BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(80), 2016, Pages 7–23
ISSN 1024–7696

Belousov's Theorem and the quantum Yang-Baxter equation

Jonathan D. H. Smith

Abstract. Quantum quasigroups are self-dual objects that provide a general framework for the nonassociative extension of quantum group techniques. Within this context, the classical theorem of Belousov on the isotopy of distributive quasigroups and commutative Moufang loops is reinterpreted to yield solutions of the quantum Yang-Baxter equation. A new concept of principal bimagma isotopy is introduced.

Mathematics subject classification: 20N05, 16T25.
Keywords and phrases: Belousov theorem, quasigroup, loop, quantum Yang-Baxter equation, quantum quasigroup, distributive, isotopy.

1 Introduction

In the 1960s, Belousov published his now classic theorem on the isotopy of distributive quasigroups and commutative Moufang loops [1, Teorema 1], [2, Teorema 8.1]. The theorem states that for each element e of a given distributive quasigroup $(Q, \cdot)$, the operation $x + y = xR(e)^{-1} \cdot yL(e)^{-1}$ defines a commutative Moufang loop on Q, with identity element e. In the following decade, Belousov's Theorem became a key step in showing how a purely group-theoretic result of Fischer [6], on the nilpotence of the derived subgroup G' of a group G generated by a class of involutions whose products have order 3, could be proved entirely by quasigroup-theoretical methods [8]. The current paper sets out to explore further new aspects of Belousov's Theorem, showing how it leads to solutions of the quantum Yang-Baxter equation within the theory of quantum quasigroups.

The self-dual concept of a quantum quasigroup was introduced recently as a far-reaching unification of Hopf algebras and quasigroups [11]. Let $(\mathbf{V}, \otimes, \mathbf{1})$ be a symmetric monoidal category (§2). For example, one might take a category of vector spaces under the usual tensor product, or the category of sets under the cartesian product. Consider a *bimagma* (A, ∇, Δ), an object of $\mathbf{V}$ equipped with morphisms providing a *magma* structure $\nabla \colon A \otimes A \to A$ and a *comagma* structure $\Delta \colon A \to A \otimes A$ such that Δ is a magma homomorphism. Then the definition of a *quantum quasigroup* requires the invertibility of two dual endomorphisms of the tensor square of the bimagma object: the *left composite*

$$\mathsf{G} \colon A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A$$

and the *right composite*

$$\reflectbox{∂}\colon A \otimes A \xrightarrow{\Delta\otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A\otimes\nabla} A \otimes A \ .$$

The quantum Yang-Baxter equation (QYBE) is

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \tag{1.1}$$

[3, §2.2C], [12]. It applies to an endomorphism

$$R\colon A \otimes A \to A \otimes A$$

of the tensor square of an object A in a symmetric, monoidal category. For a given integer $n > 1$, the notation R^{ij}, for $1 \leq i < j \leq n$, means applying R to the i-th and j-th factors in the n-th tensor power of A. Since the left and right composite morphisms are also endomorphisms of tensor squares, it is natural to seek conditions under which they satisfy the QYBE. Then, as anticipated by B. B. Venkov working in the category of sets with cartesian product [5, §9], the QYBE corresponds generally to various distributivity conditions on the products $\nabla\colon A \otimes A \to A$ appearing in the left and right composites. If the left (or right) composite of a bimagma satisfies the QYBE, then the bimagma is said to possess the property of left (or right) *quantum distributivity*.

The plan of the paper is as follows. Section 2 presents the requisite background on symmetric monoidal categories and bimagmas; Section 3 gives the background on quasigroups and quantum quasigroups. Section 4 then examines quantum distributivity of bimagmas, in particular furnishing both necessary and sufficient conditions (Theorem 4.5, Corollary 4.6) that correspond to sufficient conditions for quantum distributivity established earlier [12]. The formal similarity of the identities (4.4) and (4.7) of these results, namely

$$x(yz) = (x^R y)(x^L z) \quad \text{and} \quad (zy)x = (zx^R)(yx^L)\,,$$

with identities that appear in connection with the proof of Belousov's Theorem [2, (8.7), (8.8)], was the first indication that Belousov's Theorem might prove relevant to quantum distributivity and solution of the QYBE.

Section 5 reformulates the well-known concept of principal isotopy in terms of magmas in a symmetric monoidal category (Definition 5.1). The new concept specializes to the classical concept in the symmetric monoidal category of sets with the cartesian product (Remark 5.2). Definition 5.3 then introduces the concept of *principal bimagma isotopy*, which combines the principal isotopy of magmas with a new *enneagon condition* on the comagma side. Corollary 5.6 notes that in the category of sets with the cartesian product, any classical principal isotopy, whose components are commuting automorphisms of the domain magma of the isotopy, may be enriched to a principal bimagma isotopy simply by defining a suitable comultiplication on the underlying set of the classical principal isotopy.

Section 6 investigates preservation of quantum distributivity under principal isotopy of magmas and bimagmas. The main results (Theorem 6.1, Corollary 6.2, Theorem 6.4) include a number of apparently restrictive conditions within their hypotheses, but these conditions are both modeled on, and implemented by, the prototype of Belousov's Theorem. Indeed, the culminating Theorem 7.2 shows how the isotopy in Belousov's Theorem from a classically distributive quasigroup to a commutative Moufang loop yields a quantum distributive quantum quasigroup with the commutative Moufang loop as its magma reduct. Thus the commutative Moufang loop yields new solutions to the quantum Yang-Baxter equation that are not apparent from the classical distributivity of the original quasigroup.

For algebraic concepts and conventions that are not otherwise discussed in this paper, readers are referred to [13]. In particular, algebraic notation is used throughout the paper, with functions to the right of, or as superfixes to, their arguments. Thus compositions are read from left to right. These conventions serve to minimize the proliferation of brackets.

2 Structures in symmetric monoidal categories

The general setting for the structures studied in this paper is a symmetric monoidal category (or "symmetric tensor category" — compare [14, Ch. 11]) $(\mathbf{V}, \otimes, \mathbf{1})$. The standard example is provided by the category $\underline{\underline{K}}$ of vector spaces over a field K, under the usual tensor product. More general concrete examples are provided by varieties $\mathbf{V}$ of entropic (universal) algebras, algebras on which each (fundamental and derived) operation is a homomorphism (compare [4, 7]). These include the categories **Set** of sets and **FinSet** of finite sets (under the cartesian product), the category of pointed sets, the category $\underline{\underline{S}}$ of unital (right) modules over a commutative, unital ring S, the category of commutative monoids, and the category of semilattices.

In a monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, there is an object $\mathbf{1}$ known as the *unit object*. For example, the unit object of $\underline{\underline{K}}$ is the vector space K, while the unit object of **Set** under the cartesian product is a terminal object $\top$, a singleton. For objects A and B in a monoidal category, a *tensor product* object $A \otimes B$ is defined. For example, if U and V are vector spaces over K with respective bases X and Y, then $U \otimes V$ is the vector space with basis $X \times Y$, written as $\{x \otimes y \mid x \in X,\ y \in Y\}$. There are natural isomorphisms with components

$$\alpha_{A,B;C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)\,,\ \rho_A \colon A \otimes \mathbf{1} \to A\,,\ \lambda_A \colon \mathbf{1} \otimes A \to A$$

satisfying certain *coherence* conditions guaranteeing that one may as well regard these isomorphisms as identities [14, p.67]. Thus the bracketing of repeated tensor products is suppressed in this paper. In the vector space example, adding a third space W with basis Z, one has

$$\alpha_{U,V;W} \colon (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$$

for $z \in Z$, along with $\rho_U : x \otimes 1 \mapsto x$ and $\lambda_U : 1 \otimes x \mapsto x$ for $x \in X$.

A monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is said to be *symmetric* if there is a given natural isomorphism with *twist* components $\tau_{A,B} : A \otimes B \to B \otimes A$ such that $\tau_{A,B}\tau_{B,A} = 1_{A\otimes B}$ [14, pp.67–8]. One uses $\tau_{U,V} : x \otimes y \mapsto y \otimes x$ with $x \in X$ and $y \in Y$ in the vector space example.

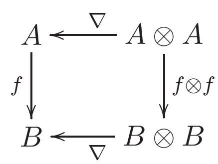
Definition 2.1. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A *magma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism

$$\nabla : A \otimes A \to A$$

known as *multiplication*.

(a.2) Let A and B be magmas in $\mathbf{V}$. Then a *magma homomorphism* $f : A \to B$ is a $\mathbf{V}$-morphism such that the diagram



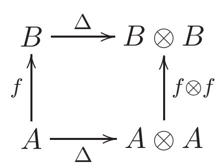
commutes.

(b.1) A *comagma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism

$$\Delta : A \to A \otimes A$$

known as *comultiplication*.

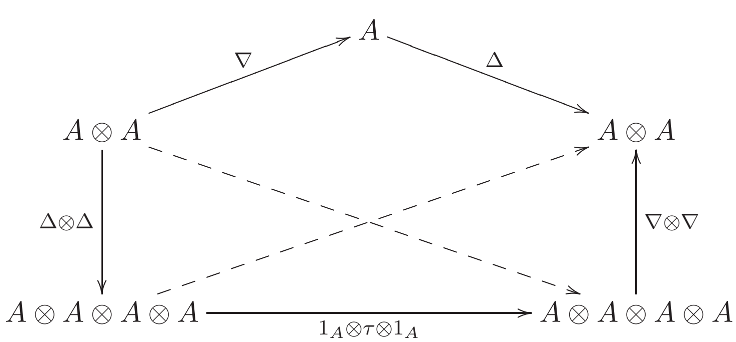
(b.2) Let A and B be comagmas in $\mathbf{V}$. A *comagma homomorphism* $f : A \to B$ is a $\mathbf{V}$-morphism such that the diagram



commutes.

(c) A *bimagma* (A, ∇, Δ) in $\mathbf{V}$ is a magma (A, ∇) and comagma (A, Δ) in $\mathbf{V}$ such that the following *bimagma diagram* commutes:

(2.1)



Remark 2.2. (a) The arrow across the bottom of the bimagma diagram (2.1) makes use of the twist isomorphism $\tau_{A,A}$ or $\tau\colon A \otimes A \to A \otimes A$.

(b) Commuting of the bimagma diagram (2.1) in a bimagma (A, ∇, Δ) means that

$$\Delta\colon (A, \nabla) \to \big(A \otimes A, (1_A \otimes \tau \otimes 1_A)(\nabla \otimes \nabla)\big)$$

is a magma homomorphism (commuting of the upper left-hand solid and dotted quadrilateral), or equivalently, that

$$\nabla\colon \big(A \otimes A, (\Delta \otimes \Delta)(1_A \otimes \tau \otimes 1_A)\big) \to (A, \Delta)$$

is a comagma homomorphism (commuting of the upper right-hand solid and dotted quadrilateral).

(c) If $\mathbf{V}$ is an entropic variety of universal algebras, the comultiplication of a comagma in $\mathbf{V}$ may be written as

$$\Delta\colon A \to A \otimes A; a \mapsto \big((a^{L_1} \otimes a^{R_1}) \dots (a^{L_{n_a}} \otimes a^{R_{n_a}})\big) w_a \tag{2.2}$$

in a universal-algebraic version of the well-known *Sweedler notation*. In (2.2), the *tensor rank* of the image of a (or any such general element of $A \otimes A$) is the smallest arity n_a of the derived word w_a expressing the image (or general element) in terms of elements of the generating set $\{b \otimes c \mid b, c \in A\}$ for $A \otimes A$. A more compact but rather less explicit version of Sweedler notation, generally appropriate within any concrete monoidal category $\mathbf{V}$, is $a\Delta = a^L \otimes a^R$, with the understanding that the tensor rank of the image is not implied to be 1.

(d) Magma multiplications on an object A of a concrete monoidal category are often denoted by juxtaposition, namely $(a \otimes b)\nabla = ab$, or with $a \cdot b$ as an infix notation, for elements a, b of A.

(e) With the notations of (c) and (d), commuting of the bimagma diagram (2.1) in a concrete bimagma (A, ∇, Δ) amounts to

$$a^L b^L \otimes a^R b^R = (ab)^L \otimes (ab)^R \tag{2.3}$$

for a, b in A.

3 Quantum quasigroups

While one- or two-sided quasigroups may be defined either equationally or combinatorially, it is actually the combinatorial definition of these structures which is "quantized" into the definition of quantum quasigroups, so here it suffices to recall the classical combinatorial definitions. Thus a *quasigroup* $(Q, \cdot)$ is defined as a set Q that is equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, where specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. Such a binary multiplication $Q \times Q \to Q; (x, y) \mapsto xy$ will often be written as a magma structure $\nabla \colon Q \otimes Q \to Q; x \otimes y \mapsto xy$ in notation for the symmetric monoidal category $(\mathbf{Set}, \times, \top)$ of sets under the cartesian product. In particular, note that tensor products of elements just correspond here to tuples. For example, $x \otimes y \otimes z$ is the ordered triple (x, y, z).

A *left quasigroup* $(Q, \cdot)$ is a set Q with a multiplication such that in the equation $a \cdot x = b$, specification of a and b determines x uniquely. The definition of *right quasigroups* is chirally dual: In the equation $x \cdot a = b$, specification of a and b determines x uniquely. If Q is a set, the right projection product $xy = y$ yields a left quasigroup structure on Q, while the left projection product $xy = x$ yields a right quasigroup structure.

Definition 3.1. Let (A, ∇, Δ) be a bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) On (A, ∇, Δ), the endomorphism

$$\mathsf{G} \colon A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \tag{3.1}$$

of $A \otimes A$ is known as the *left composite* morphism.

(b) On (A, ∇, Δ), the endomorphism

$$\reflectbox{G} \colon A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes 1_A} A \otimes A \tag{3.2}$$

of $A \otimes A$ is known as the *right composite* morphism.

Definition 3.2. Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) A *left quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ for which the left composite morphism G is invertible.

(b) A *right quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ for which the right composite morphism $\reflectbox{$\mathsf{G}$}$ is invertible.

(c) A *quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma in $\mathbf{V}$ where both G and $\reflectbox{$\mathsf{G}$}$ are invertible.

Since these basic definitions are expressed entirely within the structure of a symmetric, monoidal category, their concepts are maintained under the so-called symmetric, monoidal functors which preserve that structure. A typical example of such a functor is given by the free monoid functor from sets under cartesian products to the category of modules over a commutative ring, with the usual tensor product.

Proposition 3.3. *Suppose that* $(\mathbf{V}, \otimes, \mathbf{1_V})$ *and* $(\mathbf{W}, \otimes, \mathbf{1_W})$ *are symmetric monoidal categories. Let* $F \colon \mathbf{V} \to \mathbf{W}$ *be a symmetric monoidal functor. If* (A, ∇, Δ) *is a left, right, or two-sided quantum quasigroup in* $(\mathbf{V}, \otimes, \mathbf{1_V})$*, the structure* (AF, ∇^F, Δ^F) *is a respective left, right, or two-sided quantum quasigroup in* $(\mathbf{W}, \otimes, \mathbf{1_W})$*.*

Theorem 3.4. *Consider the symmetric, monoidal category* $(\mathbf{FinSet}, \times, \top)$ *of finite sets under the cartesian product.*

(a) *Left quantum quasigroups* $(Q, \nabla, \Delta \colon x \mapsto x^L \otimes x^R)$ *in* $(\mathbf{FinSet}, \times, \top)$ *are equivalent to triples* (A, L, R) *that consist of a left quasigroup* (Q, ∇) *with an automorphism* L *and endomorphism* R *[10].*

(b) *The quantum quasigroups* $(Q, \nabla, \Delta \colon x \mapsto x^L \otimes x^R)$ *in* $(\mathbf{FinSet}, \times, \top)$ *are equivalent to triples* (Q, L, R) *consisting of a quasigroup* (Q, ∇) *equipped with automorphisms* L *and* R *[11].*

Corollary 3.5. *[10] Given a left quasigroup* $(Q, \cdot)$ *with an automorphism* L *and endomorphism* R*, define* $\nabla \colon Q \otimes Q \to Q; x \otimes y \mapsto xy$ *as a multiplication and* $\Delta \colon Q \to Q \otimes Q; x \mapsto x^L \otimes x^R$ *as a comultiplication. Then* (Q, ∇, Δ) *is a left quantum quasigroup in* $(\mathbf{Set}, \times, \top)$*.*

Corollary 3.6. *[11] Suppose that* $(Q, \cdot)$ *is a quasigroup equipped with two automorphisms* L *and* R*. Define* $\nabla \colon Q \otimes Q \to Q; x \otimes y \mapsto xy$ *as a multiplication and* $\Delta \colon Q \to Q \otimes Q; x \mapsto x^L \otimes x^R$ *as a comultiplication. Then* (Q, ∇, Δ) *is a quantum quasigroup in* $(\mathbf{Set}, \times, \top)$*.*

4 Quantum distributivity

Definition 4.1 ([12])**.** Suppose that (A, ∇, Δ) is a bimagma in a symmetric, monoidal category.

(a) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum left distributivity* if the left composite G of (A, ∇, Δ) satisfies the quantum Yang-Baxter equation (1.1).

(b) The bimagma (A, ∇, Δ) is said to satisfy the condition of *quantum right distributivity* if the right composite $\reflectbox{$\mathsf{G}$}$ of (A, ∇, Δ) satisfies the quantum Yang-Baxter equation (1.1).

(c) The bimagma (A, ∇, Δ) satisfies the condition of *quantum distributivity* if it has both the left and right quantum distributivity properties.

Since the quantum distributivity concepts of Definition 4.1 are written entirely in the language of symmetric monoidal categories, one immediately obtains the following analogue of Proposition 3.3.

Proposition 4.2. *Suppose that* $(\mathbf{V}, \otimes, \mathbf{1_V})$ *and* $(\mathbf{W}, \otimes, \mathbf{1_W})$ *are symmetric monoidal categories. Let* $F \colon \mathbf{V} \to \mathbf{W}$ *be a symmetric monoidal functor. If* (A, ∇, Δ) *is a left, right, or two-sided distributive bimagma in* $(\mathbf{V}, \otimes, \mathbf{1_V})$*, then the structure* (AF, ∇^F, Δ^F) *becomes a respective left, right, or two-sided quantum distributive bimagma in* $(\mathbf{W}, \otimes, \mathbf{1_W})$*.*

The explicit results of this paper are predominantly concerned with obtaining quantum distributivity, and therefore solutions of the QYBE, in the symmetric monoidal category $(\mathbf{Set}, \otimes, \top)$ of sets with the cartesian product. On the other hand, one is generally interested in solving the QYBE within the symmetric monoidal category $(\underline{\underline{K}}, \otimes, K)$ of unital modules over a commutative, unital ring K. To this end, one may apply Proposition 4.2, with the symmetric, monoidal free monoid functor $F\colon \mathbf{Set} \to \underline{\underline{K}}$, to the quantum distributive structures in $(\mathbf{Set}, \otimes, \top)$ that are exhibited in the paper.

The terminology of Definition 4.1 is justified by the following result.

Proposition 4.3 ([5, 12])**.** *Let (Q, ∇) be a magma in the category of sets with the cartesian product. Define $\Delta\colon Q \to Q \otimes Q; x \mapsto x \otimes x$. Then the bimagma (Q, ∇, Δ) is quantum left distributive if and only if the magma (Q, ∇) is left distributive, in the classical sense that the identity*

$$x(yz) = (xy)(xz) \tag{4.1}$$

is satsified.

Proposition 4.4 ([12], Prop. 6.4)**.** *Let (Q, ∇, Δ) be a bimagma in $(\mathbf{Set}, \times, \top)$, equipped with the comultiplication $\Delta\colon Q \to Q \otimes Q; x \mapsto x^L \otimes x^R$.*

(a) *The bimagma (Q, ∇, Δ) is quantum left distributive if $LR = RL$ and the identity*

$$x^R(y^R z) = (x^{RR} y^R)(x^{RL} z) \tag{4.2}$$

is satisfied.

(b) *The bimagma (Q, ∇, Δ) is quantum right distributive if $LR = RL$ and the identity*

$$(zy^L)x^L = (zx^{LR})(y^L x^{LL}) \tag{4.3}$$

is satisfied.

Theorem 4.5. *Let (Q, ∇, Δ) be a bimagma in $(\mathbf{Set}, \times, \top)$, with comultiplication $\Delta\colon Q \to Q \otimes Q; x \mapsto x^L \otimes x^R$. Suppose that the following conditions are satisfied:*

(a) *The magma (Q, ∇) is a right quasigroup;*

(b) *The second comultiplication component $R\colon Q \to Q$ is surjective.*

Then the bimagma (Q, ∇, Δ) is quantum left distributive if and only if $LR = RL$ and the identity

$$x(yz) = (x^R y)(x^L z) \tag{4.4}$$

is satisfied.

Proof. Note that the identity (4.4) implies the identity (4.2). Then the sufficiency of (4.4), together with the commutation condition, follows by Proposition 4.4(a).

Conversely, suppose that (Q, ∇, Δ) is quantum left distributive. Consider an element $x \otimes y \otimes z$ of $Q \otimes Q \otimes Q$. Then

$$\begin{aligned} & x^{LL} \otimes x^{LR} y^L \otimes x^R(y^R z) && (4.5)\\ &= \big(x^L \otimes y^L \otimes x^R(y^R z)\big)\mathsf{G}^{12} \\ &= (x \otimes y^L \otimes y^R z)\mathsf{G}^{13}\mathsf{G}^{12} \\ &= (x \otimes y \otimes z)\mathsf{G}^{23}\mathsf{G}^{13}\mathsf{G}^{12} \\ &= (x \otimes y \otimes z)\mathsf{G}^{12}\mathsf{G}^{13}\mathsf{G}^{23} \\ &= (x^L \otimes x^R y \otimes z)\mathsf{G}^{13}\mathsf{G}^{23} \\ &= (x^{LL} \otimes x^R y \otimes x^{LR} z)\mathsf{G}^{23} \\ &= x^{LL} \otimes (x^R y)^L \otimes (x^R y)^R (x^{LR} z)\,. && (4.6) \end{aligned}$$

Since (Q, ∇, Δ) is a bimagma, (2.3) implies that the maps L and R are endomorphisms of the right quasigroup (Q, ∇). Since the respective middle factors of (4.5) and (4.6) agree, one has $x^{LR}y^L = x^{RL}y^L$. Canceling y^L in the right quasigroup (Q, ∇) then yields $x^{LR} = x^{RL}$, so the commutation condition $LR = RL$ holds. Since the respective final factors of (4.5) and (4.6) agree, the identity (4.2) is satisfied. The surjectivity of $R \colon Q \to Q$ then implies that (4.4) is satisfied. ☐

The chiral dual of Theorem 4.5 is formulated as follows.

Corollary 4.6. *Let (Q, ∇, Δ) be a bimagma in $(\mathbf{Set}, \times, \top)$, with comultiplication $\Delta \colon Q \to Q \otimes Q; x \mapsto x^L \otimes x^R$. Suppose that the following conditions are satisfied:*

(a) *The magma (Q, ∇) is a left quasigroup;*

(b) *The first comultiplication component $L \colon Q \to Q$ is surjective.*

Then the bimagma (Q, ∇, Δ) is quantum right distributive if and only if $LR = RL$ and the identity

$$(zy)x = (zx^R)(yx^L) \tag{4.7}$$

is satisfied.

Extracting details from the necessity proof in Theorem 4.5 yields the following.

Corollary 4.7. *Let (Q, ∇, Δ) be a bimagma in $(\mathbf{Set}, \times, \top)$, with comultiplication $\Delta \colon Q \to Q \otimes Q; x \mapsto x^L \otimes x^R$ satisfying $LR = RL$.*

(a) *Suppose that the second comultiplication component $R \colon Q \to Q$ is surjective. Then the identity (4.4) is satisfied if the bimagma (Q, ∇, Δ) is quantum left distributive.*

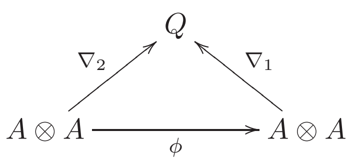
(b) *Suppose that the first comultiplication component $L \colon Q \to Q$ is surjective. Then the identity (4.7) is satisfied if the bimagma (Q, ∇, Δ) is quantum right distributive.*

5 Principal isotopy

Definition 5.1. Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. Let (A, ∇_i) be a magma in $(\mathbf{V}, \otimes, \mathbf{1})$, for $i = 1, 2$. Then an automorphism ϕ of the object $A \otimes A$ of $\mathbf{V}$ is a *principal magma isotopy*

$$\phi\colon (A, \nabla_1) \rightsquigarrow (A, \nabla_2)$$

in $(\mathbf{V}, \otimes, \mathbf{1})$ if the diagram



(5.1)

commutes in $\mathbf{V}$.

Remark 5.2. Suppose that $(Q, \nabla_i\colon x \otimes y \mapsto x \circ_i y)$ are magmas on an object Q of $(\mathbf{Set}, \times, \top)$ for $i = 1, 2$. Then if $f \otimes g\colon (Q, \nabla_1) \rightsquigarrow (Q, \nabla_2)$ is a principal isotopy in $(\mathbf{Set}, \times, \top)$ for automorphisms f, g of Q, one has

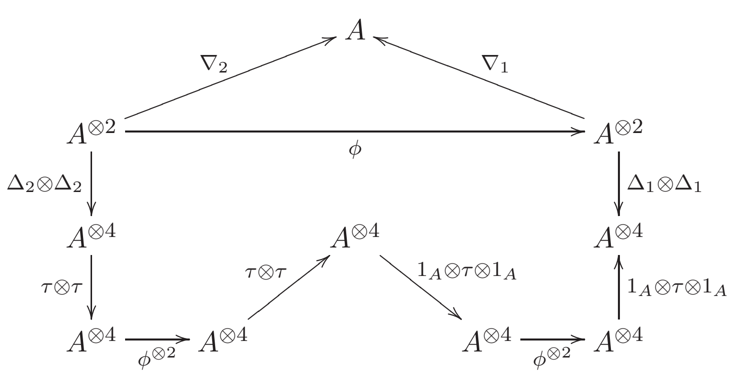
$$x \circ_2 y = x^f \circ_1 y^g \tag{5.2}$$

for $x, y \in Q$, as usual for a classical principal isotopy [2, p.13], [9, p.5].

Definition 5.3. Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. Let (A, ∇_i, Δ_i) be a bimagma in $(\mathbf{V}, \otimes, \mathbf{1})$, for $i = 1, 2$. Then an automorphism ϕ of the object $A \otimes A$ of $\mathbf{V}$ is a *principal bimagma isotopy*

$$\phi\colon (A, \nabla_1, \Delta_1) \rightsquigarrow (A, \nabla_2, \Delta_2)$$

in $(\mathbf{V}, \otimes, \mathbf{1})$ if the *decagon diagram*



(5.3)

commutes in $\mathbf{V}$. Here, the superfix $^{\otimes r}$ is used for the r-th tensor power of an object or morphism in $\mathbf{V}$. The lower part of the decagon diagram is called the *nonagon diagram* or *enneagon diagram*.

Remark 5.4. Note that the upper triangle in the decagon diagram is just the diagram (5.1). Thus a principal bimagma isotopy

$$\phi\colon (A,\nabla_1,\Delta_1) \rightsquigarrow (A,\nabla_2,\Delta_2)$$

includes a principal magma isotopy $\phi\colon (A,\nabla_1) \rightsquigarrow (A,\nabla_2)$.

Proposition 5.5. *Suppose that $(Q,\nabla_i,\Delta_i\colon x \mapsto xL_i \otimes xR_i)$, for $i=1,2$, are bimagmas in $(\mathbf{Set},\times,\top)$, with permutations $f\colon Q \to Q$ and $g\colon Q \to Q$. Then the validity of the equations*

$$L_2gf = fL_1\,,\quad L_2g^2 = gL_1\,,\quad R_2f^2 = fR_1\,,\quad R_2fg = gR_1 \tag{5.4}$$

is equivalent to the commuting of the enneagon diagram (5.3) *for a principal bimagma isotopy*

$$f \otimes g\colon (Q,\nabla_1,\Delta_1) \rightsquigarrow (Q,\nabla_2,\Delta_2)$$

composed from $f\colon Q \to Q$ and $g\colon Q \to Q$.

Proof. For an element x of Q, the commuting of the enneagon diagram (5.3) gives

$$\begin{array}{ccc}
x \otimes y & \xmapsto{f\otimes g} & xf \otimes yg \\
\downarrow{\scriptstyle \Delta_2\otimes\Delta_2} & & \downarrow{\scriptstyle \Delta_1\otimes\Delta_1} \\
xL_2 \otimes yR_2 \otimes xL_2 \otimes yR_2 & & xfL_1 \otimes xfR_1 \otimes ygL_1 \otimes ygR_1 \\
\downarrow{\scriptstyle \tau\otimes\tau} & & \downarrow{\scriptstyle 1_Q\otimes\tau\otimes 1_Q} \\
yR_2 \otimes xL_2 \otimes yR_2 \otimes xL_2 & & xfL_1 \otimes ygL_1 \otimes xfR_1 \otimes ygR_1 \\
\downarrow{\scriptstyle (f\otimes g)^{\otimes 2}} & & \| \\
yR_2f \otimes xL_2g \otimes yR_2f \otimes xL_2g & & xL_2gf \otimes xL_2g^2 \otimes yR_2f^2 \otimes yR_2fg \\
\downarrow{\scriptstyle \tau\otimes\tau} & & \uparrow{\scriptstyle (f\otimes g)^{\otimes 2}} \\
xL_2g \otimes yR_2f \otimes xL_2g \otimes yR_2f & \xmapsto[1_Q\otimes\tau\otimes 1_Q]{} & xL_2g \otimes xL_2g \otimes yR_2f \otimes yR_2f
\end{array}$$

which is equivalent to the equations

$$L_2gf = fL_1\,,\quad L_2g^2 = gL_1\,,\quad R_2f^2 = fR_1\,,\quad R_2fg = gR_1$$

holding in the endomorphism monoid of the set Q. $\square$

Corollary 5.6. *Consider a magma $(Q,\nabla_1\colon x \otimes y \mapsto x \cdot y)$ in $(\mathbf{Set},\times,\top)$, with commuting automorphisms f and g.*

(a) *There is a bimagma*

$$(Q,\nabla_1\colon x \otimes y \mapsto x \cdot y, \Delta_1\colon x \mapsto x \otimes x)$$

in $(\mathbf{Set},\times,\top)$.

(b) *There is a bimagma*

$$(Q, \nabla_2 \colon x \otimes y \mapsto xf \cdot yg, \Delta_2 \colon x \mapsto xg^{-1} \otimes xf^{-1})$$

in $(\mathbf{Set}, \times, \top)$.

(c) *There is a principal bimagma isotopy*

$$f \otimes g \colon (Q, \nabla_1, \Delta_1) \rightsquigarrow (Q, \nabla_2, \Delta_2)\,.$$

Proof. (a) Trivially, (2.3) holds for $L = R = 1$.

(b) Applying (2.3) to (Q, ∇_2, Δ_2), note that

$$\begin{aligned}(xg^{-1} \otimes & yg^{-1})\nabla_2 \otimes (xf^{-1} \otimes yf^{-1})\nabla_2 \\ &= (xg^{-1}f \cdot yg^{-1}g) \otimes (xf^{-1}f \cdot yf^{-1}g) \\ &= (xf \cdot yg)g^{-1} \otimes (xf \cdot yg)f^{-1} \\ &= (x \otimes y)\nabla_2 g^{-1} \otimes (x \otimes y)\nabla_2 f^{-1}\end{aligned}$$

for x, y in Q, so that (Q, ∇_2, Δ_2) is a bimagma.

(c) With $L_1 = R_1 = 1_Q$ and $L_2 = g^{-1}$, $R_2 = f^{-1}$, the equations (5.4) hold. □

Remark 5.7. In Corollary 5.6(b), duality interchanges and inverts the respective components f, g of the magma isotopy to yield the corresponding components g^{-1}, f^{-1} of the comultiplication.

6 Isotopy of quantum distributive structures

This section investigates the preservation of quantum distributivity under principal isotopy. The conditions assembled in the hypotheses of its theorems are seen to appear naturally in the following section, within the context of Belousov's Theorem.

Theorem 6.1. *Suppose that* $(Q, \nabla_i \colon x \otimes y \mapsto x \circ_i y, \Delta_i \colon x \mapsto x^{L_i} \otimes x^{R_i})$ *are bimagmas on an object* Q *of* $(\mathbf{Set}, \times, \top)$ *for* $i = 1, 2$*, such that the following conditions are satisfied:*

(a) *The comultiplication component* $R_1 \colon Q \to Q$ *is surjective;*

(b) *The comultiplication components* L_i *and* R_i *commute for* $i = 1, 2$*;*

(c) *The bimagma* (Q, ∇_1, Δ_1) *is quantum left distributive;*

(d) *There is a principal magma isotopy*

$$f \otimes g \colon (Q, \nabla_1) \rightsquigarrow (Q, \nabla_2)$$

whose components f, g *are commuting automorphisms of* (Q, ∇_1)*;*

(e) *The equations* $L_2gf = fL_1$ *and* $R_2f^2 = fR_1$ *hold.*

Then (Q, ∇_2, Δ_2) *is quantum left distributive.*

Proof. By the conditions (a)–(c), Corollary 4.7 implies that the identity (4.4) holds in (Q, ∇_1, Δ_1). For $x, y, z \in Q$, one then has

$$\begin{aligned} x \circ_2 (y \circ_2 z) &= x^f \circ_1 (y \circ_2 z)^g = x^f \circ_1 (y^f \circ_1 z^g)^g \\ &= x^f \circ_1 (y^{fg} \circ_1 z^{g^2}) \\ &= (x^{fR_1} \circ_1 y^{fg}) \circ_1 (x^{fL_1} \circ_1 z^{g^2}) \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} (x^{R_2} \circ_2 y) \circ_2 (x^{L_2} \circ_2 z) &= (x^{R_2f} \circ_1 y^g) \circ_2 (x^{L_2f} \circ_1 z^g) \\ &= (x^{R_2f} \circ_1 y^g)^f \circ_1 (x^{L_2f} \circ_1 z^g)^g \\ &= (x^{R_2f^2} \circ_1 y^{gf}) \circ_1 (x^{L_2fg} \circ_1 z^{g^2}) \end{aligned} \tag{6.2}$$

by (5.2), the identity (4.4) in (Q, ∇_1, Δ_1), and condition (d) saying that the principal isotopy components f and g are automorphisms of (Q, ∇_1). Now the commutation condition in (d) and the equations of (e) imply the equality of (6.1) and (6.2). Thus the identities (4.4) and (4.2) hold in (Q, ∇_2, Δ_2). By the condition (b) for $i = 2$, Proposition 4.4 implies that (Q, ∇_2, Δ_2) is quantum left distributive. □

The chiral dual of Theorem 6.1 may be formulated as follows.

Corollary 6.2. *Suppose that* $(Q, \nabla_i \colon x \otimes y \mapsto x \circ_i y, \Delta_i \colon x \mapsto x^{L_i} \otimes x^{R_i})$ *are bimagmas on an object* Q *of* $(\mathbf{Set}, \times, \top)$ *for* $i = 1, 2$*, such that the following conditions are satisfied:*

(a) *The comultiplication component* $L_1 \colon Q \to Q$ *is surjective;*

(b) *The comultiplication components* L_i *and* R_i *commute for* $i = 1, 2$*;*

(c) *The bimagma* (Q, ∇_1, Δ_1) *is quantum right distributive;*

(d) *There is a principal magma isotopy*

$$f \otimes g \colon (Q, \nabla_1) \rightsquigarrow (Q, \nabla_2)$$

whose components f, g *are commuting automorphisms of* (Q, ∇_1)*;*

(e) *The equations* $L_2g^2 = gL_1$ *and* $R_2fg = gR_1$ *hold.*

Then (Q, ∇_2, Δ_2) *is quantum right distributive.*

Remark 6.3. The equations of Theorem 6.1(e) may be written as

$$\Delta_2(g \otimes f)(f \otimes f) = f\Delta_1$$

within the symmetric monoidal category $(\mathbf{Set}, \times, \top)$. Similarly, the chirally dual equations of Corollary 6.2(e) may be written as

$$\Delta_2(g \otimes f)(g \otimes g) = g\Delta_1$$

within $(\mathbf{Set}, \times, \top)$. According to Proposition 5.5, the enneagon diagram elegantly captures the conjunction of all four equations, using the isotopy $f \otimes g$. On the other hand, there does not appear to be an equally elegant or natural way to capture the respective pairs of individual equations that appear in the theorem and corollary separately.

The diverse conditions of Theorem 6.1 and Corollary 6.2 are simplified and unified in the context of quantum quasigroups within the symmetric monoidal category of finite sets and cartesian products.

Theorem 6.4. *Suppose that* $(Q, \nabla_i \colon x \otimes y \mapsto x \circ_i y, \Delta_i \colon x \mapsto x^{L_i} \otimes x^{R_i})$ *are quantum quasigroups on an object* Q *of* $(\mathbf{FinSet}, \times, \top)$ *for* $i = 1, 2$*, such that the following conditions are satisfied:*

(a) *The comultiplication components* L_i *and* R_i *commute for* $i = 1, 2$*;*

(b) *The quantum quasigroup* (Q, ∇_1, Δ_1) *is quantum distributive;*

(c) *There is a principal bimagma isotopy*

$$f \otimes g \colon (Q, \nabla_1, \Delta_1) \rightsquigarrow (Q, \nabla_2, \Delta_2)$$

whose components f, g *are commuting automorphisms of* (Q, ∇_1)*.*

Then (Q, ∇_2, Δ_2) *is quantum distributive.*

Proof. The bimagmas (Q, ∇_i, Δ_i), for $i = 1, 2$, satisfy the conditions (a)–(e) of Theorem 6.1 and Corollary 6.2:

(a) The mappings R_1 and L_1 are bijective, by Theorem 3.4(b);

(b) This is condition (a) of the current theorem;

(c) This is condition (b) of the current theorem;

(d) By Remark 5.4, this is part of condition (c) of the current theorem;

(e) The equations

$$L_2 gf = fL_1\,, \quad L_2 g^2 = gL_1\,, \quad R_2 f^2 = fR_1\,, \quad R_2 fg = gR_1$$

hold by Proposition 5.5, given condition (c) of the current theorem.

Then by Theorem 6.1, (Q, ∇_2, Δ_2) is quantum left distributive. By Corollary 6.2, it is quantum right distributive. □

7 Belousov's Theorem and quantum distributivity

If e is an element of a magma $(Q, \cdot)$, one has the *left multiplication*

$$L(e)\colon Q \to Q; x \mapsto e \cdot x$$

and *right multiplication*

$$R(e)\colon Q \to Q; x \mapsto x \cdot e\,.$$

If $(Q, \cdot)$ is a left quasigroup, $L(e)$ is bijective. Similarly, if $(Q, \cdot)$ is a right quasigroup, $R(e)$ is bijective.

Lemma 7.1. *Let e be an element of a distributive quasigroup $(Q, \cdot)$.*

(a) *The left and right multiplications $L(e)$ and $R(e)$ are automorphisms of $(Q, \cdot)$ [2, p.131];*

(b) *The element e is idempotent: $ee = e$ [2, p.131];*

(c) *The multiplications $L(e)$ and $R(e)$ commute [2, (8.3)].*

Proof. (a) By distributivity, one has

$$e(xy) = (ex)(ey) \quad \text{and } (xy)e = (xe)(ye)$$

for elements x, y of Q.

(b) Note that $e(ee) = (ee)(ee)$, so $e = ee$.

(c) Let x be an element of Q. Then $xL(e)R(e) = (ex)e = (ee)(xe) = e(xe) = xR(e)L(e)$ by respective application of (a) and (b). □

Theorem 7.2. *Suppose that (Q, ∇_1) is a distributive quasigroup with an element e. Set $f = R(e)^{-1}$ and $g = L(e)^{-1}$. Let (Q, ∇_2) be the commutative Moufang loop with multiplication*

$$\nabla_2\colon Q \otimes Q \to Q; x \otimes y \mapsto (xf \otimes yg)\nabla_1 \tag{7.1}$$

given by Belousov's Theorem [1, Teorema 1], [2, Teorema 8.1]. Then with the comultiplication

$$\Delta_2\colon Q \to Q \otimes Q; x \mapsto x^{L(e)} \otimes x^{R(e)}\,, \tag{7.2}$$

there is a quantum distributive quantum quasigroup (Q, ∇_2, Δ_2) in the symmetric monoidal category $(\mathbf{Set}, \times, \top)$.

Proof. Lemma 7.1(a) implies that the respective components $L(e)$ and $R(e)$ of the comultiplication (7.2) are automorphisms of (Q, ∇_1). Lemma 7.1(c) yields

$$(x \otimes y)\nabla_2 L(e) = (xf \otimes yg)\nabla_1 L(e) = \big(xfL(e) \otimes ygL(e)\big)\nabla_1$$

$$= \big(xL(e)f \otimes yL(e)g\big)\nabla_1 = (xL(e) \otimes yL(e))\nabla_2$$

and

$$\begin{aligned}(x \otimes y)\nabla_2 R(e) = (xf \otimes yg)\nabla_1 R(e) &= \big(xfR(e) \otimes ygR(e)\big)\nabla_1 \\ &= \big(xR(e)f \otimes yR(e)g\big)\nabla_1 = (xR(e) \otimes yR(e))\nabla_2\,,\end{aligned}$$

so that $L(e)$ and $R(e)$ are automorphisms of (Q, ∇_2). Corollary 3.6 then implies that (Q, ∇_2, Δ_2) is a quantum quasigroup in $(\mathbf{Set}, \times, \top)$.

Define $\Delta_1 \colon Q \to Q \otimes Q; x \mapsto x \otimes x$. Corollary 3.6 implies that (Q, ∇_1, Δ_1) is a quantum quasigroup in $(\mathbf{Set}, \times, \top)$. Then the bimagmas (Q, ∇_i, Δ_i), for $i = 1, 2$, satisfy the conditions (a)–(e) of Theorem 6.1 and Corollary 6.2:

(a) The identity mapping $R_1 = L_1 = 1_Q$ is surjective;

(b) Note $R_2L_2 = R(e)L(e) = L(e)R(e) = L_2R_2$ by Lemma 7.1(c), while of course $R_1L_1 = L_1R_1 = 1_Q$;

(c) Apply Proposition 4.3 and its chiral dual;

(d) The definition (7.1) yields a principal magma isotopy $f \otimes g$; by Lemma 7.1(c), its components commute;

(e) The equations

$$\begin{aligned} L_2gf = f = fL_1\,, &\quad R_2f^2 = f = fR_1\,, \\ L_2g^2 = g = gL_1\,, &\quad R_2fg = g = gR_1 \end{aligned}$$

hold since $L_2 = g^{-1}$, $R_2 = f^{-1}$, and $L_1 = R_1 = 1_Q$.

By Theorem 6.1, (Q, ∇_2, Δ_2) is quantum left distributive, while by Corollary 6.2, it is quantum right distributive. □

Corollary 7.3. *In the context of Theorem 7.2, with*

$$\Delta_1 \colon Q \to Q \otimes Q; x \mapsto x \otimes x\,,$$

there is a principal bimagma isotopy

$$f \otimes g \colon (Q, \nabla_1, \Delta_1) \rightsquigarrow (Q, \nabla_2, \Delta_2)\,.$$

Proof. Apply Corollary 5.6. □

Remark 7.4. The left composite morphism of (Q, ∇_1, Δ_1) is

$$x \otimes y \mapsto x \otimes xy\,,$$

while the left composite morphism of (Q, ∇_2, Δ_2) is

$$x \otimes y \mapsto xe \otimes x \cdot yL(e)^{-1}\,.$$

These two distinct automorphisms of $Q \otimes Q$ give solutions of the QYBE in $(\mathbf{Set}, \times, \top)$. For each commutative, unital ring S, Proposition 4.2 shows that applying the free S-module functor then yields solutions of the QYBE in the module category $(\underline{\underline{S}}, \otimes, S)$.

References

[1] Belousov V. D. *On the structure of distributive quasigroups.* Mat. Sbornik, 1960, **50**, 267–298 (in Russian).

[2] Belousov V. D. *Foundations of the Theory of Quasigroups and Loops.* Moscow, Nauka, 1967 (in Russian).

[3] Chari V., Pressley A. *A Guide to Quantum Groups.* Cambridge University Press, Cambridge, 1994.

[4] Davey B. A., Davis G. *Tensor products and entropic varieties.* Algebra Universalis, 1985, **21**, 68–88.

[5] Drinfeld V. G. "On some unsolved problems in quantum group theory", p. 1–8 in *Quantum Groups* (P. P. Kulish, ed.), Springer Lecture Notes in Mathematics 1510, Springer-Verlag, Berlin, 1992.

[6] Fischer B. *Distributive Quasigruppen endlicher Ordnung.* Math. Z., 1964, **83**, 267–303.

[7] Romanowska A. B., Smith J. D. H. *On Hopf algebras in entropic Jónsson-Tarski varieties.* Bull. Korean Math. Soc., 2015, **52**, 1587–1606. DOI: 10.4134/BKMS.2015.52.5.1587.

[8] Smith J. D. H. *Finite distributive quasigroups.* Math. Proc. Camb. Phil. Soc., 1976, **80**, 37–41.

[9] Smith J. D. H. *An Introduction to Quasigroups and Their Representations.* Chapman and Hall/CRC, Boca Raton, FL, 2007.

[10] Smith J. D. H. *One-sided quantum quasigroups and loops.* Demonstr. Math., 2015, **48**, 620–636. DOI: 10.1515/dema-2015-0043.

[11] Smith J. D. H. *Quantum quasigroups and loops.* J. Alg., 2016, **456**, 46–75. DOI: 10.1016/j.jalgebra.2016.02.014.

[12] Smith J. D. H. *Quantum idempotence, distributivity, and the Yang-Baxter equation.* Preprint, 2015.

[13] Smith J. D. H., Romanowska A. B. *Post-Modern Algebra.* Wiley, New York, NY, 1999.

[14] Street R. *Quantum Groups.* Cambridge University Press, Cambridge, 2007.

Jonathan D. H. Smith
Dept. of Math., Iowa State Univ.
Ames, IA 50011, U.S.A.
E-mail: *jdhsmith@iastate.edu*

Received August 23, 2015