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One subfamily of cubic systems with invariant lines of total multiplicity eight and with two distinct real infinite singularities

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Abstract. In this article we classify a subfamily of differential real cubic systems possessing eight invariant straight lines, including the line at infinity and including their multiplicities. This subfamily of systems is characterized by the existence of two distinct infinite singularities, defined by the linear factors of the polynomial $C_3(x,y) = yp_3(x,y) - xq_3(x,y)$, where p_3 and q_3 are the cubic homogeneities of these systems. Moreover we impose additional conditions related with the existence of triplets and/or couples of parallel invariant lines. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of affine invariant polynomials. The invariant polynomials allow one to verify for any given real cubic system whether or not it has invariant straight lines of total multiplicity eight, and to specify its configuration of straight lines endowed with their corresponding real singularities of this system. The calculations can be implemented on computer and the results can therefore be applied for any family of cubic systems in this class, given in any normal form.

Mathematics subject classification: 34G20, 34A26, 14L30, 34C14.
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1 Introduction and the statement of the Main Theorem

We consider here real polynomial differential systems

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where $P,\ Q$ are polynomials in $x,\ y$ with real coefficients, i. e. $P,\ Q \in \mathbb{R}[x,y]$. We say that systems (1) are *cubic* if $\max(\deg(P), \deg(Q)) = 3$.

Let

$$\mathbf{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

be the polynomial vector field associated to systems (1).

A straight line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ satisfies

$$\mathbf{X}(f) = uP(x,y) + vQ(x,y) = (ux + vy + w)R(x,y)$$

for some polynomial $R(x, y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients u, v, w of an invariant straight line belong to $\mathbb{C}\setminus\mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real.* Note that, since systems (1) are real, if a system has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

To a line $f(x, y) = ux + vy + w = 0$, $(u, v) \neq (0, 0)$ we associate its projective completion $F(X, Y, Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbf{P}_2(\mathbb{C})$, $(x, y) \mapsto [x : y : 1]$. The line $Z = 0$ in $\mathbf{P}_2(\mathbb{C})$ is called *the line at infinity* of the affine plane $\mathbb{C}^2$. It follows from the work of Darboux (see, for instance [10]) that each system of differential equations of the form (1) over $\mathbb{C}$ yields a differential equation on the complex projective plane $\mathbf{P}_2(\mathbb{C})$ which is the compactification of the differential equation $Qdx - Pdy = 0$ in $\mathbb{C}^2$. The line $Z = 0$ is an invariant manifold of this complex differential equation.

Definition 1 (see [27])**.** We say that an invariant affine straight line $f(x, y) = ux + vy + w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\mathbf{X}$ has multiplicity m if there exists a sequence of real cubic vector fields X_k converging to $\mathbf{X}$, such that each $\mathbf{X}_k$ has m (respectively $m-1$) distinct invariant affine straight lines $f_k^j = u_k^j x + v_k^j y + w_k^j = 0$, $(u_k^j, v_k^j) \neq (0, 0)$, $(u_k^j, v_i^k, w_k^j) \in \mathbb{C}^3$ $(j \in \{1, \ldots m\})$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m + 1$ (respectively m).

We mention here some references on polynomial differential systems possessing invariant straight lines. For quadratic systems see [11, 24, 25, 27–30] and [31]; for cubic systems see [15–18, 26, 34] and [35]; for quartic systems see [33] and [36]; for some more general systems see [13, 21, 22] and [23].

According to [2] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line. This bound is always reached if we consider the real and the complex invariant straight lines, see [9].

So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities have been made in [16]. We also remark that a subclass of the family of cubic systems with eight invariant lines was discussed in [34] and [35].

It is well known that for a cubic system (1) with finite number of infinite singularities there exist at most 4 different slopes for invariant affine straight lines, for more information about the slopes of invariant straight lines for polynomial vector fields, see [1].

Definition 2 (see [31])**.** Consider a planar cubic system (1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

Remark 1. In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [24]. Thus we denote by '(a, b)' the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of the multiple point.

Suppose that a cubic system (1) possesses 8 distinct invariant straight lines (including the line at infinity). We say that these lines form a *configuration of type* $(3, 3, 1)$ if there exist two triplets of parallel lines and one additional line, every set with different slopes. And we say that these lines form a *configuration of type* $(3, 2, 1, 1)$ if there exist one triplet and one couple of parallel lines and two additional lines, every set with different slopes. Similarly *configurations of types* $(3, 2, 2)$ and $(2, 2, 2, 1)$ are defined and these four types of the configurations exhaust all possible configurations formed by 8 invariant lines for a cubic system.

Note that in all configurations the invariant straight line which is omitted is the infinite one.

Suppose a cubic system (1) possesses 8 invariant straight lines, including the infinite one, and taking into account their multiplicities. We say that these lines form a *potential configuration of type* $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$) if there exists a sequence of vector fields $\mathbf{X}_k$ as in Definition 1 having 8 distinct lines of type $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$).

It is well known that the infinite singularities (real or complex) of cubic systems are determined by the linear factors of the polynomial $C_3(x, y) = yp_3(x, y)-xq_3(x, y)$ where p_3 and q_3 are the cubic homogeneities of these systems.

In this paper we consider the family of cubic systems possessing two distinct infinite singularities defined by one triple and one simple factors of the invariant polynomial $C_3(x, y)$. This family univocally is determined by affine invariant criteria (see Lemma 7). Moreover we impose some additional conditions related with the existence of triplets and/or couples of parallel invariant lines of these systems (see Theorem 1 and Main Theorem). As a result we investigate the obtained subfamily of cubic systems and determine necessary and sufficient affine invariant conditions for the existence of eight invariant straight lines, including the line at infinity and taking into account their multiplicities.

Our results are stated in the following theorem.

Main Theorem. *We consider here the family of cubic systems for which the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$, $\mathcal{D}_2 \neq 0$ hold, i.e. the infinite singularities of these systems are determined by one triple and one simple factors of the invariant polynomial $C_3(x, y)$. Moreover we assume in addition that for this family the condition $\mathcal{V}_1 = \mathcal{V}_3 = 0$ is satisfied. Then:*

(A) *This family of cubic systems could be brought via an affine transformation and time rescaling to the systems*

$$\dot{x} = a + cx + dy + 2hxy + ky^2 + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + ny^2, \tag{2}$$

which could possess one of the 16 possible configurations Config. 8.23 – Config. 8.38 of invariant lines given in Figure 1.

(B) *The condition* $\mathcal{K}_5 = N_1 = 0$ *is necessary for a system* (2) *to have invariant lines of total multiplicity 8, including the line at infinity. Assuming this condition to be satisfied, a system* (2) *possesses the specific configuration Config. 8.j* ($j \in \{23, 24, \ldots, 38\}$) *if and only if the corresponding additional conditions included below are fulfilled. Moreover this system can be brought via an affine transformation and time rescaling to the canonical form, written below next to the configuration:*

- *Config.8.23* $\Leftrightarrow N_2N_3 \neq 0,\ N_4{=}N_5{=}N_6 = N_7{=}0$: $\begin{cases} \dot{x}{=}(x{-}1)x(1{+}x), \\ \dot{y}{=}x{-}y{+}x^2{+}3xy; \end{cases}$

- *Config. 8.24 - 8.27* $\Leftrightarrow\ N_2 \neq 0, N_3 = 0,\ N_4 = N_6 = N_8 = 0,\ N_9 \neq 0$:

$$\begin{cases} \dot{x} = x(r + 2x + x^2), \\ \dot{y} = (r + 2x)y,\ r(9r - 8) \neq 0; \end{cases} \quad \begin{cases} \textit{Config.8.24} \Leftrightarrow N_{11} < 0\ (r < 0); \\ \textit{Config.8.25} \Leftrightarrow N_{10} > 0, N_{11} > 0\ (0 < r < 1); \\ \textit{Config.8.26} \Leftrightarrow N_{10} = 0\ (r = 1); \\ \textit{Config.8.27} \Leftrightarrow N_{10} < 0\ (r > 1); \end{cases}$$

- *Config. 8.28 - 8.30* $\Leftrightarrow\ N_2 \neq 0, N_3 = 0,\ N_5 = N_8 = N_{12} = 0,\ N_{13} \neq 0$:

$$\begin{cases} \dot{x}{=}x(r{-}2x{+}x^2), (9r{-}8) \neq 0 \\ \dot{y}{=}2y(x{-}r), r(r-1) \neq 0; \end{cases} \quad \begin{cases} \textit{Config.8.28} \Leftrightarrow N_{15} < 0\ (r < 0); \\ \textit{Config.8.29} \Leftrightarrow N_{14} < 0, N_{15} > 0\ (0 < r < 1); \\ \textit{Config.8.30} \Leftrightarrow N_{14} > 0\ (r > 1); \end{cases}$$

- *Config. 8.31, 8.32* $\Leftrightarrow\ N_2 = N_3 = 0,\ N_{17} = N_{18} = 0,\ N_{10}N_{16} \neq 0$:

$$\begin{cases} \dot{x} = x(r + x^2), \\ \dot{y} = x - 2ry,\ r \in \{-1, 1\}; \end{cases} \quad \begin{cases} \textit{Config.8.31} \Leftrightarrow N_{10} < 0\ (r = -1); \\ \textit{Config.8.33} \Leftrightarrow N_{10} > 0,\ (r = 1); \end{cases}$$

- *Config. 8.33* $\Leftrightarrow\ N_2{=}N_3{=}0,\ N_{10}{=}N_{17}{=}N_{18}{=}0, N_{16} \neq 0$: $\begin{cases} \dot{x} = x^3, \\ \dot{y} = 1 + x; \end{cases}$

- *Config.8.34 - 8.38* $\Leftrightarrow\ N_2 = N_3 = 0,\ N_{16} = N_{19} = 0,\ N_{18} \neq 0$:

$$\begin{cases} \dot{x} = x(r + x + x^2), \\ \dot{y} = 1 + ry, (9r{-}2) \neq 0; \end{cases} \quad \begin{cases} \textit{Config. 8.34} \Leftrightarrow N_{21} < 0\ (r < 0); \\ \textit{Config. 8.35} \Leftrightarrow N_{20} > 0, N_{21} > 0\ (0 < r < 1/4); \\ \textit{Config. 8.36} \Leftrightarrow N_{20} = 0\ (r = 1/4); \\ \textit{Config. 8.37} \Leftrightarrow N_{20} < 0\ (r > 1/4); \\ \textit{Config. 8.38} \Leftrightarrow N_{21} = 0\ (r = 0). \end{cases}$$

Remark 2. If in a configuration an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is in bold face. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. We indicate next to the real singular points of the system, located on the invariant straight lines, their corresponding multiplicities.

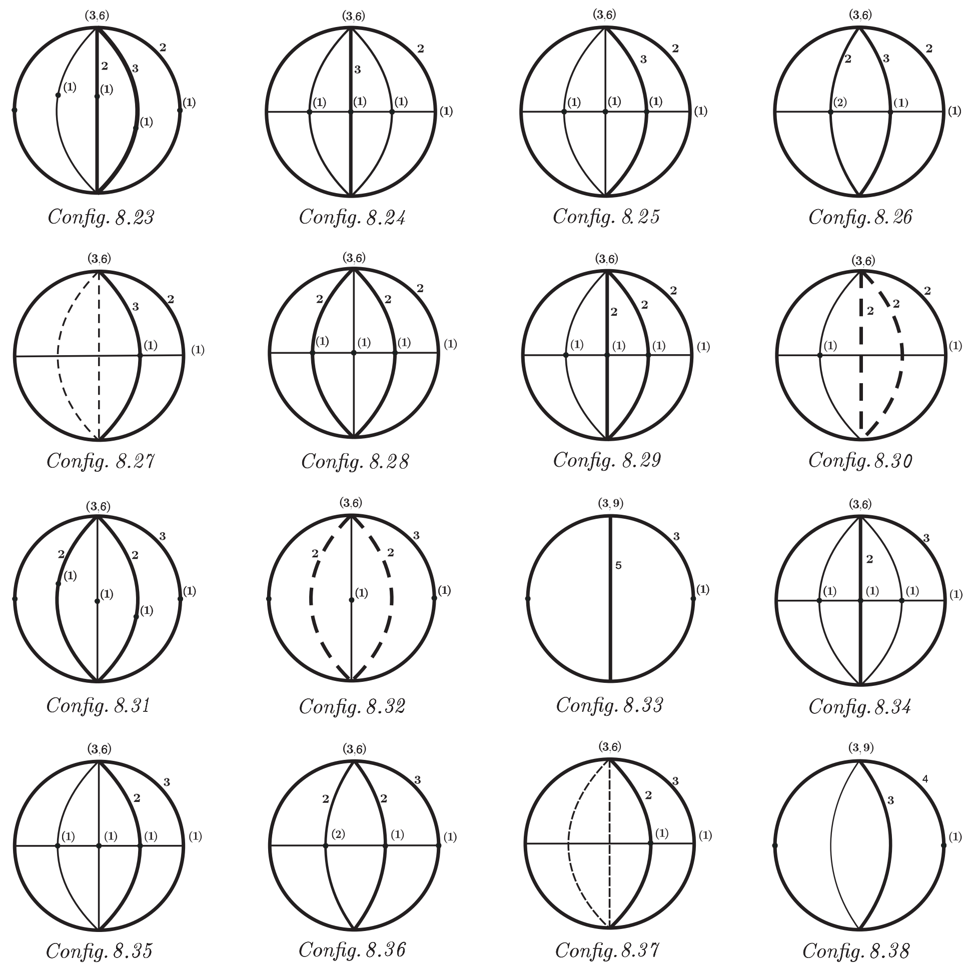


Figure 1. **The configurations of invariant straight lines of cubic systems** (2)

2 Preliminaries

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv p(x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv q(x,y) \end{aligned} \tag{3}$$

with real coefficients and variables x and y. The polynomials p_i and q_i $(i = 0, 1, 2, 3)$ are homogeneous polynomials of degree i in x and y:

$$\begin{aligned} &p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ &p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\ &q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (3) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y]$.

2.1 The main invariant polynomials associated to configurations of invariant lines

It is known that on the set **CS** of all cubic differential systems (3) the group $Aff(2,\mathbb{R})$ of affine transformations acts on the plane [27]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **CS**. We can identify the set **CS** of systems (3) with a subset of $\mathbb{R}^{20}$ via the map **CS** $\longrightarrow \mathbb{R}^{20}$ which associates to each system (3) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03},\ b_{00}, b_{10}, b_{01}, \dots, b_{03})$ of its coefficients.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [27]. Here we shall only construct the necessary $T-$ and $CT-$comitants associated to configurations of invariant lines for the family of cubic systems mentioned in the statement of Main Theorem.

Let us consider the polynomials

$$C_i(a,x,y) = yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ \ i = 0,1,2,3,$$
$$D_i(a,x,y) = \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ \ i = 1,2,3,$$

which in fact are GL-comitants, see [32]. Let $f,\ g \in \mathbb{R}[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called *the transvectant* of index k of (f,g) (cf. [12],[19])

We apply a translation $x = x' + x_0,\ y = y' + y_0$ to the polynomials $p(a,x,y)$ and $q(a,x,y)$ and we obtain $\tilde{p}(\tilde{a}(a,x_0,y_0),x',y') = p(a,x'+x_0,y'+y_0)$, $\tilde{q}(\tilde{a}(a,x_0,y_0),x',y') = q(a,x'+x_0,y'+y_0)$. Let us construct the following polynomials:

$$\Omega_i(a,x_0,y_0) \equiv \mathrm{Res}_{\,x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\tilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y]\ \ (i=1,2,3).$$

Remark 3. We note that the constructed polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ are affine comitants of systems (3) and are homogeneous polynomials in the coefficients $a_{00},\dots,b_{02}$ and non-homogeneous in x,y and

$$\deg_a \mathcal{G}_1 = 3, \qquad \deg_a \mathcal{G}_2 = 4, \qquad \deg_a \mathcal{G}_3 = 5,$$
$$\deg_{(x,y)} \mathcal{G}_1 = 8, \qquad \deg_{(x,y)} \mathcal{G}_2 = 10, \qquad \deg_{(x,y)} \mathcal{G}_3 = 12.$$

Notation 1. *Let* $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ *be the homogenization of* $\tilde{\mathcal{G}}_i(a,x,y)$*, i.e.*

$$\mathcal{G}_1(a,X,Y,Z)=Z^8\tilde{\mathcal{G}}_1(a,X/Z,Y/Z),\quad \mathcal{G}_2(a,X,Y,Z)=Z^{10}\tilde{\mathcal{G}}_2(a,X/Z,Y/Z),$$
$$\mathcal{G}_3(a,X,Y,Z)=Z^{12}\tilde{\mathcal{G}}_3(a,X/Z,Y/Z),$$

and $\mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ *in* $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of the above defined affine comitants is given by the two following lemmas (see [16]):

Lemma 1. *The straight line* $f(x,y)\equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ *is an invariant line for a cubic system* (3) *if and only if the polynomial* $f(x,y)$ *is a common factor of the polynomials* $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ *and* $\tilde{\mathcal{G}}_3(a,x,y)$ *over* $\mathbb{C}$*, i.e.* $\tilde{\mathcal{G}}_i(a,x,y)=(ux+vy+w)\widetilde{W}_i(x,y)$ $(i=1,2,3)$*, where* $\widetilde{W}_i(x,y)\in\mathbb{C}[x,y]$.

Lemma 2. *Consider a cubic system* (3) *and let* $a\in\mathbb{R}^{20}$ *be its 20-tuple of coefficients.*

1) If $f(x,y)\equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ *is an invariant straight line of multiplicity* k *for this system then* $[f(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1,\tilde{\mathcal{G}}_2,\tilde{\mathcal{G}}_3)$ *in* $\mathbb{C}[x,y]$*, i.e. there exist* $W_i(a,x,y)\in\mathbb{C}[x,y]$ $(i=1,2,3)$ *such that*

$$\tilde{\mathcal{G}}_i(a,x,y)=(ux+vy+w)^kW_i(a,x,y),\quad i=1,2,3. \tag{4}$$

2) If the line $l_\infty: Z=0$ *is of multiplicity* $k>1$ *then* $Z^{k-1}\mid\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$*, i.e. we have* $Z^{k-1}\mid H(a,X,Y,Z)$.

Consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ constructed in [4] and acting on $\mathbb{R}[a,x,y]$, where

$$\begin{aligned}\mathbf{L}_1 =&3a_{00}\frac{\partial}{\partial a_{10}}+2a_{10}\frac{\partial}{\partial a_{20}}+a_{01}\frac{\partial}{\partial a_{11}}+\frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}}+\frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}}+a_{20}\frac{\partial}{\partial a_{30}}+\\ &3b_{00}\frac{\partial}{\partial b_{10}}+2b_{10}\frac{\partial}{\partial b_{20}}+b_{01}\frac{\partial}{\partial b_{11}}+\frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}}+\frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}}+b_{20}\frac{\partial}{\partial b_{30}},\\ \mathbf{L}_2 =&3a_{00}\frac{\partial}{\partial a_{01}}+2a_{01}\frac{\partial}{\partial a_{02}}+a_{10}\frac{\partial}{\partial a_{11}}+\frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}}+\frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}}+a_{02}\frac{\partial}{\partial a_{03}}+\\ &3b_{00}\frac{\partial}{\partial b_{01}}+2b_{01}\frac{\partial}{\partial b_{02}}+b_{10}\frac{\partial}{\partial b_{11}}+\frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}}+\frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}}+b_{02}\frac{\partial}{\partial b_{03}}.\end{aligned}$$

Using this operator and the affine invariant $\mu_0=\text{Resultant}_x\big(p_3(a,x,y),q_3(a,x,y)\big)/y^9$ we construct the following polynomials

$$\mu_i(a,x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,9,$$

where $\mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0)=\mu_0$.

These polynomials are in fact comitants of systems (3) with respect to the group $GL(2,\mathbb{R})$ (see [4]). The polynomial $\mu_i(a,x,y)$, $i\in\{0,1,\dots,9\}$ is homogeneous of degree 6 in the coefficients of systems (3) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomial is revealed in the next lemma.

Lemma 3 (see [3, 4]). *Assume that a cubic system (S) with coefficients $\tilde{a}$ belongs to the family* (3). *Then:*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i \in \{0,1,\dots,k-1\}$ we have $\mu_i(\tilde{a},x,y)=0$ in the ring $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y)\neq 0$. In this case the factorization $\mu_k(\tilde{a},x,y)=\prod_{i=1}^{k}(u_i x - v_i y)\neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of those finite singularities of the system (S) which "have gone" to infinity. Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have collapsed with the infinite singular point $[v_i : u_i : 0]$.

(ii) The system (S) is degenerate (i.e. $\gcd(P,Q)\neq \text{const}$) if and only if $\mu_i(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ for every $i=0,1,\dots,9$.

In order to define the needed invariant polynomials we first construct the following comitants of second degree with respect to the coefficients of the initial system:

$$
\begin{array}{lll}
S_1=(C_0,C_1)^{(1)}, & S_{10}=(C_1,C_3)^{(1)}, & S_{19}=(C_2,D_3)^{(1)},\\
S_2=(C_0,C_2)^{(1)}, & S_{11}=(C_1,C_3)^{(2)}, & S_{20}=(C_2,D_3)^{(2)},\\
S_3=(C_0,D_2)^{(1)}, & S_{12}=(C_1,D_3)^{(1)}, & S_{21}=(D_2,C_3)^{(1)},\\
S_4=(C_0,C_3)^{(1)}, & S_{13}=(C_1,D_3)^{(2)}, & S_{22}=(D_2,D_3)^{(1)},\\
S_5=(C_0,D_3)^{(1)}, & S_{14}=(C_2,C_2)^{(2)}, & S_{23}=(C_3,C_3)^{(2)},\\
S_6=(C_1,C_1)^{(2)}, & S_{15}=(C_2,D_2)^{(1)}, & S_{24}=(C_3,C_3)^{(4)},\\
S_7=(C_1,C_2)^{(1)}, & S_{16}=(C_2,C_3)^{(1)}, & S_{25}=(C_3,D_3)^{(1)},\\
S_8=(C_1,C_2)^{(2)}, & S_{17}=(C_2,C_3)^{(2)}, & S_{26}=(C_3,D_3)^{(2)},\\
S_9=(C_1,D_2)^{(1)}, & S_{18}=(C_2,C_3)^{(3)}, & S_{27}=(D_3,D_3)^{(2)}.
\end{array}
$$

We shall use here the following invariant polynomials constructed in [16] to characterize the family of cubic systems possessing the maximal number of invariant straight lines:

$$
\begin{aligned}
\mathcal{D}_1(a) &= 6S_{24}^3-\Big[(C_3,S_{23})^{(4)}\Big]^2,\ \mathcal{D}_2(a,x,y)=-S_{23},\\
\mathcal{D}_3(a,x,y) &= (S_{23},\,S_{23})^{(2)}-6C_3(C_3,\,S_{23})^{(4)},\ \mathcal{D}_4(a,x,y)=(C_3,\mathcal{D}_2)^{(4)},\\
\mathcal{V}_1(a,x,y) &= S_{23}+2D_3^2,\ \mathcal{V}_2(a,x,y)=S_{26},\ \mathcal{V}_3(a,x,y)=6S_{25}-3S_{23}-2D_3^2,\\
\mathcal{V}_4(a,x,y) &= C_3\Big[(C_3,S_{23})^{(4)}+36\,(D_3,S_{26})^{(2)}\Big],\\
\mathcal{V}_5(a,x,y) &= 6T_1(9A_5-7A_6)+2T_2(4T_{16}-T_{17})-3T_3(3A_1+5A_2)+3A_2T_4+\\
&\quad +36T_5^2-3T_{44},\\
\mathcal{L}_1(a,x,y) &= 9C_2\,(S_{24}+24S_{27})-12D_3\,(S_{20}+8S_{22})-12\,(S_{16},D_3)^{(2)}\\
&\quad -3\,(S_{23},C_2)^{(2)}-16\,(S_{19},C_3)^{(2)}+12\,(5S_{20}+24S_{22},C_3)^{(1)},
\end{aligned}
$$

$$\begin{aligned}
\mathcal{L}_2(a,x,y) = {} & 32\,(13S_{19}+33S_{21},D_2)^{(1)}+84\,(9S_{11}-2S_{14},D_3)^{(1)}+\\
& +8D_2\,(12S_{22}+35S_{18}-73S_{20})-448\,(S_{18},C_2)^{(1)}-\\
& -56\,(S_{17},C_2)^{(2)}-63\,(S_{23},C_1)^{(2)}+756D_3S_{13}-1944D_1S_{26}+\\
& +112\,(S_{17},D_2)^{(1)}-378\,(S_{26},C_1)^{(1)}+9C_1\,(48S_{27}-35S_{24}),\\
\mathcal{U}_1(a) = {} & T_{31}-4T_{37},\\
\mathcal{U}_2(a,x,y) = {} & 6\,(T_{30}-3T_{32},T_{36})^{(1)}-3T_{30}\,(T_{32}+8T_{37})-\\
& -24T_{36}^2+2C_3\,(C_3,T_{30})^{(4)}+24D_3\,(D_3,T_{36})^{(1)}+24D_3^2T_{37}.\\
\mathcal{K}_2(a,x,y) = {} & T_{74},\quad \mathcal{K}_4(a,x,y)=T_{13}-2T_{11},\\
\mathcal{K}_1(a,x,y) = {} & \left(3223T_2^2T_{140}+2718T_4T_{140}-829T_2^2T_{141},T_{133}\right)^{(10)}/2,\\
\mathcal{K}_5(a,x,y) = {} & 45T_{42}-T_2T_{14}+2T_2T_{15}+12T_{36}+45T_{37}-45T_{38}+30T_{39},\\
\mathcal{K}_6(a,x,y) = {} & 4T_1T_8(2663T_{14}-8161T_{15})+6T_8(178T_{23}+70T_{24}+555T_{26})+\\
& +18T_9(30T_2T_8-488T_1T_{11}-119T_{21})+5T_2(25T_{136}+16T_{137})-\\
& -15T_1(25T_{140}-11T_{141})-165T_{142},\\
\mathcal{K}_8(a,x,y) = {} & 10A_4T_1-3T_2T_{15}+4T_{36}-8T_{37}.
\end{aligned}$$

However these invariant polynomials are not sufficient to characterize the cubic systems with invariant lines of the total multiplicity 8. So we construct here the following new invariant polynomials:

$$\begin{aligned}
N_1(a,x,y) = {} & S_{13},\quad N_2(a,x,y)=C_2D_3+3S_{16},\quad N_3(a,x,y)=T_9,\\
N_4(a,x,y) = {} & -S_{14}^2-2D_2^2(3S_{14}-8S_{15})-12D_3(S_{14},C_1)^{(1)}+\\
& +D_2(-48D_3S_9+16(S_{17},C_1)^{(1)}),\\
N_5(a,x,y) = {} & 36D_2D_3(S_8-S_9)+D_1(108D_2^2D_3-54D_3(S_{14}-8S_{15}))+\\
& +2S_{14}(S_{14}-22S_{15})-8D_2^2(3S_{14}+S_{15})-9D_3(S_{14},C_1)^{(1)}-16D_2^4,\\
N_6(a,x,y) = {} & 40D_3^2(15S_6-4S_3)-480D_2D_3S_9-20D_1D_3(S_{14}-4S_{15})+\\
& +160D_2^2S_{15}-35D_3(S_{14},C_1)^{(1)}+8\big((S_{23},C_2)^{(1)},C_0\big)^{(1)},\\
N_7(a,x,y) = {} & 18C_2D_2(9D_1D_3-S_{14})-2C_1D_3(8D_2^2-3S_{14}-74S_{15})-\\
& -432C_0D_3S_{21}+48S_7(8D_2D_3+S_{17})-51S_{10}S_{14}+\\
& +6S_{10}(12D_2^2+151S_{15})-162D_1D_2S_{16}+864D_3(S_{16},C_0)^{(1)},\\
N_8(a,x,y) = {} & -32D_3^2S_2-108D_1D_3S_{10}+108C_3D_1S_{11}-18C_1D_3S_{11}-\\
& -27S_{10}S_{11}+4C_0D_3(9D_2D_3+4S_{17})+108S_4S_{21},\\
N_9(a,x,y) = {} & 11S_{14}^2-2592D_1^2S_{25}+88D_2(S_{14},C_2)^{(1)}-\\
& -16D_1D_3(16D_2^2+19S_{14}-152S_{15})-8D_2^2(7S_{14}+32S_{15}),\\
N_{10}(a,x,y) = {} & -24D_1D_3+4D_2^2+S_{14}-8S_{15},\\
N_{11}(a,x,y) = {} & S_{14}^2+8D_1D_3[2D_2^2-(S_{14}-8S_{15})]-2D_2^2(5S_{14}-8S_{15})+\\
& +8D_2(S_{14},C_2)^{(1)},
\end{aligned}$$

$$
\begin{aligned}
N_{12}(a,x,y) =\ & 135D_1D_3[8D_2^2-(S_{14}-20S_{15})]-5D_2^2(39S_{14}-32S_{15})+\\
& +5S_{14}^2-160D_2^4-1620D_3^2S_3+85D_2(S_{14},C_2)^{(1)}+\\
& +81\big((S_{23},C_2)^{(1)},C_0\big)^{(1)},\\
N_{13}(a,x,y) =\ & 2(136D_3^2S_2-126D_2D_3S_4+60D_2D_3S_7+63S_{10}S_{11})-\\
& -18C_3D_1(S_{14}-28S_{15})-12C_1D_3(7S_{11}-20S_{15})+\\
& +4C_0D_3(21D_2D_3+17S_{17})+3C_2(S_{14},C_2)^{(1)}-192C_2D_2S_{15},\\
N_{14}(a,x,y) =\ & -6D_1D_3-15S_{12}+2S_{14}+4S_{15},\\
N_{15}(a,x,y) =\ & 216D_1D_3(63S_{11}-104D_2^2-136S_{15})+4536D_3^2S_6+\\
& +4096D_2^4+120S_{14}^2+992D_2(S_{14},C_2)^{(1)}+\\
& +135D_3\big[28(S_{17},C_0)^{(1)}+5(S_{14},C_1)^{(1)}\big],\\
N_{16}(a,x,y) =\ & 2C_1D_3+3S_{10},\ \ N_{17}(a,x,y)=\ 6D_1D_3-2D_2^2-(C_3,C_1)^{(2)},\\
N_{18}(a,x,y) =\ & 2D_2^3-6D_1D_2D_3-12D_3S_5+3D_3S_8,\\
N_{19}(a,x,y) =\ & C_1D_3(18D_1^2-S_6)-3C_0D_3(4D_1D_2+6S_5-3S_8)+\\
& +6C_2D_1S_82D_2(9D_3S_1-4D_2S_2)+2D_1(12D_3S_2-9C_3S_6)+\\
& +4C_0D_2^3-18D_3(S_4,C_0)^{(1)},\\
N_{20}(a,x,y) =\ & 3D_2^4-8D_1D_2^2D_3-8D_3^2S_6-16D_1D_3S_{11}+16D_2D_3S_9,\\
N_{21}(a,x,y) =\ & 2D_1D_2^2D_3-4D_3^2S_6+D_2D_3S_8+D_1(S_{23},C_1)^{(1)}
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 =\ & S_{24}/288,\quad A_2=S_{27}/72,\quad A_3=(72D_1A_2+(S_{22},D_2)^{(1)}/24,\\
A_4 =\ & [9D_1(S_{24}-288A_2)+4(9S_{11}-2S_{14},D3)^{(2)}+8(3S_{18}-S_{20}-4S_{22},D_2)^{(1)}]/2^73^3
\end{aligned}
$$

are affine invariants, whereas the polynomials

$$
\begin{aligned}
T_1 &= C_3,\quad T_2=\ D_3,\quad T_3=\ S_{23}/18,\quad T_4=\ S_{25}/6,\quad T_5=\ S_{26}/72,\\
T_6 &= \big[2C_3(2D_2^2-S_{14}+8S_{15})-3C_1M_1-2C_2M_2\big]/2^4/3^2,\\
T_8 &= \big[5D_2(D_3^2+27T_3-18T_4)+20D_3S_{19}+12\big(S_{16},D_3\big)^{(1)}-8D_3S_{17}\big]/5/2^5/3^3,\\
T_9 &= \big[9D_1M_1+2D_2(D_2D_3-3S_{17}-S_{19}-9S_{21})+18\big(S_{15},C_3\big)^{(1)}-\\
&\quad -6C_2(2S_{20}-3S_{22})+18C_1S_{26}+2D_3S_{14}\big]/2^4/3^3,\\
T_{11} &= \big[6\big(M_1,D_2\big)^{(1)}-\big(M_1,C_2\big)^{(2)}-12\big(S_{26},C_2\big)^{(1)}+12D_2S_{26}+\\
&\quad +432(A_1-5A_2)C_2\big]/2^7/3^4,\\
T_{13} &= \big[27(T_3,C_2)^{(2)}-18(T_4,C_2)^{(2)}+48D_3S_{22}-216(T_4,D_2)^{(1)}+36D_2S_{26}-\\
&\quad -1296C_2A_1-7344C_2A_2+(D_3^2,C_2)^{(2)}\big]/2^7/3^4,\\
T_{14} &= \big[\big(8S_{19}+9S_{21},D_2\big)^{(1)}-D_2(8S_{20}+3S_{22})+18D_1S_{26}+1296C_1A_2\big]/2^4/3^3,
\end{aligned}
$$

$$
\begin{aligned}
T_{15} = {} & 8\big(9S_{19}+2S_{21}, D_2\big)^{(1)}+3\big(M_1, C_1\big)^{(2)}-4\big(S_{17}, C_2\big)^{(2)}+\\
& +4\big(S_{14}-17S_{15}, D_3\big)^{(1)}-8\big(S_{14}+S_{15}, C_3\big)^{(2)}+432C_1(5A_1+11A_2)+\\
& +36D_1S_{26}-4D_2(S_{18}+4S_{22})\big]/2^6/3^3,\\
T_{21} = {} & \big(T_8, C_3\big)^{(1)}, \quad T_{23} = \big(T_6, C_3\big)^{(2)}/6, \quad T_{24} = \big(T_6, D_3\big)^{(1)}/6,\\
T_{26} = {} & \big(T_9, C_3\big)^{(1)}/4, \quad T_{30} = \big(T_{11}, C_3\big)^{(1)}, \quad T_{31} = \big(T_8, C_3\big)^{(2)}/24,\\
T_{32} = {} & \big(T_8, D_3\big)^{(1)}/6, \quad T_{36} = \big(T_6, D_3\big)^{(2)}/12, \quad T_{37} = \big(T_9, C_3\big)^{(2)}/12,\\
T_{38} = {} & \big(T_9, D_3\big)^{(1)}/12, \quad T_{39} = \big(T_6, C_3\big)^{(3)}/2^4/3^2, \quad T_{42} = \big(T_{14}, C_3\big)^{(1)}/2,\\
T_{44} = {} & \big((S_{23}, C_3)^{(1)}, D_3\big)^{(2)}/5/2^6/3^3,\\
T_{74} = {} & \big[27C_0M_1^2 - C_1\big(2^83^5T_{11}C_3 + 3M_1M_2\big) + 2^83^4T_{11}C_2^2+\\
& + C_2M_1(8D_2^2 + 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15})-\\
& - 54D_1M_1S_{16} - 54C_3M_1(2D_1D_2 - S_8 + 2S_9) - 2^63^2T_6M_2\big]/2^8/3^4,\\
T_{133} = {} & (T_{74}, C_3)^{(1)}, \quad T_{136} = \big(T_{74}, C_3\big)^{(2)}/24, \quad T_{137} = \big(T_{74}, D_3\big)^{(1)}/6,\\
T_{140} = {} & \big(T_{74}, D_3\big)^{(2)}/12, \; T_{141} = \big(T_{74}, C_3\big)^{(3)}/36, \; T_{142} = \big((T_{74}, C_3)^{(2)}, C_3\big)^{(1)}/72
\end{aligned}
$$

where $M_1 = 9T_3-18T_4-D_3^2,\ M_2 = 2D_2D_3-S_{17}+2S_{19}-6S_{21}$ and $T_i,\ i = 1, ..., 142$ are T-comitants of cubic systems (3) (see for details [27]). We note that these invariant polynomials are the elements of the polynomial basis of T-comitants up to degree six constructed by Iu. Calin [8].

2.2 Preliminary results

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a, x, y)$ for $i = 1, 2, 3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance [14, 19]).

Following [16] we consider two polynomials $f(z) = a_0z^n+a_1z^{n-1}+\cdots+ a_n,\ g(z) = b_0z^m + b_1z^{m-1} + \cdots + b_m$, in the variable z of degree n and m, respectively. Thus the k–th *subresultant* with respect to variable z of the two polynomials $f(z)$ and $g(z)$ will be denoted by $R_z^{(k)}(f, g)$.

We say that the k–th *subresultant* with respect to variable z of the two polynomials $f(z)$ and $g(z)$ is the $(m + n - 2k) \times (m + n - 2k)$ determinant

$$
R_z^{(k)}(f,g) = \left|\begin{array}{cccc}
a_0a_1 & a_2\dots & \dots a_{m+n-2k-1}\\
0a_0 & a_1\dots & \dots a_{m+n-2k-2}\\
00 & a_0\dots & \dots a_{m+n-2k-3}\\
\dots\dots & \dots\dots & \dots\dots\ \dots\ \dots\\
00 & b_0\dots & \dots b_{m+n-2k-3}\\
0b_0 & b_1\dots & \dots b_{m+n-2k-2}\\
b_0b_1 & b_2\dots & \dots b_{m+n-2k-1}
\end{array}\right|
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(m-k)-times\\
\\
\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(n-k)-times
\end{array}
\qquad (5)
$$

in which there are $m-k$ rows of a's and $n-k$ rows of b's, and $a_i = 0$ for $i > n$, and $b_j = 0$ for $j > m$.

For $k = 0$ we obtain the standard resultant of two polynomials. In other words we can say that the k–th subresultant with respect to the variable z of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last k rows and the first and the last k columns from its resultant written in the form (5) when $k = 0$.

The geometrical meaning of the subresultant is based on the following lemma.

Lemma 4 (see [14, 19])**.** *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g) = R_z^{(1)}(f,g) = R_z^{(2)}(f,g) = \cdots = R_z^{(k-1)}(f,g) = 0 \neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 4 the following result.

Lemma 5. *Two polynomials $\tilde{f}(x_1, x_2, ..., x_n)$ and $\tilde{g}(x_1, x_2, ..., x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde{f},\tilde{g}) = R_{x_j}^{(1)}(\tilde{f},\tilde{g}) = R_{x_j}^{(2)}(\tilde{f},\tilde{g}) = \cdots = R_{x_j}^{(k-1)}(\tilde{f},\tilde{g}) = 0 \neq R_{x_j}^{(k)}(\tilde{f},\tilde{g}),$$

where $R_{x_j}^{(i)}(\tilde{f},\tilde{g}) = 0$ in $\mathbb{R}[x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_n]$.

In paper [16] 23 configurations of invariant lines (one more configuration is constructed in [5]) are determined in the case, when the total multiplicity of these lines (including the line at infinity) equals nine. For this purpose in [16] the authors proved some lemmas concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. In [6] these results have been completed.

Theorem 1 (see [6])**.** *If a cubic system* (3) *possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

(i)	*2 triplets*	$\Rightarrow$	$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;
(ii)	*1 triplet and 2 couples*	$\Rightarrow$	$\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$;
(iii)	*1 triplet and 1 couple*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;
(iv)	*one triplet*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{U}_2 = 0$;
(v)	*3 couples*	$\Rightarrow$	$\mathcal{V}_3 = 0$;
(vi)	*2 couples*	$\Rightarrow$	$\mathcal{V}_5 = 0$.

In papers [6] and [7] all the possible configurations of invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity are determined for cubic systems with at least three distinct infinite singularities. In particular the next result is obtained.

Lemma 6 (see [6]). *A cubic system with four distinct infinite singularities could not possess configuration of invariant lines of type* $(3,2,2)$. *And it possesses a configuration or potential configuration of a given type if and only if the following conditions are satisfied, respectively*

$$\begin{array}{lcl} (3,3,1) & \Leftrightarrow & \mathcal{V}_1=\mathcal{V}_2=\mathcal{L}_1=\mathcal{L}_2=\mathcal{K}_1=0,\ \mathcal{K}_2\neq 0; \\ (3,2,1,1) & \Leftrightarrow & \mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=0,\quad \mathcal{D}_4\neq 0; \\ (2,2,2,1) & \Leftrightarrow & \mathcal{V}_3=\mathcal{K}_4=\mathcal{K}_2=\mathcal{K}_8=0,\quad \mathcal{D}_4\neq 0. \end{array}$$

Let $L(x,y)=Ux+Vy+W=0$ be an invariant straight line of the family of cubic systems (3). Then, we have

$$UP(x,y)+VQ(x,y)=(Ux+Vy+W)(Ax^2+2Bxy+Cy^2+Dx+Ey+F),$$

and this identity provides the following 10 relations:

$$\begin{aligned} Eq_1 =&(a_{30}-A)U+b_{30}V=0,\ Eq_2=(3a_{21}-2B)U+(3b_{21}-A)V=0,\\ Eq_3 =&(3a_{12}-C)U+(3b_{12}-2B)V=0,\ Eq_4=(a_{03}-C)U+b_{03}V=0,\\ Eq_5 =&(a_{20}-D)U+b_{20}V-AW=0,\\ Eq_6 =&(2a_{11}-E)U+(2b_{11}-D)V-2BW=0,\\ Eq_7 =&a_{22}U+(b_{22}-E)V-CW=0,\ Eq_8=(a_{10}-F)U+b_{10}V-DW=0,\\ Eq_9 =&a_{01}U+(b_{01}-F)V-EW=0,\ Eq_{10}=a_{00}U+b_{00}V-FW=0. \end{aligned} \tag{6}$$

As it was mentioned earlier, the infinite singularities (real or complex) of systems (3) are determined by the linear factors of the polynomial C_3. So in the case of two distinct infinite singularities they are determined either by one triple and one simple real or two double real (or complex) factors of the polynomial $C_3(x,y)$. We consider here the first case.

Lemma 7 (see [20]). *A cubic system* (3) *possesses the infinite singularities determined by one triple and one simple factors of the invariant polynomial* $C_3(x,y)$ *if and only if the conditions* $\mathcal{D}_1=\mathcal{D}_3=\mathcal{D}_4=0,\ \mathcal{D}_2\neq 0$ *hold. Moreover the cubic homogeneities of this system could be brought via a linear transformation to the canonical form*

$$\begin{aligned} x' =&(u+1)x^3+vx^2y+rxy^2,\\ y' =&ux^2y+vxy^2+ry^3, \quad \textit{with} \quad C_3=x^3y. \end{aligned} \tag{7}$$

3 The proof of the Main Theorem

Assume that a cubic system possesses two distinct infinite singularities which are determined by one simple and one triple real factors of the polynomial C_3. Then considering Lemma 7 we obtain that systems (3) via a linear transformation become:

$$\begin{aligned} x' &= p_0+p_1(x,y)+p_2(x,y)+(u+1)x^3+vx^2y+rxy^2,\\ y' &= q_0+q_1(x,y)+q_2(x,y)+ux^2y+vxy^2+ry^3 \end{aligned} \tag{8}$$

with $C_3 = x^3y$. Hence, the infinite singular points are located at the "ends" of the following straight lines: $x = 0$ and $y = 0$.

The proof of the Main Theorem proceeds in 4 steps.

First we construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems for which the corresponding necessary conditions provided by Theorem 1 in order to have the given number of triplets or/and couples of invariant parallel lines in the respective directions are satisfied.

Secondly, taking cubic systems $\dot{x} = \tilde{P}_3,\ \dot{y} = \tilde{Q}_3$ we add all quadratic, linear and constant terms and using the equations (6) we determine these terms in order to get the needed number of invariant lines in the needed configuration. Thus the second step ends with the construction of the canonical systems possessing the needed configuration.

The third step consists in the determination of the affine invariant conditions necessary and sufficient for a cubic system to belong to the family of systems (constructed at the second step) which possess the corresponding configuration of invariant lines.

And finally, in the case of the existence of multiply invariant lines in a potential configuration we construct the corresponding perturbed systems possessing 8 distinct invariant lines (including the line at infinity).

3.1 Construction of the corresponding cubic homogeneities

In what follows we construct the cubic homogeneous parts of systems (8) for each one of the possible configurations mentioned in Lemma 6.

a) The case of the configuration $(3, 3, 1)$. In this case we have two triplets of parallel invariant straight lines and according to Theorem 1 the condition $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ is necessary for systems (8). A straightforward computation of the value of $\mathcal{V}_1$ provides $\mathcal{V}_1 = 16\sum_{j=0}^{4} \mathcal{V}_{1j}x^{4-j}y^j$, where

$$\mathcal{V}_{10} = u(2u + 3), \quad \mathcal{V}_{12} = 4ru + 3r + 2v^2,$$
$$\mathcal{V}_{11} = v(4u + 3), \quad \mathcal{V}_{13} = 4vr, \quad \mathcal{V}_{14} = 2r^2.$$

Therefore from $\mathcal{V}_1 = 0$ it results $v = r = 0$ and $u(2u + 3) = 0$, and we consider two subcases: $u = 0$ and $u = -3/2$. For $u = 0$ we get the cubic homogeneous system:

$$\dot{x} = x^3, \quad \dot{y} = 0 \tag{9}$$

whereas for $u = -3/2$, after the time rescaling $t \to -2t$, we have

$$\dot{x} = x^3, \quad \dot{y} = 3x^2y. \tag{10}$$

It has to be underlined that for systems (9) and (10) the relation $\mathcal{V}_2 = \mathcal{U}_1 = 0$ holds.

b) The case of the configuration $(3,2,1,1)$. According to Theorem 1, if a cubic system possesses 7 invariant straight lines in the configuration $(3,2,1,1)$, then necessarily the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ hold.

We consider again systems (8). A straightforward computation of the value of $\mathcal{V}_5$ yields: $\mathcal{V}_5 = \dfrac{9}{32}\sum_{j=0}^{4} \mathcal{V}_{5j} x^{4-j} y^j$, where

$$\mathcal{V}_{50} = -u(3r + ru - v^2), \quad \mathcal{V}_{52} = 6r^2 u,$$
$$\mathcal{V}_{51} = 4ruv, \quad \mathcal{V}_{53} = 0, \quad \mathcal{V}_{54} = -r^3.$$

Hence $r = 0$ which gives $\mathcal{V}_4 = 0$ and $\mathcal{U}_2 = -12288 v^2 x^2 (ux + vy)^2$. So the condition $\mathcal{U}_2 = 0$ is equivalent to $v = 0$ and in this case we have $\mathcal{V}_5 = \mathcal{V}_4 = \mathcal{U}_2 = 0$. As a result we get the family of systems

$$\dot{x} = (u+1)x^3, \quad \dot{y} = ux^2 y \tag{11}$$

if $u \neq 0$, whereas if $u = 0$ we arrive at system (9).

c) The case of the configuration $(2,2,2,1)$. According to Theorem 1 if a cubic system possesses 7 invariant straight lines in the configuration (2, 2, 2, 1), then necessarily the condition $\mathcal{V}_3 = 0$ holds.

So we shall consider the family of systems (8) and we force the condition $\mathcal{V}_3 = 0$ to be satisfied. We have:

$$\mathcal{V}_{30} = u(3+u), \quad \mathcal{V}_{31} = 2uv, \quad \mathcal{V}_{32} = -3r + 2ru + v^2, \quad \mathcal{V}_{33} = 2rv, \quad \mathcal{V}_{34} = r^2$$

where $\mathcal{V}_{3j}$ are the elements of $\mathcal{V}_3 = -32\sum_{j=0}^{4} \mathcal{V}_{3j} x^{4-j} y^j$. So the condition $\mathcal{V}_{34} = 0$ is equivalent to $r = 0$ and, in consequence, $\mathcal{V}_{33} = \mathcal{V}_{34} = 0$ and $\mathcal{V}_{32} = v^2$. So $v = 0$ and the condition $\mathcal{V}_{30} = 0$ gives $u(u+3) = 0$. Therefore if $u = -3$ due to the time rescaling $t \to -t$ we arrive at the cubic homogeneities

$$\dot{x} = 2x^3, \quad \dot{y} = 3x^2 y \tag{12}$$

whereas in the case $u = 0$ we get system (9).

So we get three specific systems (9), (10) and (12) and one-parameter family of systems (11). As it can be observed, the first three systems belong to this family for some values of the parameter u: system (9) for $u = 0$, system (10) for $u = -3/2$ (after the time rescaling $t \to -2t$) and system (12) for $u = -3$ (after the time rescaling $t \to -t$).

On the other hand for systems (11) we have $\mathcal{V}_1 = 16u(3+2u)x^4$, $\mathcal{V}_3 = -32u(3+u)x^4$ and therefore we arrive at the next proposition.

Proposition 1. *Assume that for a cubic homogeneous system the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0$ and $\mathcal{D}_2 \neq 0$ hold. Then this system can be brought to one of the canonical systems indicated below if and only if the corresponding conditions are satisfied, respectively:*

(i) $\mathcal{V}_1 = \mathcal{V}_3 = 0, \Rightarrow$ *system* (9), *(ii)* $\mathcal{V}_1 = 0,\ \mathcal{V}_3 \neq 0 \Rightarrow$ *system* (10),

(iii) $\mathcal{V}_1 \neq 0,\ \mathcal{V}_3 = 0 \Rightarrow$ *system* (12), *(iv)*$\mathcal{V}_1\mathcal{V}_3 \neq 0,\ \mathcal{V}_5 = \mathcal{U}_2 = 0 \Rightarrow$ *system* (11).

Thus for the further investigation four different homogeneous systems remain: (9), (10), (11) and (12). However in this article, we will consider only the cubic systems with cubic homogeneities of the form (9), as in the statement of the Main Theorem we assume the additional condition $\mathcal{V}_1 = \mathcal{V}_3 = 0$.

We observe that if for perturbed systems some condition $K(x, y) = 0$ holds, where $K(x, y)$ is an invariant polynomial, then this condition must hold also for the initial (unperturbed) systems. So considering Lemma 6 we arrive at the next remark.

Remark 4. Assume that a cubic system with two distinct infinite singularities possesses a potential configuration of a given type. Then for this system the following conditions must be satisfied, respectively:

$$\begin{array}{lll} (a_1) & (3,3,1) & \Rightarrow \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0; \\ (a_2) & (3,2,1,1) & \Rightarrow \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0; \\ (a_3) & (2,2,2,1) & \Rightarrow \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0. \end{array}$$

3.2 Construction of the configurations and of the corresponding normal forms

In this case, considering (8) and (9) via a translation of the origin of coordinates we can consider $g = 0$ and hence we get the cubic systems

$$\dot{x} = a + cx + x^3 + dy + 2hxy + ky^2, \quad \dot{y} = b + ex + lx^2 + fy + 2mxy + ny^2 \quad (13)$$

for which we have $H(X, Y, Z) = Z$ (see Notation 1).

Now we force the necessary conditions given in Remark 4 which correspond to each type of configuration. We claim that if any of the conditions (a_1), (a_2) or (a_3) are satisfied for a system (13) then $k = h = n = 0$ and this condition is equivalent to $\mathcal{K}_5 = 0$. We divide the proof of this claim in three subcases defined by (a_1)–(a_3).

(a_1). For systems (13) we calculate: $\mathcal{L}_1 = 0$ and

$$\begin{aligned} &\text{Coefficient}[\mathcal{L}_2, xy] = -20736(12h^2 + 7km - 6hn + 3n^2) = 0, \\ &\text{Coefficient}[\mathcal{K}_1, y^2] = 3967 \cdot 2^{18}3^9 5^4 7^3 19 k^6 = 0. \end{aligned}$$

Therefore we get $k = 0$ and as the discriminant of the binary form $4h^2 - 2hn + n^2$ is negative we obtain $h = n = 0$ (and this implies $\mathcal{L}_2 = \mathcal{K}_1 = 0$).

(a_2). In the same manner in the case of the configuration $(3, 2, 1, 1)$ we determine $\mathcal{K}_4 = \mathcal{K}_6 = 0$ and

$$\mathcal{K}_5 = -180m(h - n)x^4 + 60(4h^2 - 3km - 2hn + n^2)x^3y - 240k^2xy^3.$$

From $\mathcal{K}_5 = 0$ it results $k = 0$ and we get the same binary form $4h^2 - 2hn + n^2$ which leads to $h = n = 0$. Consequently $\mathcal{K}_5 = 0$ if and only if $k = h = n = 0$.

(a_3). We calculate $\mathcal{K}_4 = 0$ and Coefficient$[\mathcal{L}_2, x^2y^7] = 2k^3 = 0$, i.e. $k = 0$. Then calculations yield

$$\text{Coefficient}[\mathcal{K}_2, x^5y^4] = -2n(h-n)^2 = 0, \ \text{Coefficient}[\mathcal{K}_8, x^3y] = 2(4h^2 + 14hn + n^2) = 0$$

and evidently we obtain $h = n = 0$ (then $\mathcal{K}_2 = \mathcal{K}_8 = 0$) and this completes the proof of the claim.

Remark 5. Since infinite singularities of systems (8) are located on the "ends" of the axis $x = 0$ and $y = 0$, the invariant affine lines must be either of the form $Ux + W = 0$ or $Vy + W = 0$. Therefore we can assume $U = 1$ and $V = 0$ (for the direction $x = 0$) and $U = 0$ and $V = 1$ (for the direction $y = 0$). In this case, considering W as a parameter, six equations among (6) become linear with respect to the parameters $\{A, B, C, D, E, F\}$ (with the corresponding non–zero determinant) and we can determine their values, which annulate some of the equations (6). So in what follows we will examine only the non-zero equations containing the last parameter W.

Since for systems (8) the condition $k = h = n = 0$ is equivalent to $\mathcal{K}_5 = 0$ we assume this condition to be fulfilled.

We begin with the examination of the direction $x = 0$ ($U = 1,\ V = 0$). So, considering (6) and Remark 5 for systems (13) we have: $Eq_9 = d,\ Eq_{10} = a - cW - W^3$. So in the direction $x = 0$ we could have three invariant lines (which could coincide) and this occurs if and only if $d = 0$. Thus we arrive at the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + ex + lx^2 + fy + 2mxy \tag{14}$$

for which we calculate

$$H(X, Z) = Z(X^3 + cXZ^2 + aZ^3). \tag{15}$$

Remark 6. Any invariant line of the form $x + \alpha = 0$ (i.e. in the direction $x = 0$) of cubic systems (3) must be a factor of the polynomials $P(x, y)$, i.e. $(x + \alpha) \mid P(x, y)$.

Indeed, according to the definition, for an invariant line $ux + vy + w = 0$ we have $uP + vQ = (ux + vy + w)R(x, y)$, where the cofactor $R(x, y)$ generically is a polynomial of degree two. In our particular case (i.e. $u = 1, v = 0, w = \alpha$) we obtain $P(x) = (x + \alpha)R(x)$, which means that $(x + \alpha)$ divides $P(x)$.

This remark could be applied for any cubic systems when we examine the direction $x = 0$. Similarly, for an invariant line $y + \beta = 0$ in the the direction $y = 0$ it is necessary $(y + \beta = 0) \mid Q(x, y)$.

Considering systems (14) we calculate

$$
\begin{aligned}
\mathcal{G}_1/H =& lX^4 + X^3\big[4mY + 2(e - lm)Z)\big] + X^2\big[(3f - 4m^2)YZ + (3b - cl - lf-\\
&+ -2em)Z^2\big] + X\big[-4fmYZ^2 + (-2al - ef - 2bm)Z^3\big] + (cf - f^2-\\
&- 2am)YZ^3 + (bc - ae - bf)Z^4 \equiv F_1(X,Y,Z),\\
\mathcal{G}_2/H =& (X^3 + cXZ^2 + aZ^3)\Big\{2lX^3 + \big[X^2(6mY + (3e - 2lm)Z\big] + X\big[(3f-\\
&- 4m^2)YZ + (3b - cl - 2em)Z^2\big] - 2fmYZ^2 + (-al - 2bm)Z^3\Big] \equiv\\
&\equiv P^*(X,Z)F_2(X,Y,Z),\\
\mathcal{G}_3/H =& 24(lX^2 + 2mXY + eXZ + fYZ + bZ^2)(X^3 + cXZ^2 + aZ^3)^2 \equiv\\
&\equiv 24Q^*(X,Y,Z)\,[P^*(X,Z)]^2\,,
\end{aligned}
\tag{16}
$$

where $P^*(X,Z)$ and $Q^*(X,Y,Z)$ are the homogenization of the polynomials $P(x)$ and $Q(x,y)$ of systems (14). It is clear that these systems are degenerate if and only if the polynomials $P(x)$ and $Q(x,y)$ have a nonconstant common factor (depending on x) and this implies the existence of such a common factor (depending on X and Z) of the polynomials $P^*(X,Z)$ and $Q^*(X,Y,Z)$. So for non-degenerate systems the condition

$$R_X^{(0)}\left(P^*(X,Z), Q^*(X,Y,Z)\right) \neq 0 \tag{17}$$

must hold. We have the next lemma.

Lemma 8. *For a non-degenerate system* (14) *the polynomial $P^*(X,Z)$ could not be a factor of $\mathcal{G}_1/H$, i.e. $P^*(X,Z)$ does not divide $F_1(X,Y,Z)$.*

Proof. Suppose the contrary that $P^*(X,Z)$ divides $F_1(X,Y,Z)$. Then considering the form of the polynomial $P^*(X,Z)$ (which contains the term X^3) by Lemma 5 the following conditions are necessary and sufficient: $R_X^{(0)}(F_1,P^*) = R_X^{(1)}(F_1,P^*) = R_X^{(2)}(F_1,P^*) = 0$. We calculate $R_X^{(2)}(F_1,P^*) = \big[(3f - 4m^2)Y + (3b - 2cl - lf - 2em)\big]Z = 0$ and this implies $f = 4m^2/3$ and $b = 2(3cl + 3em + 2lm^2)/9$. Then we obtain

$$R_X^{(1)}(F_1,P^*) = \frac{Z^4}{81}\big[12m(3c+4m^2)Y + (27al + 18ce - 6clm + 24em^2 + 8lm^3)Z\big]^2 = 0$$

and we consider two cases: $m \neq 0$ and $m = 0$.

1) If $m \neq 0$ then we may assume $m = 1$ and $e = 0$ due to the change $(x,y,t) \to (mx, y - e/2m, t/m^2)$ and in this case the above condition gives us $c = -4/3$ and $a = -16/27$. However in this case we have $R_X^{(0)}(P^*,Q^*) = 0$, i.e. we get a contradiction with the condition (17).

2) Assume now $m = 0$. In this case we obtain

$$
\begin{aligned}
&R_X^{(1)}(F_1,P^*) = (3al + 2ce)^2 Z^6 = 0,\\
&R_X^{(0)}(F_1,P^*) = (27a^2 + 4c^3)\big[27a^2l^3 + 27ae(cl^2 - e^2) + 2c^2l(cl^2 + 9e^2)\big]Z^{12}/27 = 0,\\
&R_X^{(0)}(P^*,Q^*) = \big[27a^2l^3 + 27ae(cl^2 - e^2) + 2c^2l(cl^2 + 9e^2)\big]/27 \neq 0
\end{aligned}
$$

and this implies $c \neq 0$, otherwise the second equality yields $a = 0$ and then $R_X^{(0)}(P^*, Q^*) = 0$. So $c \neq 0$ and the first equation gives $e = -3al/(2c)$ and then we arrive at the contradiction:

$$R_X^{(0)}(F_1, P^*) = \frac{l^3 Z^{12}}{216c^3}(27a^2 + 4c^3)^3 = 0, \quad R_X^{(0)}(P^*, Q^*) = \frac{l^3 Z^6}{216c^3}(27a^2 + 4c^3)^2 \neq 0.$$

This completes the proof of the lemma. □

Now we examine the direction $y = 0$. The following proposition holds.

Proposition 2. *For the existence of an invariant line of systems* (14) *in the direction $y = 0$ it is necessary and sufficient*

$$l = 0, \quad ef - 2bm = 0, \quad f^2 + m^2 \neq 0. \tag{18}$$

Proof. Indeed, considering the equations (6) for a system (14) we obtain

$$Eq_5 = l, \;\; Eq_8 = e - 2mW, \;\; Eq_{10} = b - fW.$$

Clearly, $Eq_5 = 0$ is equivalent to $l = 0$. On the other hand in order to have a line in the direction $y = 0$ the condition $f^2 + m^2 \neq 0$ is necessary. Therefore the condition $\text{Res}_W(Eq_8, Eq_{10}) = ef - 2bm = 0$ is necessary and sufficient for the existence of a common solution $W = W_0$ of the equations $Eq_8 = 0$ and $Eq_{10} = 0$. This completes the proof of the proposition. □

3.2.1 The case $m \neq 0, \; l \neq 0$

By Proposition 2 we could not have invariant line in the direction $y = 0$. So after the transformation $(x, y, t) \to (mx, -e/2m + ly, t/m^2)$ we can consider $l = m = 1$ and $e = 0$. As a result we arrive at the family of systems

$$\dot{x} = a + cx + x^3 \equiv P(x), \quad \dot{y} = \;\; b + x^2 + fy + 2xy \equiv Q(x, y). \tag{19}$$

Proposition 3. *Systems* (19) *possess invariant lines of total multiplicity 8 if and only if*

$$a = 0, \quad f = c = -\frac{4}{9}, \;\; b = \frac{4}{27}. \tag{20}$$

Proof. Sufficiency. Assume that (20) are satisfied. Then for the system (19) we calculate $H(X, Y, Z) = -3^{-8}X^2(3X - 2Z)^3 Z(3X + 2Z)$ and hence, we have 8 invariant straight lines (including the line at infinity).

Necessity. Consider systems (19) for which the polynomial H has the form (15). The degree of this polynomial equals four, but should be seven . Therefore we have to find out the conditions to increase the degree of the polynomial H up to seven, namely we have to find out additionally a common factor of degree three of the polynomials $\mathcal{G}_i, \; i = 1, 2, 3$ (see Lemma 2 and Notation 1).

Considering (16) for systems (19) we obtain $\mathcal{G}_1/H|_{Z=0} = X^3(X+4Y)$. Therefore we conclude that all three polynomials could only have common factors of the form

$X+\alpha=0$, which by Remark 6 must be factors of the polynomial $P^*(X,Z)$. We observe that $P^*(X,Z)$ is a common factor of the polynomials $\mathcal{G}_2/H$ and $\mathcal{G}_3/H$ and, moreover, in the last one this factor is of the second degree.

According to Lemma 8 the polynomial $P^*(X,Z)$ could not be a factor of $\mathcal{G}_1/H$, i.e. of the polynomial $F_1(X,Y,Z)$. Thus not all the factors of the polynomial $P^*(X,Z)$ are also the factors in $F_1(X,Y,Z)$. This leads us to the conclusion that the polynomial $F_2(X,Y,Z)$ must have a common factor with $P^*(X,Z)$, i.e. the condition

$$R_X^{(0)}(F_2,P^*)=(8+27a+18c)Z^3R_X^{(0)}(P^*,Q^*)=0$$

has to be fulfilled. Due to (17) this gives $c=-(8+27a)/18$ and we obtain that the polynomial $\psi=(3X-2Z)$ is a common factor of the polynomials $F_2(X,Y,Z)$ and $P^*(X,Z)$. On the other hand it must be a factor in $F_1(X,Y,Z)$. We calculate

$$\begin{aligned}R_X^{(0)}(F_1,\psi)=&-(8+27a+18f)Z^3(12Y+9fY+4Z+9bZ)/2=0,\\ R_X^{(0)}(P^*,Q^*)=&(12Y+9fY+4Z+9bZ)\Psi(Y,Z)\neq 0,\end{aligned}$$

where $\Psi(Y,Z)$ is a polynomial. So the above conditions give us the equality $a=-2(4+9f)/27$ and then we obtain $f=c$. In this case calculations yield

$$\begin{aligned}\mathcal{G}_1/H=&\frac{1}{27}(3X-2Z)\big[9X^3+12X^2(3Y-Z)+3(9c-4)XYZ+\\ &+(27b-18c-8)XZ^2-2(4+9c)YZ^2\big]\equiv\frac{1}{27}(3X-2Z)F_1'(X,Y,Z),\\ \mathcal{G}_2/H=&\frac{1}{729}(3X-2Z)^2\big[18X^2+54XY-6XZ+27cYZ+(27b-9c-4)Z^2\big]\times\\ &\times(9X^2+6XZ+4Z^2+9cZ^2)\equiv\frac{1}{729}(3X-2Z)^2F_2'(X,Y,Z)\tilde{P}(X,Y,Z)\end{aligned}$$

and we obtain

$$\begin{aligned}R_X^{(0)}\left(F_1',\tilde{F}_2'\right)=&-729Z^2\big[36Y^2-3(4+9c)YZ+(4-27b+9c)Z^2\big]\Gamma(Y,Z),\\ R_X^{(0)}\left(F_1',\tilde{P}\right)=&729(4+9c)Z^4\Gamma(Y,Z),\\ R_X^{(0)}(P^*,Q^*)=&\frac{1}{729}Z^3(12Y+9cY+4Z+9bZ)\Gamma(Y,Z),\end{aligned}$$

where $\Gamma(Y,Z)$ is a polynomial. Since $R_X^{(0)}\left(F_1',\tilde{F}_2'\right)\neq 0$ due to $R_X^{(0)}(P^*,Q^*)\neq 0$, we deduce that for the existence of a common factor of degree 3 of the polynomials $\mathcal{G}_1/H$ and $\mathcal{G}_2/H$ the condition $R_X^{(0)}\left(F_1',\tilde{P}\right)=0$ is necessary, i.e. $c=-4/9$ and we get $c=f=-4/9$ and $a=0$. In this case we obtain

$$\begin{aligned}&\mathcal{G}_1/H=\frac{1}{9}X(3X-2Z)(3X^2+12XY-4XZ-8YZ+9bZ^2)\equiv\frac{1}{9}X(3X-2Z)F_1'',\\ &P^*(X,Z)=X(3X-2Z)(3X+2Z)/9\end{aligned}$$

and since X could not be a factor of $F_1''(X,Y,Z)$ and, moreover, as it was proved earlier the polynomial $P^*(X,Z)$ could not divide $\mathcal{G}_1/H$, we deduce that the factor of $F_1''(X,Y,Z)$ must be $3X-2Z$. So the condition

$$R_X^{(0)}\left(F_1'', 3X-2Z\right) = 3(27b-4)Z^2 = 0$$

is necessary and this implies $b = 4/27$, i.e. we arrive at the conditions (20) and this completes the proof of Proposition 3. □

Considering the conditions (20) we obtain the family of systems which after the suitable transformation $(x,y,t) \to (2x/3, y+1/3, 9t/4)$ becomes

$$\dot{x} =(x-1)x(1+x), \quad \dot{y} = x - y + x^2 + 3xy \tag{21}$$

with $H(X,Y,Z) = -X^2(X-Z)^3Z(X+Z)$. We observe that these systems possess 3 finite singularities: $(0,0)$, $(1,-1)$ and $(-1,0)$. On the other hand considering Lemma 3 for systems (21) we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = 8x^6 \neq 0.$$

So by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$.

Thus this system possesses 3 real distinct invariant affine lines (besides the double infinite line) and namely: one triple, one double and one simple, all real and distinct. Therefore we obtain the configuration *Config. 8.23*.

3.2.2 The case $m \neq 0$, $l = 0$

As it was mentioned earlier we may assume $m=1$ and $e=0$ due to the change $(x,y,t) \to (mx, y-e/2m, t/m^2)$. So we get the family of systems

$$\dot{x} =a + cx + x^3, \quad \dot{y} = b + fy + 2xy \tag{22}$$

which by Proposition 2 possess invariant line in the direction $y=0$ if and only if $b=0$.

1) The subcase $b \neq 0$. We claim that in this case the above systems could not have invariant lines of total multiplicity 8. Indeed, due to the rescaling $y \to by$ we can consider $b=1$ and we obtain that for systems (22) the polynomial H of the form (15) has the degree 4, but should be 7. Moreover we have $\mathcal{G}_1/H|_{Z=0} = 4X^3Y$ and hence the polynomials $\mathcal{G}_k/H$, $k=1,2,3$ (see their values (16) for $m=b=1$ and $l=e=0$) could have only the common factors of the form $X+\alpha Z$.

Considering Remark 6 and Lemma 8 we arrive again at the conclusion that the polynomial $F_2(X,Y,Z)$ must have a common factor with $P^*(X,Z)$. We determine that for systems (22) $F_2(X,Y,Z) = (3X-2Z)P^*(X,Z)Q^*(X,Y,Z)$ and hence due to the condition (17) and according to Lemma 8 (which says that $P^*(X,Z)$ could not

divide $\mathcal{G}_1/H$) we conclude that $3X-2Z$ must be a double factor in $\mathcal{G}_1/H$. However we obtain
$$R_X^{(1)}\big((3X-2Z)^2, \mathcal{G}_1/H\big) = 162Z^3 \neq 0,$$
i. e. for systems (22) we could not increase the degree of $H(X,Y,Z)$ up to 7 and this completes the proof of our claim.

2) The subcase $b = 0$. We obtain the family of systems
$$\dot{x} = a + cx + x^3 \equiv P(x), \quad \dot{y} = y(f+2x) \equiv y\tilde{Q}(x). \tag{23}$$

Proposition 4. *Systems* (23) *possess invariant lines of total multiplicity 8 if and only if one of the following sets of conditions holds:*
$$f = c,\ a = -\frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0; \tag{24}$$
$$f = \frac{-2(3c+2)}{3},\ a = \frac{2(4+9c)}{27},\ \ (4+3c)(4+9c) \neq 0. \tag{25}$$

Proof. Sufficiency. Assume that (24) (respectively (25)) are satisfied. Then considering systems (23) we calculate $H(X,Y,Z) = 3^{-8}Y(3X-2Z)^3Z(9X^2+6XZ+4Z^2+9cZ^2)$ (respectively $H(X,Y,Z) = 3^{-9}2YZ(3X+2Z)(9X^2-6XZ+4Z^2+9cZ^2)^2$) and hence, we have 8 invariant straight lines, including the line at infinity. Moreover for the corresponding systems we calculate $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 3^{11}2(4+3c)^2(4+9c)Z^3$ (respectively $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = -3^{15}(4+3c)^2(4+9c)Z^3$) and this leads to the condition $(4+3c)(4+9c) \neq 0$ which does not allow us to have 9 invariant lines.

Necessity. For systems (23) we have $H(X,Y,Z) = YZ(X^3 + cXZ^2 + aZ^3)$. Thus according to Lemma 2 we conclude that we need additionally a non-constant factor of the second degree of H. For systems (23) we calculate (see Notation 1)
$$\begin{aligned}
\mathcal{G}_1/H =& 4X^3 - (4-3f)X^2Z - 4fXZ^2 - (2a - cf + f^2)Z^3,\\
\mathcal{G}_2/H =& (3X-2Z)(2X+fZ)(X^3+cXZ^2+aZ^3) \equiv (3X-2Z)\tilde{Q}^*(X,Z)P^*(X,Z),\\
\mathcal{G}_3/H =& 24(2X+fZ)(X^3+cXZ^2+aZ^3)^2 \equiv \tilde{Q}^*(X,Z)\left[P^*(X,Z)\right]^2,
\end{aligned}$$
where $P^*(X,Z)$ and $\tilde{Q}^*(X,Z)$ are the homogenization of the polynomial $P(x)$ and $\tilde{Q}(x)$ from (23).

We observe that $\mathcal{G}_1/H|_{Z=0} = 4X^3$ and we conclude that all three polynomials could not have as a common factor Z. On the other hand these polynomials do not depend on Y. So common factors of the above polynomials could be only factors of the form $X + \alpha Z$, which by Remark 6 must be also factors in $P^*(X,Z)$. So considering this remark and Lemma 8 we arrive at the two possibilities: the linear form $3X-2Z$ either is a common factor of the polynomials $\mathcal{G}_1/H$ and $P^*(X,Z)$ or it is not.

a) Assume first that $3X-2Z$ is a common factor of $\mathcal{G}_1/H$ and $P^*(X,Z)$. Then the following condition must be satisfied:
$$R_X^{(0)}(3X-2Z, P^*) = (8+27a+18c)Z^3 = 0$$

and this implies $a = -2(4+9c)/27$. Herein we have

$$\begin{aligned} R_X^{(0)}\,(3X-2Z,\mathcal{G}_1/H,) =& 9(c-f)(4+3f)Z^3 = 0,\\ R_X^{(0)}\,(P^*(X,Z),Q^*(X,Z)) =& (4+3f)(16+36c-12f+9f^2)Z^3/27 \neq 0 \end{aligned}$$

and hence the condition $f = c$ must hold, which leads to the first two conditions (24).

b) Suppose now that $3X-2Z$ is not a common factor of $\mathcal{G}_1/H$ and $P^*(X,Z)$. Then clearly these polynomials must have a common factor of the second degree. So the conditions

$$\begin{aligned} R_X^{(0)}\,(P^*,\mathcal{G}_1/H) =& (8a-4cf-f^3)\Phi_1(a,c,f)Z^9 = 0,\ \ R_X^{(1)}\,(P^*,\mathcal{G}_1/H) = \Phi_2(a,c,f)^4 = 0,\\ R_X^{(0)}\,(P^*,Q^*) =& (4cf+f^3-8a)Z^3 \neq 0 \end{aligned}$$

must hold, where $\Phi_1 = 8a+27a^2+4c^2+4c^3+18af-f^3-cf(4+3f)$, $\Phi_2 = 16c^2+2c(8+6f+3f^2)+3(6af-8a+4f^2+f^3)$. Due to $R_X^{(0)}\,(P^*,Q^*) \neq 0$ we must have $\Phi_1 = \Phi_2 = 0$ and we calculate

$$R_a^{(0)}\,(\Phi_1,\Phi_1) = 3(4+6c+3f)^2(4c+3f^2)(16+16c+3f^2) = 0.$$

We claim that the condition $4+6c+3f = 0$ has to be satisfied for non-degenerate systems (23). Indeed assuming $c = -3f^2/4$ (respectively $c = -(16+3f^2)/16)$) we get that $4a+f^3$ (respectively $32a+16f-f^3$) is a common factor of Φ_1 and Φ_2, however in this case the polynomial $R_X^{(0)}\,(P^*,Q^*)$ gives the value $-2(4a+f^3)Z^3 \neq 0$ (respectively $-(32a+16f-f^3)Z^3/4 \neq 0$).

So $4+6c+3f = 0$, i.e $f = -2(2+3c)/3$ and in this case the common factor of Φ_1 and Φ_2 is $(8-27a+18c)$. Hence the condition $\Phi_1 = \Phi_2 = 0$ implies $a = 2(4+9c)/27$ and this leads to the conditions (25). ☐

Next we construct the respective canonical forms of systems (23) when either the conditions (24) or (25) of Proposition 4 are satisfied.

(i) Conditions (24). We observe that in this case due to a translation and an additional notation, namely $r = (4+3c)/3$, we arrive at the family of systems

$$\dot{x} = x(r+2x+x^2), \quad \dot{y} = (r+2x)y \tag{26}$$

for which we have $H(X,Y,Z) = X^3YZ(X^2+2XZ+rZ^2)$. So the polynomial $H(X,Y,Z)$ has the degree 7 and by Lemma 2 the above systems possess invariant lines of total multiplicity 8 (including the line at infinity, which is double). Now we need an additional condition under the parameter r which conserves the degree of the polynomial $H(X,Y,Z)$. For systems (26) we calculate $R_X^{(0)}\,(\mathcal{G}_3/H,\mathcal{G}_1/H) = 48r^3(8-9r)^2Z^5 \neq 0$. Consequently we get the condition $r(8-9r) \neq 0$ which for systems (23) is equivalent to $(4+3c)(4+9c) \neq 0$ (see the last condition from (24)).

Besides the infinite line $Z = 0$ (which is double) systems (26) possess six affine invariant lines, namely:

$$L_{1,2,3} = x,\ L_4 = y,\ L_{5,6} = r + 2x + x^2.$$

We detect that the lines $L_{5,6} = 0$ are either complex or real distinct or real coinciding, depending on the sign of the discriminant of the polynomial x^2+2x+r, which equals $\Delta = 4(1-r)$. We also observe that systems (26) possess 3 finite singularities: $(0,0)$ and $(-1 \pm \sqrt{1-r}, 0)$ which are located on the invariant line $y = 0$. On the other hand considering Lemma 3 for systems (26) we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = r^3x^6 \neq 0.$$

So by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$. Moreover by this lemma systems (26) became degenerate only if $r = 0$, and we observe that in this case the system indeed is degenerate.

We consider the three possibilities given by the value of the discriminant Δ.

a) *The possibility* $\Delta > 0$. Then $1 - r > 0$, i.e. $r < 1$. We set the notation $1 - r = u^2$ (i.e. $r = 1 - u^2$) which leads to the systems

$$\dot{x} = (1 - u + x)x(1 + u + x), \quad \dot{y} = (1 - u^2 + 2x)y$$

possessing one triple and three simple distinct real invariant lines. Comparing the line $x = \mp u - 1$ with $x = 0$ we conclude that if $|u| > 1$ (i.e. $r < 0$) then in the direction $x = 0$ the triple invariant line is situated in the domain between two simple ones, whereas in the case $|u| < 1$ (i.e. $0 < r < 1$) the triple line is located outside this domain. As a result we get *Config. 8.24* in the case of $r < 0$ and *Config. 8.25* in the case of $0 < r < 1$.

b) *The possibility* $\Delta = 0$. Then $r = 1$ and we obtain the configuration *Config. 8.26.*

c) *The possibility* $\Delta < 0$. In this case $r > 1$ and we get systems possessing two complex, one simple and one triple real all distinct invariant lines and this leads to the configuration *Config. 8.27.*

(ii) Conditions (25). In this case after the translation of the origin of coordinates to the singular point $(-2/3, -e/2)$ and setting a new parameter $r = (4 + 3c)/3$ we obtain the systems

$$\dot{x} = (r - 2x + x^2)x, \quad \dot{y} = 2(x - r)y. \tag{27}$$

For these systems we have $H(X,Y,Z) = 2XYZ(X^2 - 2XZ + rZ^2)^2$. Besides the double infinite line systems (27) possess 4 affine invariant lines:

$$L_1 = x,\ L_2 = y,\ L_{3,4} = x^2 - 2x + r,$$

where the lines $L_{3,4} = 0$ are double ones. We denote by $\Delta = 4(1-r)$ the discriminant of the polynomial $x^2 - 2x + r$ and we observe that for $\Delta = 0$ (i.e. $r = 1$) the systems become degenerate.

We also observe that systems (27) possess 3 finite singularities: $(0,0)$ and $(1 \pm \sqrt{1-r}, 0)$ which are located on the invariant line $y = 0$. On the other hand considering Lemma 3 for systems (26) we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = 8(1-r)r^2x^6.$$

If $r(r-1) \neq 0$ by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$. Moreover by this lemma systems (27) became degenerate only if either $r = 0$ or $r = 1$ and in both cases we get degenerate systems.

Thus we have the following two possibilities:

a) *The possibility* $\Delta > 0$. Then $r < 1$ and denoting $r = 1 - v^2$ we obtain the systems

$$\dot{x} =(1+v-x)x(1-v-x), \quad \dot{y} = 2(v^2-1+x)y \tag{28}$$

with $H(X,Y,Z) = 2XYZ(X-Z-vZ)^2(X-Z+vZ)^2$. Examining the lines $x = 1 \pm v$ and $x = 0$ we conclude that if $|v| > 1$ then we get a simple invariant line between two double real lines in the directions $x = 0$ and consequently we arrive at *Config. 8.28*. In the case of $|v| < 1$ these two double real lines are located on the right–hand side of the simple invariant line. So we get *Config. 8.29*.

b) *The possibility* $\Delta > 0$. In this case $r > 1$ and systems (27) possess 2 real simple, 2 complex double invariant lines, all distinct $\Rightarrow$ *Config. 8.30*.

3.2.3 The case $m = 0, \ l \neq 0$

We claim that in this case systems (14) could not possess invariant lines of total multiplicity 8.

Indeed, since $l \neq 0$ by Proposition 2 we could not have a line in the direction $y = 0$. Via the rescaling $(x \to x, y \to ly, t \to t)$ we can consider $l = 1$ and therefore we arrive at the systems

$$\dot{x} =a+cx+x^3, \quad \dot{y} = \quad b+ex+x^2+fy \tag{29}$$

for which the polynomial H has the form (15) and $\mathcal{G}_i/H$ $(i = 1,2,3)$ are the polynomials (16) for the particular case $m = 0$ and $l = 1$. We observe that $\mathcal{G}_1/H|_{Z=0} = X^4$ and hence Z could not be a common factor of these polynomials. Since we have no invariant lines in the direction $y = 0$, in what follows we shall examine only the conditions given by resultants with respect to X. According to (16) and condition (17) the polynomial $F_1(X,Y,Z)$ must have a common factor of degree 3 with $\left[P^*(X,Y)\right]^2$. For systems (29) we calculate Coefficient$[R_X^{(2)}\left(F_1, \left[P^*\right]^2\right), Y^4Z^4] = 81f^4$. Clearly the condition $f = 0$ is necessarily to get a common factor of the degree 3. Then we have

$$R_X^{(0)}\left(F_1, \left[P^*\right]^2\right) = (27a^2+4c^3)^2[\Phi(a,b,c,e)]^2Z^{24} = 0, \ R_X^{(0)}(Q^*, P^*) = \Phi(a,b,c,e)Z^6 \neq 0$$

where $\Phi(a,b,c,e)$ is a polynomial. So the above conditions imply $27a^2+4c^3=0$. First we examine the possibility $a=0$ and we get $c=0$. Then we calculate

$$R_X^{(0)}(Q^*,P^*)=b^3Z^6\neq 0,\ R_X^{(2)}\left(F_1,[P^*]^2\right)=81b^4Z^8=0$$

and we arrive at the contradictory condition $(0\neq b=0)$. So it remains to examine the case when $a\neq 0$. Since in this case $c\neq 0$ we denote $a=2a_1c$ which implies $c=-27a_1^2$. We calculate

$$\begin{aligned}R_X^{(0)}(Q^*,P^*)=&(9a_1^2+b-3a_1e)^2(36a_1^2+b+6a_1e)Z^6\neq 0,\\ R_X^{(1)}\left(F_1,[P^*]^2\right)=&2^33^{10}a_1^5(9a_1^2+b-3a_1e)^3(36a_1^2+b+6a_1e)^2Z^{15}=0\end{aligned}$$

and we also get a contradiction which completes the proof of our claim.

3.2.4 The case $m=0,\ l=0$

We divide our examination in two subcases: $e\neq 0$ and $e=0$.

1) The subcase $e\neq 0$. Then due to the rescaling $(x,y,t)\to(ex,y,t/e^2)$ we can consider $e=1$ and therefore we arrive at the systems

$$\dot{x}=a+cx+x^3,\quad \dot{y}=b+x+fy. \tag{30}$$

Proposition 5. *Systems* (30) *possess invariant lines of total multiplicity 8 if and only if the following conditions hold:*

$$f=-2c,\quad a=0. \tag{31}$$

Proof. *Sufficiency.* Assume that (31) is satisfied. Then considering systems (30) we calculate $H(X,Y,Z)=XZ^2(X^2+cZ^2)^2$ and hence, we have invariant straight lines of total multiplicity 8 (including the line at infinity). On the other hand we could not have 9 lines, because $R_X^{(0)}(G_2/H,G_1/H)=-27(2cY-bZ)^3=0$ if and only if $b=c=0$. However in this case we get a degenerate system.

Necessity. For systems (30) we have $H(X,Y,Z)=Z^2(X^3+cXZ^2+aZ^3)$ and we observe that the degree of the polynomial H is 5. So we have to increase the degree of H up to 7. In other words we have to determine the conditions under which the three polynomials G_1/H, G_2/H and G_3/H have a common factor of degree 2. For these systems we calculate

$$\begin{aligned}\mathcal{G}_1/H=&2X^3+3fX^2Y+3bX^2Z-fXZ^2+f(c-f)YZ^2+(bc-a-bf)Z^3,\\ \mathcal{G}_2/H=&3X(X+fY+bZ)(X^3+cXZ^2+aZ^3)\equiv 3XQ^*P^*,\\ \mathcal{G}_3/H=&24(X+fY+bZ)(X^3+cXZ^2+aZ^3)^2\equiv 24Q^*[P^*]^2.\end{aligned}$$

We observe that $G_1/H|_{Z=0}=2X^3+3fX^2Y$ and hence Z could not be a common factor of these polynomials. For systems (30) we get $R_Y^{(0)}(G_3/H,G_1/H)=-24f(X^3+cXZ^2+aZ^3)^3$ which vanishes if and only if $f=0$ and since $m=0$, considering Proposition 2, we conclude that in this case we could not have a line in the direction $y=0$. Thus all three mentioned polynomials could only have common factors of the form $X+\alpha=0$, which by Remark 6 must be factors of the polynomial $P^*(X,Z)$. So considering this remark and Lemma 8 we arrive at the two possibilities: the linear form X either is not a common factor of the polynomials $\mathcal{G}_1/H = F_1(X,Y,Z)$ and $P^*(X,Z)$ (i.e. $a\neq 0$) or it is (i.e. $a=0$).

a) Assume first that X is not a factor of $P^*(X,Z)$, i.e. we have to consider $a\neq 0$. According to (16) and condition (17) the polynomial $F_1(X,Y,Z)$ must have a common factor of degree 2 with $P^*(X,Y)$. Then considering systems (30) the following conditions must be satisfied:

$$R_X^{(0)}(F_1,P^*) = [27a^2+(c-f)(2c+f)^2]Z^6\Psi(Y,Z)=0,\ R_X^{(0)}(Q^*,P^*)=\Psi(Y,Z)\neq 0$$

where $\Psi(Y,Z)$ is a polynomial. So the condition $27a^2+(c-f)(2c+f)^2=0$ is necessary for the existence of a common factor of the polynomials F_1 and P^*. Then $(c-f)(2c+f)\neq 0$ (due to $a\neq 0$) and denoting $u=2c+f\neq 0$ (i.e. $f=u-2c$) we obtain $c=u/3-9a^2/u^2$ and $f=u-2c=(54a^2+u^3)/(3u^2)$. In this case we obtain

$$F_1=(uX+3aZ)F_1^*(X,Y,Z)/(3u^4),\ P^*=(uX+3aZ)(3uX^2-9aXZ+u^2Z^2)/(3u^2)$$

where $F_1^*(X,Y,Z)$ is a polynomial of the second degree. Assume first that $uX+3aZ$ is a factor in F_1^*. In this case it must be a factor in $3uX^2-9aXZ+u^2Z^2$ and therefore the following condition must hold:

$$R_X^{(0)}(uX+3aZ,3uX^2-9aXZ+u^2Z^2)=u(54a^2+u^3)Z^2=0.$$

Since $u\neq 0$ we can set $a=a_1u$ and thus, we get $u=-54a_1^2$. Then

$$R_X^{(0)}(F_1^*,uX+3aZ)=18a_1(3a_1-b)Z^2=0,\ R_X^{(0)}(P^*,Q^*)=(b-3a_1)^2(6a_1+b)Z^3\neq 0$$

and we arrive at the contradiction.

Now we consider that $uX+3aZ$ is not a factor in F_1^*. Then the polynomials F_1^* and $3uX^2-9aXZ+u^2Z^2$ must have a common factor, i.e. the following conditions hold:

$$R_X^{(0)}(F_1^*,3uX^2-9aXZ+u^2Z^2)=27u^5Z^2F_1^{**}(Y,Z)=0,$$

$$R_X^{(0)}(P^*,Q^*)=[(3a-bu)3uZ-(54a^2+u^3)Y]F_1^{**}(Y,Z)/(27u^6)\neq 0$$

where $F_1^{**}(Y,Z)$ is a polynomial of the second degree. Since $c\neq 0$ in this case we also arrive at the contradictory condition.

b) Assume now that X is a common factor of $P^*(X,Z)$, i.e. we have the condition $a=0$ which implies $\mathcal{G}_2/H=3X^2(X^2+cZ^2)Q^*$. Therefore either X^2 or X^2+cZ^2 must be a factor of F_1. In order to have X^2 as a common factor of the mentioned polynomial the condition $R_X^{(0)}(X^2,F_1)=R_X^{(1)}(X^2,F_1)=0$ must be satisfied. We calculate

$$R_X^{(1)}(X^2,F_1)=-fZ^2=0,\ R_X^{(0)}(X^2,F_1)=(c-f)^2Z^4(fY+bZ)^2=0$$

and $R_X^{(0)}(P^*, Q^*) \mid_{\{c=f=0\}} = -b(b^2+c)Z^3$. It is evident that in order to have X^2 as a factor of the polynomial F_1 it is necessary the conditions $f = c = 0$ and $b \neq 0$ to be satisfied, i.e. we get a particular case of the conditions (31). Since $b \neq 0$, due to the rescaling $\{x \to bx, y \to y/b, t \to t/b^2\}$ we can consider $b = 1$. So we arrive at the system

$$\dot{x} = x^3, \quad \dot{y} = \ 1 + x \tag{32}$$

for which $H(X,Z) = X^5Z^2$. This system possesses the affine invariant line of the multiplicity 5 in the direction $x = 0$ and the infinite invariant line is of the multiplicity 3. Considering Lemma 3 for these systems we get

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = 0, \quad \mu_9 = 9x^9 \neq 0.$$

Therefore by Lemma 3 all 9 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$. Consequently we get the configuration *Config. 8.33.*

Now we assume that $X^2 + cZ^2$ is a factor of the polynomial F_1, i.e. the condition $R_X^{(0)}\big(X^2 + cZ^2, F_1 \mid_{\{a=0\}}\big) = R_X^{(1)}\big(X^2 + cZ^2, F_1 \mid_{\{a=0\}}\big) = 0$ must hold. We calculate

$$R_X^{(1)}\big(X^2 + cZ^2, F_1 \mid_{\{a=0\}}\big) = -(2c+f)Z^2 = 0$$

from which it results $f = -2c \neq 0$ and we obtain the conditions (31). Since $c \neq 0$ we may assume $b = 0$ (applying the translation of the origin of coordinates at the point $x_0 = 0,\ y_0 = b/2c$). Therefore we arrive at non-degenerate systems depending on the parameter $c = \{-1, 1\}$ (applying a rescaling)

$$\dot{x} = x(c + x^2), \quad \dot{y} = \ x - 2cy. \tag{33}$$

For the above systems we have $H(X,Z) = XZ^2(X^2 + cZ^2)^2$. Thus beside the triple infinite invariant line systems (37) possess 5 invariant affine lines. More precisely, we have one simple and two double, all real and distinct if $c = -1$ and one simple real and two double complex if $c = 1$.

On the other hand we observe that systems (33) possess 3 finite singularities: $(0,0)$ and $\big(\pm\sqrt{-c}, \mp 1/(2\sqrt{-c})\big)$. Considering Lemma 3 for these systems we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = -8c^3x^6 \neq 0.$$

Therefore by Lemma 3 all other 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x = 0$. Thus we get *Config. 8.31* if $c = -1$ and *Config. 8.32* if $c = 1$. □

2) The subcase $e = 0$. Then we get the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = \ b + fy \tag{34}$$

for which $H(a, X, YZ) = Z^2(fY + bZ)(X^3 + cXZ^2 + aZ^3)$. So the degree of H is six but should be seven. Therefore we need an additional common factor of $\mathcal{G}_i,\ i = 1, 2, 3$. We calculate

$$\mathcal{G}_1/H = 3X^2 + cZ^2 - fZ^2, \quad \mathcal{G}_2/H = 3X(X^3 + cXZ^2 + aZ^3),$$
$$\mathcal{G}_3/H = 24(X^3 + cXZ^2 + aZ^3)^2$$

and we observe that these polynomials could not have as a common factor neither Z nor Y. So we examine their resultants with respect to X. We calculate

$$R_X^{(0)}(\mathcal{G}_1/H, P^*) = [27a^2 + (c - f)(2c + f)^2]Z^6 = 0,$$
$$R_X^{(0)}(P^*, Q^*) = (fY + bZ)^3 \neq 0,$$

which implies $27a^2 + (c - f)(2c + f)^2 = 0$. We observe that $(c - f)(2c + f) \neq 0$, otherwise we get $a = 0$ and this leads to systems with invariant lines of total multiplicity 9.

Denoting $u = 2c + f \neq 0$ (i.e. $f = u - 2c$) we obtain $c = u/3 - 9a^2/u^2$ and $f = u - 2c = (54a^2 + u^3)/(3u^2)$. So we get the family of systems

$$\dot{x} = \frac{1}{3u^2}(3a + ux)(u^2 - 9ax + 3ux^2), \quad \dot{y} = b + \frac{54a^2 + u^3}{3u^2}y. \tag{35}$$

Without loss of generality we may assume $b \neq 0$, because in the case $b = 0$ we must have $54a^2 + u^3 \neq 0$ (otherwise we get degenerate systems) and then via a translation $y \to y + y_0$ (with $y_0 \neq 0$) we obtain $b \neq 0$. So applying the translation of the origin of coordinates at the point $(-3a/u, 0)$, after the suitable rescaling $\{x \to -(9ax)/u, y \to bu^2y/(81a^2), t \to tu^2/81a^2\}$ systems (35) become

$$\dot{x} = rx + x^2 + x^3, \quad \dot{y} = 1 + ry, \tag{36}$$

where $r = (54a^2 + u^3)/(243a^2)$. For these systems we calculate $H = X^2(rY + Z)Z^2(X^2 + XZ + rZ^2)$ and $R_X^{(0)}(\mathcal{G}_2/H, \mathcal{G}_1/H) = 3(9r - 2)Z^3 \neq 0$ and this leads to the condition $9r - 2 \neq 0$ which guarantee the non-existence of nine invariant lines. We observe that the infinite invariant line Z=0 is triple if $r \neq 0$ and it has multiplicity four in the case $r = 0$.

a) *The possibility* $r \neq 0$. In this case the geometry of the configuration depends on the sign of the discriminant Δ of the polynomial $x^2 + x + r$, i.e. $\Delta = 1 - 4r$. Accordingly we conclude that besides the double infinite invariant line the systems (35) possess 5 affine lines which are as follows:

$\Delta > 0$ (i.e. $0 \neq r < 1/4$) $\Rightarrow$ 3 simple, 1 double, all real and distinct,
$\Delta = 0$ (i.e. $r = 1/4$) $\Rightarrow$ 1 simple, 2 double, all real and distinct,
$\Delta < 0$ (i.e. $r > 1/4$) $\Rightarrow$ 2 real simple, 1 complex double.

On the other hand considering Lemma 3 we calculate:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0,\ \mu_6 = r^3x^6,\ \mu_7 = r^2x^6(3x - ry),$$
$$\mu_8 = rx^6(3x^2 - 2rxy + r^3y^2),\ \mu_9 = 9x^7(x^2 - rxy + r^3y^2).$$

Since $r \neq 0$ by Lemma 3 only 6 finite singular points have gone to infinity and collapsed with the singular point $[0,1,0]$ located on the "end" of the invariant line $x=0$. Other three finite points are $(0,-1/r)$ and $\big((-1\pm\sqrt{1-4r})/2, -1/r\big)$ (located on the invariant line $ry+1=0$).

Moreover, in the case of $\Delta > 0$, denoting $1-4r = v^2$ (i.e. $r=(1-v^2)/4$) we obtain the systems

$$\dot{x} =(1-v+2x)x(1+v+2x)/4, \quad \dot{y} = \ 1+(1-v^2)y/4.$$

We compare the lines $x=(-1\pm v)/2$ with $x=0$ and conclude that if $|v|>1$, i.e. $r<0$ (respectively $0<|v|<1/4$, i.e. $0<r<1/4$) then the double real invariant line is located (respectively is not located) between two simple ones and we arrive at the configuration *Config. 8.34* (respectively *Config. 8.35.*).

Additionally, we have the configuration *Config. 8.36* in the case of $\Delta = 0$ (i.e. $r=1/4$) and *Config. 8.37* in the case of $\Delta<0$ (i.e. $r>1/4$).

b) *The possibility* $r=0$. In this case we get the system

$$\dot{x} = x^2(x+1), \quad \dot{y} = 1 \tag{37}$$

with $H(X,Z) = X^3Z^3(X+Z)$. Therefore besides the infinite line of the multiplicity four this system possesses 2 distinct affine invariant lines (one of the multiplicity 3 and one simple), and namely: $L_{1,2,3} = x, \quad L_4 = x+1.$

Since in this case we obtain $\mu_i = 0$ $(i=0,1,\dots,8)$ and $\mu_9 = 9x^9 \neq 0$, by Lemma 3 all 9 finite singular points have gone to infinity and collapsed with the the same singular point $[0,1,0]$. As a result we get the configuration *Config. 8.38* Thus considering the above results we arrive at the following proposition.

Proposition 6. *The systems* (34) *possess invariant lines of total multiplicity eight if and only if*

$$27a^2 + (c-f)(2c+f)^2 = 0, \quad a \neq 0. \tag{38}$$

3.3 Invariant conditions for the configurations *Config. 8.23*–*Config. 8.38*

By Lemma 7 the conditions $\mathcal{D}_1 = \mathcal{D}_3 = \mathcal{D}_4 = 0,\ \mathcal{D}_2 \neq 0$ are necessary and sufficient for a cubic system to have two real distinct infinite singularities, and namely they are determined by one triple and one simple factors of $C_3(x,y)$. After a linear transformation a cubic system could be brought to the form (8). According to Proposition 1, the condition $\mathcal{V}_1 = \mathcal{V}_3 = 0$ gives systems (13) (via a linear transformation and time rescaling). At the beginning of Subsection 3.2 it was proved that for these systems the condition $\mathcal{K}_5 = 0$ is equivalent to $k=h=n=0$. Moreover, for the existence of invariant lines in the direction $x=0$ the additional condition $d=0$ has to be satisfied. So considering the condition $\mathcal{K}_5 = 0$ (i.e. $k=h=n=0$)

for systems (13) we calculate $N_1 = 12d$ and evidently $N_1 = 0$ is equivalent to $d = 0$ and we arrive at systems (14). For these systems we calculate

$$N_2 = -m^2x^4, \ N_3 = -12lx^5.$$

We remark that in the previous subsections the examination of systems (14) was divided in the cases determined by the parameters m and l. In addition it was proved (see Subsection 3.2.3) that in the case $m = 0$ and $l \neq 0$ (i.e. $N_2 = 0$ and $N_3 \neq 0$) systems (14) could not have invariant lines of total multiplicity 8. So in what follows we split our examination here in three cases, defined by the invariant polynomials N_2 and N_3:

$$(i)\ N_2N_3 \neq 0; \quad (ii)\ N_2 \neq 0, N_3 = 0; \quad (iii)\ N_2 = N_3 = 0.$$

3.3.1 The case $N_2N_3 \neq 0$

Then $l \cdot m \neq 0$ and as it was shown earlier systems (14) could be brought via an affine transformation to systems (19). According to Proposition 3 the last systems possess invariant lines of total multiplicity 8 if and only if the conditions (3) are satisfied. We prove that these conditions are equivalent to $N_4 = N_5 = N_6 = N_7 = 0$, i.e.

$$a = 0, \ f = c = -\frac{4}{9}, \ b = \frac{4}{27} \quad \Leftrightarrow \quad N_4 = N_5 = N_6 = N_7 = 0.$$

Indeed, for systems (19) we calculate

$$N_4 = 5184(c - f)x^4 \quad \text{and} \quad N_5 = 2592(4 + 6c + 3f)x^4$$

and clearly the condition $N_4 = N_5 = 0$ is equivalent to $f = c = -4/9$. Then considering the last conditions we calculate $N_6 = 8640ax^4$ and hence $N_6 = 0$ gives $a = 0$. It remains to determine the invariant condition which governs the parameter b. Considering the obtained conditions for systems (19) we calculate $N_7 = 288(27b - 4)x^6 = 0$ which is equivalent to $b = \frac{4}{27}$. So if for systems (14) the conditions $N_2N_3 \neq 0, \ N_4 = N_5 = N_6 = N_7 = 0$ are satisfied then we arrive at the system (21) possessing the configuration *Config. 8.23*.

3.3.2 The case $N_2 \neq 0, \ N_3 = 0$

These conditions imply $m \neq 0$ and $l = 0$, and as it was proved in Subsection 3.2.2 the condition $ef - 2bm = 0$ is necessary to be fulfilled for systems (17) in order to have invariant lines of total multiplicity 8. On the other hand for these systems we calculate $N_8 = 1296(ef - 2bm)x^6$ and the last condition is equivalent to $N_8 = 0$. Due to a rescaling we may assume $m = 1$ and then we get $b = ef/2$ and this leads to systems (23). By Proposition 4 these systems possess invariant lines of total multiplicity 8 if and only if either the conditions (24) or (25) are satisfied.

In what follows we consider each one of these sets of conditions and construct the corresponding equivalent invariant conditions as well as the additional invariant conditions for the realization of the respective configurations.

(a) Conditions (24). We clam that for a system (23) the following conditions are equivalent:

$$f = c,\ a = -\frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0 \quad \Leftrightarrow \quad N_4 = N_6 = 0,\ N_9 \neq 0.$$

Indeed, for systems (23) we calculate $N_4 = 5184(c-f)x^4$ and therefore $N_4 = 0$ gives $f = c$. Then we have $N_6 = 320(27a + 18c + 8)x^4$ and $N_9 = 2304(4 + 3c)(4 + 9c)x^4$ which imply the condition $a = -\dfrac{2(4+9c)}{27}$ if $N_6 = 0$ and $(4 + 3c)(4 + 9c) \neq 0$ if $N_9 \neq 0$.

Thus if the conditions $N_4 = N_6 = 0$ are satisfied then systems (23) via a translation and a suitable notation can be brought to systems (26), for which the condition $N_9 = 6912r(9r - 8)x^4 \neq 0$ holds. Now for these systems we need to determine the invariant polynomials which govern the conditions under parameter r in order to get different configurations of invariant straight lines.

We calculate $N_{10} = 144(1 - r)x^2$ and $N_{11} = 3456rx^4$. Therefore, considering the obtained earlier for systems (26) configurations (see page 70) we conclude that if for a system (14) the conditions $N_3 = N_4 = N_6 = N_8 = 0,\ N_2N_9 \neq 0$ are satisfied then we get the configuration *Config. 8.24* if $N_{11} < 0$; *Config. 8.25* if $N_{10} > 0$ and $N_{11} > 0$; *Config. 8.26* if $N_{10} = 0$ and *Config. 8.27* in the case $N_{10} < 0$.

(b) Conditions (25). We clam that for a system (23) the next conditions are equivalent:

$$f = -\frac{2(2+3c)}{3},\ a = \frac{2(4+9c)}{27},\ (4+3c)(4+9c) \neq 0 \quad \Leftrightarrow \quad N_5 = N_{12} = 0,\ N_{13} \neq 0.$$

Indeed, for (23) we calculate $N_5 = 2592(4+6c+3f)x^4$ and hence $N_5 = 0$ implies $f = -\dfrac{2(2+3c)}{3}$. Then we have $N_{12} = 3240(27a - 18c - 8)x^4$ and, clearly, $N_{12} = 0$ gives $a = \dfrac{2(4+9c)}{27}$. For $N_5 = N_{12} = 0$ we calculate $N_{13} = 1008(4 + 3c)(4 + 9c)x^5y$ and therefore $N_{13} \neq 0 \Leftrightarrow (4 + 3c)(4 + 9c) \neq 0$.

So, considering the above relations among the parameters a, c and f of systems (23) it was shown earlier that these systems can be brought (via a translation and additional notation) to systems (27).

It remains to determine the invariant polynomial which gives the expression of the discriminant $\Delta = 4(1 - r)$. For these systems we calculate $N_{14} = 288(r - 1)x^2$ and $N_{15} = 2^9 3^7 rx^4$.

Therefore if for a system (14) the conditions $N_3 = N_5 = N_8 = N_{12} = 0,\ N_2N_{13} \neq 0$ are satisfied then we get the configuration *Config. 8.28* if $N_{15} < 0$; the configuration *Config. 8.29* if $N_{14} < 0,\ \ N_{15} > 0$ and the configuration *Config. 8.30* if $N_{14} > 0$.

3.3.3 The case $N_2 = N_3 = 0$

Then $l = m = 0$ and we get systems for which we calculate $N_{16} = -12ex^4$. In what follows we split our examination here in two subcases, defined by the polynomial N_{16}.

1) *The subcase* $N_{16} \neq 0$. Then $e \neq 0$ and systems (14) could be brought via a rescaling (i.e. assuming $e = 1$) to systems (30). According to Proposition 5 the last systems possess invariant lines of total multiplicity 8 if and only if the conditions (31) are satisfied. We prove that these conditions are equivalent to $N_{17} = N_{18} = 0$, i.e.

$$f = -2c, \quad a = 0 \ \Leftrightarrow \ N_{17} = N_{18} = 0.$$

Indeed, for the corresponding systems we calculate $N_{17} = 12(2c + f)x^2 = 0,\ N_{18} = 216ax^3 = 0$ and evidently, the above equalities are equivalent to $f = -2c,\ a = 0$.

It remains to determine the invariant condition which governs the value of c. For the last systems we determine $N_{10} = 72cx^2$. Next we split our examinations according to the parameter c.

a) *The possibility* $N_{10} \neq 0$. Then $c \neq 0$ and assuming $b = 0$ after a translation we arrive at the system (33). So, if for systems (14) the conditions $N_2 = N_3 = N_{17} = N_{18} = 0,\ N_{10}N_{16} \neq 0$ are satisfied then we get the configuration *Config. 8.31* if $N_{10} < 0$ and *Config. 8.32* if $N_{10} > 0$.

b) *The possibility* $N_{10} = 0$. Then $f = c = 0$ and after a rescaling we assume $b = 1$ and we get the systems (32). So, if for systems (14) the conditions $N_2 = N_3 = N_{10} = N_{17} = N_{18} = 0,\ N_{16} \neq 0$ are satisfied then we get the configuration *Config. 8.33*.

2) *The subcase* $N_{16} = 0$. Then $e = 0$ and systems (14) became of the form (34).

According to Proposition 6 the last systems possess invariant lines of total multiplicity 8 if and only if the conditions (38) hold. We prove that these conditions are equivalent to $N_{19} = 0,\ N_{18} \neq 0$, i.e.

$$27a^2 + (c - f)(2c + f)^2 = 0,\ a \neq 0 \ \Leftrightarrow N_{19} = 0,\ N_{18} \neq 0.$$

Indeed, for systems (34) we have $N_{19} = 24[27a^2 + (c - f)(2c + f)^2]x^3y$ and, evidently, $N_{19} = 0$ implies $27a^2 + (c - f)(2c + f)^2 = 0$. On the other hand we have $N_{18} = 216ax^3$ and thus, the condition $N_{18} \neq 0$ is equivalent to $a \neq 0$. Therefore if the conditions $N_{19} = 0,\ N_{18} \neq 0$ are satisfied then systems (34) via a transformation and a suitable notation (see page 76) can be brought to systems (36). For these systems we calculate $N_{20} = 48(1 - 4r)x^4,\quad N_{21} = 48rx^4$.
Therefore if for a system (14) the conditions $N_2 = N_3 = N_{16} = N_{19} = 0$ and $N_{18} \neq 0$ hold then we obtain the configuration *Config. 8.34* if $N_{21} < 0$; *Config. 8.35* if $N_{20} > 0, N_{21} > 0$; *Config. 8.36* if $N_{20} = 0$ and *Config. 8.37* in the case $N_{20} < 0$. Moreover if $N_{21} = 0$, i.e. $r = 0$ we obtain *Config. 8.38*.

3.3.4 Perturbations of normal forms

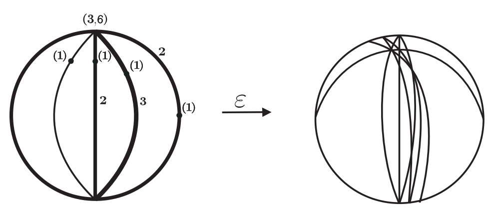
To finish the proof of the Main Theorem it remains to construct for the normal forms given in this theorem the corresponding perturbations, which prove that the respective invariant straight lines have the indicated multiplicities. In this section we construct such perturbations and for each configuration *Configs. 8.j*, $j = 23, 24, \dots, 38$ we give: (i) the corresponding normal form and its invariant straight lines; (ii) the respective perturbed normal form with its invariant straight lines and (iii) the configuration *Configs.* $8.j_\varepsilon$, $j = 23, 24, \dots, 38$ corresponding to the perturbed system.

$$\textit{Config. 8.23}\ \begin{cases}\dot{x} = (x-1)x(1+x),\\ \dot{y} = x - y + x^2 + 3xy;\end{cases}$$

Invariant lines: $L_{1,2} = x,\ L_{3,4,5} = x - 1,\ L_6 = x + 1,\ L_7:\ Z = 0;$

$$\textit{Config. } 8.23_\varepsilon\text{:}\ \begin{cases}\dot{x} = x(1+x)(x + 3x\varepsilon - 1),\\ \dot{y} = (1 + 3\varepsilon y)(x + x^2 - y + 3xy - 3\varepsilon y + 3\varepsilon xy - 6\varepsilon y^2 - 9\varepsilon^2 y^2);\end{cases}$$

$$\textit{Invariant lines:}\ \begin{cases}L_1 = x,\quad L_2 = x - 3\varepsilon y,\quad L_3 = x + 3\varepsilon x - 1,\ L_4 = x - 3\varepsilon y - 1,\\ L_5 = x - 3\varepsilon - 6\varepsilon y - 9\varepsilon^2 y - 1,\ L_6 = 1 + x,\ L_7 = 1 + 3\varepsilon y.\end{cases}$$



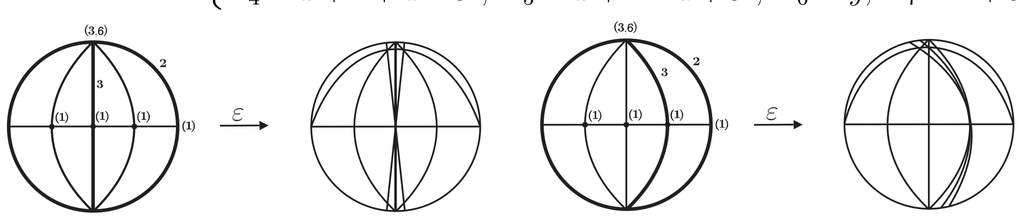
Config. 8.23 *Config.* 8.23_ε

$$\textit{Config. 8.24-8.26}\ \begin{cases}\dot{x} = x(1 - u + x)(1 + u + x),\\ \dot{y} = (1 - u^2 + 2x)y,\quad |u| \neq 1,\end{cases}\quad \begin{cases}|u| > 1 \ \Rightarrow\ \textit{Config. 8.24};\\ |u| < 1 \ \Rightarrow\ \textit{Config. 8.25};\\ \ u = 0 \ \Rightarrow\ \textit{Config. 8.26};\end{cases}$$

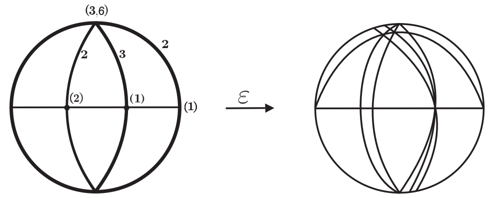
Invariant lines: $L_{1,2,3} = x,\ L_4 = x + 1 + u,\ L_5 = x + 1 - u,\ L_6 = y,\ L_7:\ Z = 0;$

$$\textit{Config. } 8.24_\varepsilon\text{-}8.26_\varepsilon\text{:}\ \begin{cases}\dot{x} = x(1 - u + \varepsilon^2 + x)(1 + u - \varepsilon^2 + x),\\ \dot{y} = y(1 + \varepsilon y)\big[1 - (u - \varepsilon^2)^2 + 2x + \big(\varepsilon^2(u - \varepsilon^2)^2 - \varepsilon^2\big)y\big];\end{cases}$$

$$\textit{Invariant lines:}\ \begin{cases}L_1 = x,\quad L_2 = x - \varepsilon(u+1)y,\quad L_3 = x - \varepsilon(u-1)y - y\varepsilon^3,\\ L_4 = x + 1 + u - \varepsilon^2,\ L_5 = x + 1 - u + \varepsilon^2,\ L_6 = y,\ L_7 = 1 + \varepsilon y.\end{cases}$$



Config. 8.24 *Config.* 8.24_ε *Config. 8.25* *Config.* 8.25_ε



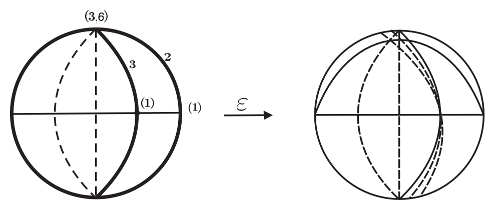
Config. 8.26 *Config. 8.26$_\varepsilon$*

Config. 8.27: $\begin{cases} \dot{x} = x\big[(x+1)^2 + u^2\big], \\ \dot{y} = (1+u^2+2x)y, \quad u \neq 0; \end{cases}$

Invariant lines: $L_{1,2,3} = x,\ L_4 = x+1+iu,\ L_5 = x+1-iu,\ L_6 = y,\ L_7:\ Z=0;$

Config. 8.27$_\varepsilon$: $\begin{cases} \dot{x} = x\big[(x+1)^2 + u^2\big], \\ \dot{y} = y(1-y\varepsilon)(1+u^2+2x+y\varepsilon+u^2y\varepsilon); \end{cases}$

Invariant lines: $L_1 = x,\ L_{2,3} = x+\varepsilon y \pm iu\varepsilon y,\ L_{4,5} = x+1 \pm iu,\ L_6 = y,\ L_7 = -1+y\varepsilon.$



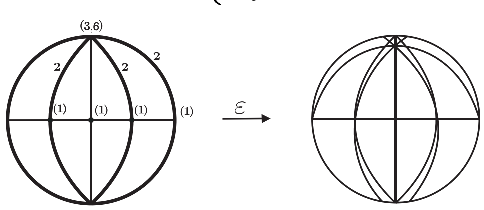
Config. 8.27 *Config. 8.27$_\varepsilon$*

Config. 8.28, 8.29 $\begin{cases} \dot{x} = (1-x+u)x(1-x-u), \\ \dot{y} = 2(u^2+x-1)y,\ |u| \neq 1, \end{cases}$ $\begin{cases} |u| > 1 \ \Rightarrow\ \textit{Config. 8.28}; \\ |u| < 1 \ \Rightarrow\ \textit{Config. 8.29}; \end{cases}$

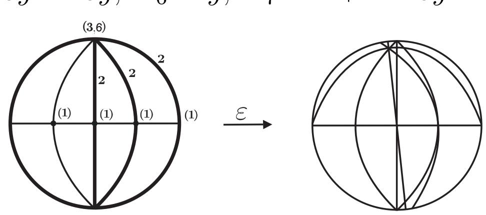
Invariant lines: $L_1 = x,\ L_{2,3} = 1-x+u,\ L_{4,5} = 1-x-u,\ L_6 = y,\ L_7:\ Z=0;$

Config. 8.28$_\varepsilon$, 8.29$_\varepsilon$: $\begin{cases} \dot{x} = (1-x+u)x(1-x-u), \\ \dot{y} = y(1+u-\varepsilon y)(2u^2+2x+\varepsilon y-u\varepsilon y-2)/(1+u); \end{cases}$

Invariant lines: $\begin{cases} L_1 = x, \quad L_2 = 1-x+u, \quad L_3 = 1-x+u-\varepsilon y,\ L_4 = 1-x-u, \\ L_5 = x-1+u^2+ux+\varepsilon y-u\varepsilon y,\ L_6 = y,\ L_7 = 1+u-\varepsilon y. \end{cases}$



Config. 8.28 *Config. 8.28$_\varepsilon$*



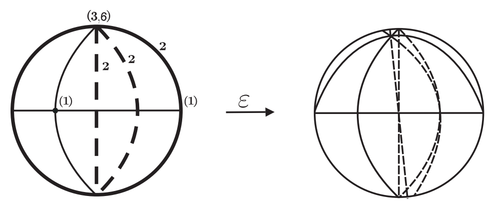
Config. 8.29 *Config. 8.29$_\varepsilon$*

Config. 8.30: $\begin{cases} \dot{x} = x(1+u^2-2x+x^2), \\ \dot{y} = 2y(x-1-u^2), \quad u \neq 0; \end{cases}$

Invariant lines: $L_1 = x,\ L_{2,3} = x-1-iu,\ L_{4,5} = x-1+iu,\ L_6 = y,\ L_7:\ Z=0;$

Config. 8.30$_\varepsilon$: $\begin{cases}\dot{x}=x(1+u^2-2x+x^2),\\ \dot{y}=y(1-\varepsilon y)(2x-2-2u^2+\varepsilon y+u^2\varepsilon y);\end{cases}$

Invariant lines: $\begin{cases}L_1=x,\ L_2=x-1-iu,\ L_3=x-1-iu+y\varepsilon+iu\varepsilon y,\\ L_4=x-1+iu,\ L_5=x-1+iu+\varepsilon y-iu\varepsilon y,\ L_6=y,\ L_7=\varepsilon y-1.\end{cases}$

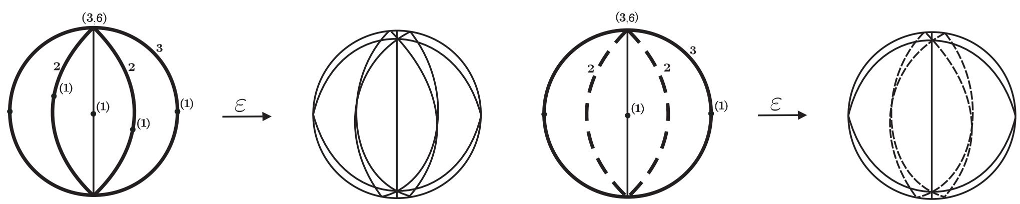


Config. 8.30 *Config. 8.30$_\varepsilon$*

Config. 8.31, 8.32 $\begin{cases}\dot{x}=x(x^2+r),\\ \dot{y}=x-2ry,\end{cases}$ $\begin{cases}r=-1 \ \Rightarrow\ \textit{Config. 8.31};\\ r=1 \ \Rightarrow\ \textit{Config. 8.32};\end{cases}$

Invariant lines: $L_1=x,\ L_{2,3}=x-\sqrt{-r},\ L_{4,5}=x+\sqrt{-r},\ L_6=y,\ L_{6,7}:Z=0;$

Config. 8.31$_\varepsilon$, 8.32$_\varepsilon$: $\begin{cases}\dot{x}=(2r-\varepsilon^4+\varepsilon^6)(4r+4x^2-4r\varepsilon^2-3\varepsilon^4+6\varepsilon^6-3\varepsilon^8)\times\\ (x-x\varepsilon+6ry\varepsilon+2ry\varepsilon^2-3y\varepsilon^5-y\varepsilon^6+3y\varepsilon^7+y\varepsilon^8)/(8r),\\ \dot{y}=(x-2ry+\varepsilon^4y-y\varepsilon^6)(4r-4r\varepsilon^2+16r^2\varepsilon^2y^2-3\varepsilon^4+\\ +6\varepsilon^6-16r\varepsilon^6y^2-3\varepsilon^8+16r\varepsilon^8y^2+4\varepsilon^{10}y^2-8\varepsilon^{12}y^2+4\varepsilon^{14}y^2)/(4r);\end{cases}$



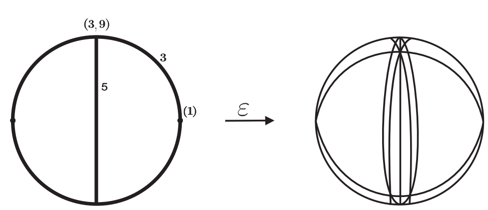
Config. 8.31 *Config. 8.31$_\varepsilon$* *Config. 8.32* *Config. 8.32$_\varepsilon$*

Config. 8.33: $\{\dot{x}=x^3,\quad \dot{y}=1+x;$

Invariant lines: $L_{1,2,3,4,5}=x,\ L_{6,7}:\ Z=0;$

Config. 8.33$_\varepsilon$: $\begin{cases}\dot{x}=x(9x-6\varepsilon+4\varepsilon^2)(9x+6\varepsilon-10\varepsilon^2+4\varepsilon^3)/81,\\ \dot{y}=(3-2\varepsilon+y\varepsilon^2)(3-2\varepsilon-y\varepsilon^2)(9+9x-15\varepsilon+6\varepsilon^2-\varepsilon^2y+\varepsilon^3y)/81;\end{cases}$

Invariant lines: $\begin{cases}L_1=x,\ L_2=x-6\varepsilon+4\varepsilon^2,\ L_3=x+6\varepsilon-10\varepsilon^2+4\varepsilon^3,\\ L_4=x-3\varepsilon+2\varepsilon^2+\varepsilon^3y,\ L_5=x+3\varepsilon-5\varepsilon^2+2\varepsilon^3-\varepsilon^3y+\varepsilon^4y,\\ L_6=3-2\varepsilon+\varepsilon^2y,\ L_7=-3+2\varepsilon+\varepsilon^2y.\end{cases}$



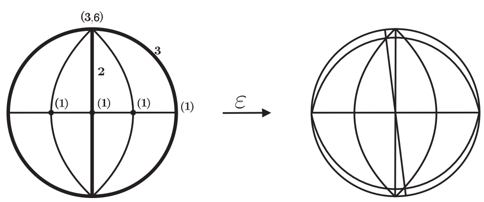
Config. 8.33 *Config. 8.33$_\varepsilon$*

Config. 8.34-8.37 $\begin{cases} \dot{x} = x(r + x + x^2), \\ \dot{y} = ry, \quad r \neq 0, \end{cases}$ $\begin{cases} r < 0 & \Rightarrow \textit{Config. } 8.34; \\ 0 < r < 1/4 & \Rightarrow \textit{Config. } 8.35; \\ r = 1/4 & \Rightarrow \textit{Config. } 8.36; \\ r > 1/4 & \Rightarrow \textit{Config. } 8.37; \end{cases}$

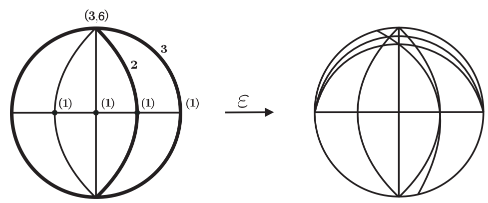
Invariant lines: $L_{1,2} = x,\ L_{3,4} = r + x + x^2,\ L_5 = y,\ L_{6,7}:\ Z = 0;$

Config. 8.34$_\varepsilon$-8.37$_\varepsilon$: $\begin{cases} \dot{x} = x(r - \varepsilon^2 + x + x^2), \\ \dot{y} = y(r - \varepsilon^2 - \varepsilon y + \varepsilon^2 y^2); \end{cases}$

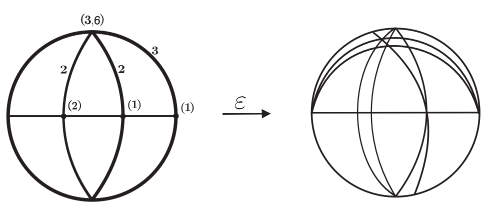
Invariant lines: $\begin{cases} L_1 = x, \quad L_2 = x - \varepsilon y, \quad L_{3,4} = r + x + x^2 - \varepsilon^2, \\ L_{5,6} = r - \varepsilon y - \varepsilon^2 + \varepsilon^2 y^2,\ L_7 = y. \end{cases}$



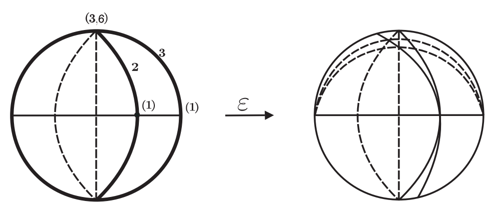
Config. 8.34 *Config. 8.34$_\varepsilon$*



Config. 8.35 *Config. 8.35$_\varepsilon$*



Config. 8.36 *Config. 8.36$_\varepsilon$*



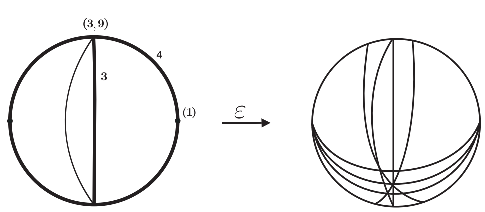
Config. 8.37 *Config. 8.37$_\varepsilon$*

Config. 8.38: $\{\, \dot{x} = x^2(x+1), \quad \dot{y} = 1;$

Invariant lines: $L_{1,2,3} = x,\ L_4 = x + 1,\ L_{5,6,7}:\ Z = 0;$

Config. 8.38$_\varepsilon$: $\begin{cases} \dot{x} = x(x - \varepsilon)(1 + x + \varepsilon - 2\varepsilon y), \\ \dot{y} = (\varepsilon y - 1)(2\varepsilon y - 1)(1 - 2\varepsilon y + 2\varepsilon^2 y); \end{cases}$

Invariant lines: $\begin{cases} L_1 = x,\ L_2 = x - \varepsilon,\ L_3 = x + \varepsilon - 2y\varepsilon^2, L_4 = 1 + x - \varepsilon - 2y\varepsilon + 2y\varepsilon^2, \\ L_5 = y\varepsilon - 1,\ L_6 = 2y\varepsilon - 1,\ L_7 = 1 - 2y\varepsilon + 2y\varepsilon^2. \end{cases}$



Config. 8.38 *Config. 8.38$_\varepsilon$*

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