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A Sumudu based algorithm for solving differential equations

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Abstract

An algorithm based on Sumudu transform is developed. The algorithm can be implemented in computer algebra systems like Maple. It can be used to solve differential equations of the following form automatically without human interaction

$$\sum_{i=0}^{m} p_i(x)y^{(i)}(x) = \sum_{j=0}^{k} q_j(x)h_j(x)$$

where $p_i(x)$ $(i = 0, 1, ..., m)$ and $q_j(x)$ $(j = 0, 1, ...k)$ are polynomials. $h_j(x)$ are non-rational functions, but their Sumudu transforms are rational. m, k are nonnegative integers.

Keywords: algorithm, differential equations, coefficient, Sumudu transform.

1 Introduction

The author exploited transforms for calculating Taylor's coefficients [1-2]. In this paper, an algorithm based on Sumudu transform for calculating Taylor's coefficients is presented and applied to solve differential equations of the following form

$$\sum_{i=0}^{m} p_i(x)y^{(i)}(x) = \sum_{j=0}^{k} q_j(x)h_j(x) \tag{1}$$

where $p_i(x)$ $(i = 0, 1..., m)$ and $q_j(x)$ $(j = 0, 1, ...k)$ are polynomials. $h_j(x)$ are non-rational functions, and their Sumudu transforms are rational. m, k are nonnegative integers.

For a given function $f(x)$, quite a few theorems exist about how to find the coefficient a_n of a general term $a_n x^n$ in the expansion, which we shall denote $[x^n]f(x)$ or $[x^n]f$. Under some conditions, we have Taylor's formula:

$$a_n = [x^n]f(x) = \frac{f^{(n)}(0)}{n!}. \tag{2}$$

This is a very nice formula and can be quite useful in finding a specific term such as $[x^3]f(x)$. However, for an arbitrary number n (usually considered to be very large), we cannot use the formula directly to determine $[x^n]f(x)$.
If $f(x)$ is a rational function, we can apply the methods in [8-10] to calculate $[x^n]f(x)$. If $f(x)$ is a non-rational function, it is difficult in practice to calculate $[x^n]f(x)$. This work is to provide an approach to calculate Maclaurin coefficients for non rational functions automatically based on Sumudu transform, and then applied to solve equation (1).

2 Sumudu Transform

There are numerous integral based transforms, one was studied and named Sumudu transform by Watugala [7] in 1993. Since then, many works dedicated on Sumudu transform were done by F. B. M. Belgacem and others recently[3-7].

Assume that function f is a function of x. We write the Sumudu transform as

$$F(z) = S[f(x)] = \int_0^\infty \frac{1}{z} e^{-x/z} f(x) dx. \tag{3}$$

We shall refer to $f(x)$ as the original function of $F(z)$ and $F(z)$ as the Sumudu transform of the function $f(x)$. We also refer to $f(x)$ as the inverse Sumudu transform of $F(z)$. The symbol S denotes the Sumudu transform. The function $\frac{1}{z}e^{-x/z}$ is called the kernel of the transform.

The following are a few basic properties of the Sumudu transform [3],

(1) *Linearity*

$$S[c_1 f(x) + c_2 g(x)] = c_1 S[f(x)] + c_2 S[g(x)]. \quad (4)$$

(2) *Convolution*

$$S[(f * g)(x)] = zS[f(x)] * S[g(x)]. \quad (5)$$

(3) *Laplace − Sumudu Duality*

$$L[f(x)] = S[f(1/x)]/z, F[f(x)] = L[f(1/x)]/z. \quad (6)$$

(4) *Derivative*

$$S[f^{(m)}(x)] = S[f(x])/z^m - f(0)/z^m - ... - f^{m-1}(0)/z. \quad (7)$$

A very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except a factor $n!$. This fact is illustrated by the following theorems:

Theorem 2.1 [11]: If

1. $f(x)$ is bounded and continuous,
2. $F(z) = S[f(x)]$, and,
3. $F(z) = \sum_{n=0}^{\infty} a_n z^n$,

then we have

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}. \quad (8)$$

Theorem 2.2 [4]: The Sumudu transform amplifies the coefficients of the power series function,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{9}$$

by sending it to the power series function,

$$S[f(z)] = \sum_{n=0}^{\infty} n! a_n z^n. \tag{10}$$

On another hand side, generating functions are a very important technic in study discrete mathematics. For a given sequence a_n, there are two classic generating functions:

$$g(x) = \sum_{n=0}^{\infty} a_n x^n, l(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n. \tag{11}$$

The first one is called the ordinary generating function of the sequence a_n and the second one is its exponential generating function. Theorem 2.1 and theorem 2.2 are inverse to each other and give a complete relationship about coefficients under transform. Based on theorem 2.1 and theorem 2.2, we have another interesting fact about Sumudu transform:

Proposition 2.3 Sumudu transform of an exponential generating function is its ordinary generating function; the inverse Sumudu transform of an ordinary generating function is its exponential generating function.
These theorems can serve as a base to calculate the general term of a Taylor series expansion.

3 Calculation theory

Theorem 3.1: Assume f has a continuous derivative of order n in some open interval I containing 0, and define $E_n(x)$ for x in I by the

equation
$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k+E_n(x).$$

Then $E_n(x)$ is called the Lagrange Remainder of order n, given by the integral
$$E_n(x)=\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}f^n(t)dt,$$

$E_n(x)$ can be written in another form,
$$E_n(x)=\frac{f^{(n)}(c)}{(n)!}x^n.$$

for some c between 0 and x.

Then we have the following:
$$[x^n]f(x)=[x^n]E_n(x).$$

Assume f and g are functions of x, a and b are constants, d is a nonnegative integer, then,

Proposition 3.2 (Linear Pairs)

$$\begin{aligned}[x^n](af+bg) &= a([x^n]f)+b([x^n]g),\\ [z^n](S[af+bg]) &= a([z^n]S[f])+b([z^n]S[g]).\end{aligned}$$

The first formula of linear pairs can be obtained from theorem 3.1 directly, and the second one from the linear property of Sumudu transform.

Theorem 3.3: Assume that the power series $\sum_{n=0}^{\infty}a_nx^n$ converges for each $x\in B(0,R)$. Then the function f defined by the equation
$$f(x)=\sum_{n=0}^{\infty}a_nx^n, \qquad x\in B(0,R),$$

has a continuous derivative $f'(x)$ for each x, $|x| < R$, given by

$$f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}. \qquad x \in B(0, R).$$

It is important to know that the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} na_n x^{n-1}$ have the same radius of convergence, as can be shown by applying the Root Test, which can be found in many books.

Proposition 3.4 (Derivative Pairs)

$$\begin{aligned} [x^n]f^{(d)} &= ((n+d)(n+d-1)...(n+1))([x^{n+d}]f), \\ [z^n]S[f^{(d)}] &= [z^{n+d}]S[f]. \end{aligned}$$

The first formula in derivative pairs can be derived by applying theorem 3.3 repeatedly for d times. The second can be easily verified by applying the Sumudu derivative property above. Obviously, we assume d is a positive integer and $n > d$ in this situation.

Suppose $p(z)$ and $g(z)$ have Taylor series expansions

$$\begin{aligned} p(z) &= p_0 + p_1 z + p_2 z^2 + \cdots + p_l z^l \\ g(z) &= b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots \end{aligned}$$

where $p_l \neq 0$ and l is a non-negative integer.

Proposition 3.5: If $f(z) = p(z)g(z)$, we can compute $[z^n]f(z)$ by comparing coefficients,

$$[z^n]f(z) = \sum_{k=0}^{m} p_k b_{n-k}. \tag{12}$$

4 The algorithm and implementation

The purpose of this paper is to solve the equations of form (1). In this paper, we assume that $h_j(x)$ are non-rational functions, but their

Sumudu transforms $S[h_j]$ are rational. There are a lot of such functions [1]. For example: $e^x, sin(x), e^x cos(x), sin^{(40)}(x)$.

Assume form (1) has a solution which can be expanded to

$$y(x) = r_0 + r_1 x + ... + r_n x^n + ... \tag{13}$$

Based on the theorems and propositions above, by comparing coefficients, we can calculate the coefficient of the general term r_n for the solution $y(x)$ as following:

(1) Calculate $c_i = [x^n]p_i y^{(i)}$ by proposition 3.4 and 3.5. for $i = 0, ..., m$.
(2) Add $s = \sum_{i=0}^{m} c_i$.
(3) Calculate the Sumudu transform $S[h_j(x)]$, for $j = 0, ..., k$.
(4) Calculate the $t_j = [x^n]q_j h_j$ by theorem 2.1 and proposition 3.5 .
(5) Add $t = \sum_{j=0}^{k} t_j$.
(6) Calculate some small r_i as initial values by comparing coefficients.
(7) Solve the linear difference equation $s = t$,with initial values from (6), return r_n.

Example Let's look at the following example. Assume

$$\begin{aligned} p_0(x) &= 1 + x, \\ p_1(x) &= x, \\ q_0(x) &= 1 + x, \\ q_1(x) &= 1 + x, \\ h_0(x) &= sinh(x), \\ h_1(x) &= cosh(x), \\ p_1 y' + p_0 y &= q_1 h_1 + q_0 h_0. \end{aligned}$$

We are going to trace the algorithm with this example.

Step (1): Calculate

$$\begin{aligned} c_0 &= [x^n]p_0 y = r_n + r_{n-1}, \\ c_1 &= [x^n]p_1 y' = n r_n, \end{aligned}$$

Step (2): Add

$$s = c_1 + c_0 = (n+1)r_n + r_{n-1},$$

Step (3): Calculate the Sumudu transforms:

$$\begin{aligned} S[h_0] &= \frac{x}{1-x^2} = \sum_{n=0}^{\infty} \frac{1-(-1)^n}{2n!}, \\ S[h_1] &= \frac{1}{1-x^2} = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2n!} \end{aligned}$$

Step (4): Calculate

$$\begin{aligned} t_0 &= [x^n]q_0 h_0 = \frac{1-(-1)^n}{2n!} + \frac{1-(-1)^{(n-1)}}{2(n-1)!}, \\ t_1 &= [x^n]q_1 h_1 = \frac{1+(-1)^n}{2n!} + \frac{1+(-1)^{(n-1)}}{2(n-1)!} \end{aligned}$$

Step (5): Add:

$$t = t_1 + t_0 = \frac{1}{n!} + \frac{1}{(n-1)!}$$

Step (6): Calculate

$$r_0 = 1$$

Step (7): Solve the recurrence equation with initial value from step 6.

$$\begin{aligned} (n+1)r_n + r_{n-1} &= \frac{1}{n!} + \frac{1}{(n-1)!} \\ r_n &= \frac{3+2n+(-1)^n}{4(n+1)!} \end{aligned}$$

So, we have:

$$y(x) = \sum_{n=0}^{\infty} \frac{3 + 2n + (-1)^n}{4(n+1)!} x^n$$

In Step 3, to calculate the Sumudu transform, one can calculate the Laplace transform which was implemented in most algebra systems, then use Laplace-Sumudu Duality to get the Sumudu transform. The Sumudu transforms are rational functions, so their Taylor's expansions can be easy calculated by the methods in [8-10]. In Step 6, the number of initial values needed is the order of the recurrence relation $s = t$. In Step 7, we need to solve a linear recurrence relation with polynomial coefficients of n, we can apply the algorithms in [12] to do the job. The rest of the Steps are straightforward.

Noticed that the algorithm can be implemented in computer algebra systems like Maple to solve such equations automatically.

5 Conclusions

A very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except a factor $n!$. As for generating functions, Sumudu transform of an exponential generating function is its ordinary generating function; the inverse Sumudu transform of the ordinary generating function is its exponential generating function.

An algorithm based on Sumudu transform is developed. The algorithm can be implemented in computer algebra systems like Maple. The algorithm can be used to solve differential equations of the following form automatically without human interaction

$$\sum_{i=0}^{m} p_i(x) y^{(i)}(x) = \sum_{j=0}^{k} q_j(x) h_j(x) \tag{14}$$

where $p_i(x)$ $(i = 0, ..., m)$ and $q_j(x)$ $(j = 0, 1, ..., k)$ are polynomials.

$h_j(x)$ are non-rational functions, but their Sumudu transforms are rational. m, k are nonnegative integers.

More algorithms can be developed similarly to solve more differential equations.

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