CR(G)$-set. The concept of cardinality–redundance was further studied in, for example, [6], in which the authors presented several extremal Problems Related to the Cardinality–redundance on graphs G with $CR(G) = 0, 1, 2$.

In this paper, we prove an upper bound for the cardinality–redundance in trees in terms of the order and the number of leaves, and characterize all trees achieving equality for the proposed bound.

2 Main result

In this section, we prove an upper bound for the cardinality-redundance of a tree in terms of the order and the number of leaves, and characterize trees achieving equality of the given bound. For this purpose, we first introduce the following family of trees $\mathcal{T}$. Let $\mathcal{T}$ be the family of all trees T that can be obtained from a sequence $T_0, T_1, ..., T_{k-1}, T_k$, where T_0 is a strong double star, and if $k \geq 1$, then T_i is obtained from T_{i-1} by the following operation for each $i = 1, 2, ..., k-1$:

Operation $\mathcal{O}$: Add a strong double star and join one of its centers to a support vertex of T_{i-1}.

To prove our main result, we need a series of lemmas.

Lemma 1. *If $T' \in \mathcal{T}$ and T is obtained from T' by Operation $\mathcal{O}$, then $CR(T) = CR(T') + 1$.*

Proof. Let $T' \in \mathcal{T}$ and T be obtained from T' by adding a $S(a, b)$ with $\deg(b) \geq 3$, and joining a to a support vertex c of T'. If S' is a $CR(T')$-set, then $S = S' \cup \{b\} \cup L(a)$ is a dominating set for T, and $CR(S) = CR(S') + 1$, since a is over-dominated by S. Thus, $CR(T) \leq CR(S) = CR(S') + 1$. Now let S be a $CR(T)$-set. Since S is a dominating set for T, we find that $|S \cap V(S(a, b))| \geq 2$. Assume that $c \in S$. If $a \notin S$, then $L(a) \subseteq S$ and a is over-dominated by S. Then $S' = S - V(S(a, b))$ is a dominating set for T' with $CR(S') \leq CR(S) - 1$, and thus, $CR(T') \leq CR(T) - 1$. Thus, assume that $a \in S$. Then both a and b are over-dominated by S, since $S \cap (\{b\} \cup L(b)) \neq \emptyset$. Now $S' = S - V(S(a, b))$ is a dominating set for T' with $CR(S') \leq CR(S) - 2$, and thus, $CR(T') \leq CR(T) - 2$.

We next assume that $c \notin S$. Then $L(c) \subseteq S$. If $a \notin S$, then $L(a) \subseteq S$ and at least one vertex of $S(a, b)$ is over-dominated by S, since $S(a, b)$ is a strong double star. Then $S' = S - V(S(a, b))$ is a dominating set for T' with $CR(S') \leq CR(S) - 1$, and thus, $CR(T') \leq CR(T) - 1$. Thus, assume that $a \in S$. Then b is over-dominated by S, since $S \cap (\{b\} \cup L(b)) \neq \emptyset$. Then $S' = S - V(S(a, b))$ is a dominating set for T' with $CR(S') \leq CR(S) - 1$, and thus, $CR(T') \leq CR(T) - 1$. We conclude that $CR(T) = CR(T') + 1$. $\square$

The following is an immediate consequence of the definition of the family of $\mathcal{T}$.

Lemma 2. *If $T \in \mathcal{T}$ has n vertices and ℓ leaves, then:*
(1) $V(T) = L(T) \cup S(T)$.
(2) $CR(T) = \frac{n-\ell}{2}$.

Proof. (1) is obvious.
(2) Let X and Y be the partite sets of T. Without lost of generally let

$$|S(T) \cap X| \leq |S(T) \cap Y|.$$

Clearly, Y is a dominating set for T and

$$CR(Y) = |S(T) \cap X|.$$

Thus,

$$CR(G) \leq CR(Y) = |S(T) \cap X| \leq \frac{|S(T)|}{2} \leq \frac{n-\ell}{2}.$$

We next prove that $CR(T) \geq \frac{n-\ell}{2}$. Note that T is obtained from a sequence $T_0, T_1, ..., T_{k-1}, T_k$, where T_0 is a strong double star, and if $k \geq 1$, then T_i is obtained from T_{i-1} by the Operation $\mathcal{O}$, for each $i = 1, 2, ..., k-1$. We use an induction on k (the number of times that the Operation $\mathcal{O}$ is performed to construct T). For the base step $k = 0$, it is clear that $CR(T) = 1 = \frac{n-\ell}{2}$. Assume the result is true for any tree $T' \in \mathcal{T}$ arisen by applying $k' < k$ operations. Now consider the tree T, and let $T' = T_{k-1}$. Assume that T is obtained from T' by adding a strong double star $S(a, b)$ and joining a to a support vertex c of T'. By the inductive hypothesis, $CR(T') \geq \frac{n'-\ell'}{2}$, where $n' = n(T')$ and $\ell' = \ell(T')$. By Lemma 1, $CR(T) = CR(T') + 1$. Then,

$$\begin{aligned} CR(T) &= CR(T') + 1 \\ &\geq \frac{n'-\ell'}{2} + 1 \\ &= \frac{n - (\deg(a)-1) - \deg(b) - (\ell - (\deg(a)-2) - (\deg(b)-1))}{2} \\ &+ 1 = \frac{n-\ell}{2}, \end{aligned}$$

as desired. □

Lemma 3. *If $T \in \mathcal{T}$ has n vertices and ℓ leaves, and S is a $CR(T)$-set, then:*
(1) S contains precisely half of members of $S(T)$.
(2) For each vertex $x \in S(T)$, if $x \in S$, then $L(x) \cap S = \emptyset$ and if $x \notin S$, then $L(x) \subseteq S$.
(3) If X and Y are partite sets of T, then $CR(X) = CR(Y) = \frac{n-\ell}{2}$.

Proof. (1) and (2) are obvious.
(3). We prove this by an induction on the number of times that the Operation $\mathcal{O}$ is performed to construct T. The result is obvious if T is a strong double star. Assume the result holds if T is obtained by applying Operation $\mathcal{O}$, $k' < k$ times, and now T is obtained from a sequence $T_0, T_1, ..., T_{k-1}, T_k$, where T_0 is a strong double star and T_i is obtained from T_{i-1} by the Operation $\mathcal{O}$, for each $i = 1, 2, ..., k-1$. Let X_{k-1} and Y_{k-1} be partite sets of T_{k-1}, and let T is obtained by adding the center a of a strong double star $S(a, b)$ to a support vertex c of T_{k-1}, and without loss of generality, assume that $c \in X$. By Lemma 1, $CR(T) = CR(T_{k-1}) + 1$. By the inductive hypothesis, $CR(X_{k-1}) = CR(Y_{k-1}) = \frac{n(T_{k-1}) - \ell(T_{k-1})}{2}$. Let $X_k = X_{k-1} \cup \{b\} \cup L(a)$, and $Y_k = Y_{k-1} \cup \{a\} \cup L(b)$. Then $X_k = X_{k-1} \cup \{b\} \cup L(a)$ is a dominating set for T with $CR(X_k) = CR(X_{k-1}) + 1 = \frac{n(T_{k-1} - \ell(T_{k-1})}{2} + 1 = \frac{n-\ell}{2}$, since a is over-dominated by X_k. Similarly, from $CR(Y_{k-1}) = \frac{n(T_{k-1}) - \ell(T_{k-1})}{2}$ we obtain that $Y_k = Y_{k-1} \cup \{a\} \cup L(b)$ is a dominating set for T with

$$CR(Y_k) = CR(Y_{k-1}) + 1 = \frac{n(T_{k-1}) - \ell(T_{k-1})}{2} + 1 = \frac{n - \ell}{2},$$

since b is over-dominated by Y_k and c is a support vertex of T_{k-1}. $\square$

The following is a direct consequence of Lemma 3, Part (3).

Corollary 1. *If $T \in \mathcal{T}$ has n vertices and ℓ leaves, then for any vertex x, there is a $CR(T)$-set S with $CR(S) = \frac{n-\ell}{2}$ and $x \notin S$.*

We are now ready to present the main result of this paper.

Theorem 1. *If T is a tree of order $n \geq 3$ with $\ell = \ell(T)$ leaves, then $CR(T) \leq \frac{n-\ell}{2}$, with equality if and only if $n = 2$ or $T \in \mathcal{T}$.*

Proof. The result is obvious for $n = 2$; thus, assume that $n \geq 3$. We prove by induction on n that $CR(T) \leq (n-\ell)/2$, and if equality holds, then $T \in \mathcal{T}$. We root T at a leaf x_0 of a diametrical path $P_0 : x_0, x_1, ..., x_d$, where d is the diameter of T. For the base step of the induction, we assume that $d = 1$. Then T is a star, and it is evident that $CR(T) = 0 < \frac{n-\ell}{2}$. If $d = 3$, then T is a double star $S(a, b)$. If $\deg(a) = \deg(b) = 2$, then the leaves of T form a dominating set implying that $CR(T) = 0 < \frac{n-\ell}{2}$. Thus, assume that $\deg(b) \geq 3$. Then $CR(T) = 1 = \frac{n-\ell}{2}$. We thus assume that $d \geq 4$.

Assume that $\deg(x_{d-2}) = 2$. Let $T' = T - T_{x_{d-2}}$. By the inductive hypothesis,

$$CR(T') \leq \frac{n'-\ell'}{2} = \frac{n - (\deg(x_{d-1}) + 1) - \ell'}{2}.$$

Observe that $\ell - (\deg(x_{d-1}) - 1) \leq \ell' \leq \ell - (\deg(x_{d-1}) - 1) + 1$. Thus,

$$\begin{aligned} CR(T') &\leq \frac{n'-\ell'}{2} \leq \frac{n - (\deg(x_{d-1}) + 1) - (\ell - (\deg(x_{d-1}) - 1))}{2} \\ &= \frac{n-\ell-2}{2}. \end{aligned}$$

If $CR(T') < \frac{n'-\ell'}{2}$ and S' is a $CR(T')$-set, then $S = S' \cup \{x_{d-1}\}$ is a dominating set for T with $CR(S) \leq CR(S') + 1$. Then $CR(T) \leq CR(S) < \frac{n'-\ell'}{2} + 1 = \frac{n-\ell-2}{2} + 1 = \frac{n-\ell}{2}$. Thus, assume that $CR(T') = \frac{n'-\ell'}{2}$. By the inductive hypothesis, $T' \in \mathcal{T}$. By Corollary 1, there is a $CR(T')$-set S' with $CR(S') = \frac{n'-\ell'}{2}$ and $x_{d-3} \notin S'$. Then $S = S' \cup \{x_{d-1}\}$ is a dominating set for T with $CR(S) = CR(S')$. Thus,

$$CR(T) \leq CR(S) = CR(S') = \frac{n'-\ell'}{2} = \frac{n-\ell-2}{2} < \frac{n-\ell}{2}.$$

We thus assume that $\deg(x_{d-2}) \geq 3$. Note that x_{d-1} is a child of x_{d-2} which is a support vertex. Suppose that x_{d-2} has at least two children which are support vertices. Let x_{d-1}, z_1,...,z_k be the children of x_{d-2} which are support vertices, where $k \geq 1$. Let $T' = T - T_{x_{d-2}}$.

By the inductive hypothesis,

$$\begin{aligned} CR(T') &\leq \frac{n'-\ell'}{2} \\ &= \frac{n-\deg(x_{d-1})-\sum_{i=1}^{k}\deg(z_i)-1-|L(x_{d-2})|-\ell'}{2}. \end{aligned}$$

Observe that $\ell' \geq \ell - (\deg(x_{d-1})-1) + \sum_{i=1}^{k}(\deg(z_i)-1) - |L(x_{d-2})|$. Thus, since $k \geq 1$, we obtain that

$$CR(T') \leq \frac{n'-\ell'}{2} \leq \frac{n-\ell-2-k}{2} \leq \frac{n-\ell-3}{2}.$$

Let S' be a $CR(T')$-set. Then $S = S' \cup L(x_{d-2}) \cup \{x_{d-1}, z_1, ..., z_k\}$ is a dominating set for T with $CR(S) = CR(S') + 1$. Thus,

$$CR(T) \leq CR(S) = CR(S') + 1 \leq \frac{n-\ell-3}{2} + 1 < \frac{n-\ell}{2}.$$

We thus assume that x_{d-1} is the only child of x_{d-2} which is a support vertex. Then $T_{x_{d-2}}$ is a double star. Let $T' = T - T_{x_{d-2}}$. By the inductive hypothesis,

$$CR(T') \leq \frac{n'-\ell'}{2} = \frac{n-(\deg(x_{d-2}-2)-(\deg(x_{d-1}+1)-\ell'}{2}.$$

Observe that $\ell' \geq \ell - \deg(x_{d-1}) - \deg(x_{d-2}) + 3$. Thus,

$$\begin{aligned} CR(T') &\leq \frac{n'-\ell'}{2} \leq \frac{n-(\deg(x_{d-2}-2)-(\deg(x_{d-1}+1)-\ell'}{2} \\ &\leq \frac{n-\ell-2}{2}. \end{aligned}$$

Let S' be a $CR(T')$-set. Then $S = S' \cup L(x_{d-2}) \cup \{x_{d-1}\}$ is a dominating set for T with $CR(S) = CR(S') + 1$. Thus,

$$CR(T) \leq CR(S) = CR(S') + 1 \leq \frac{n-\ell-2}{2} + 1 \leq \frac{n-\ell}{2}. \quad (1)$$

We thus proved that $CR(T) \leq \frac{n-\ell}{2}$. Assume the equality holds. Following the above proof, we find that if $d = 3$, then T is a strong double star that belongs to $\mathcal{T}$, or equality in (1) holds. If $CR(T') < \frac{n'-\ell'}{2}$, then considering $S = S' \cup L(x_{d-2}) \cup \{x_{d-1}\}$, we find that $CR(T) \leq CR(S) = CR(S') + 1 < \frac{n-\ell-2}{2} + 1 \leq \frac{n-\ell}{2}$, a contradiction. Thus, $CR(T') = \frac{n'-\ell'}{2}$. By the inductive hypothesis, $T' \in \mathcal{T}$. If $\deg(x_{d-3}) = 2$, then $\ell' = \ell - (\deg(x_{d-1}) - 1) - (\deg(x_{d-2}) - 2) + 1$, and so $CR(T') \leq \frac{n-\ell-3}{2}$, and we obtain that $CR(T) < \frac{n-\ell}{2}$, a contradiction. Thus, $\deg(x_{d-3}) \geq 3$, that is, x_{d-3} is a support vertex of T'. Thus, T is obtained from T' by Operation $\mathcal{O}$. Consequently, $T \in \mathcal{T}$.

The converse follows by Lemma 2, Part (2). □

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