II. Inecuatii trigonometrice

Metoda principala de rezolvare a inecuatiilor trigonometrice consta in reducerea lor la inecuatii de forma

$$\sin x \vee a, \quad \cos x \vee a, \quad \operatorname{tg} x \vee a, \quad \operatorname{ctg} x \vee a, \tag{1}$$

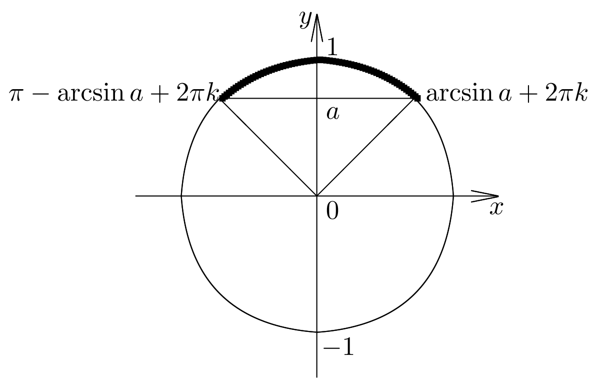
unde $a \in \mathbf{R}$, semnul " $\vee$ " desemneaza semnul compararii si inlocuieste oricare din semnele " $>$ ", " $\geq$ ", " $<$ ", " $\leq$ " si utilizarea afirmatiilor ce urmeaza.

Afirmatia 1. Multimea solutiilor inecuatiei

$$\sin x > a \tag{2}$$

este

1. $\mathbf{R}$, daca $a < -1$;
2. $\bigcup_{k \in \mathbf{Z}} (\arcsin a + 2\pi k; \pi - \arcsin a + 2\pi k)$, daca $-1 \leq a < 1$;
3. Multimea vida, daca $a \geq 1$.

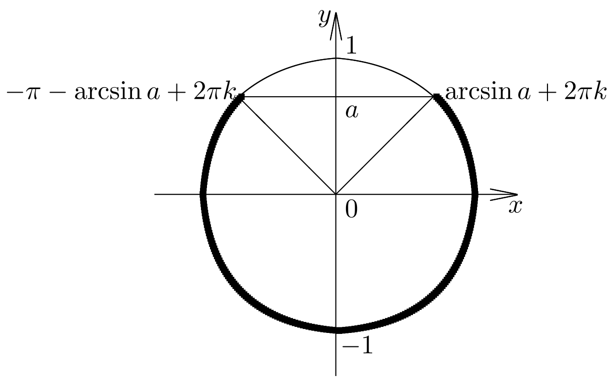


Afirmatia 2. Multimea solutiilor inecuatiei

$$\sin x < a \tag{3}$$

este

1. $\mathbf{R}$, daca $a > 1$;
2. $\bigcup_{k \in \mathbf{Z}} (-\pi - \arcsin a + 2\pi k; \arcsin a + 2\pi k)$, daca $-1 < a \leq 1$;
3. Multimea vida, daca $a \leq -1$.

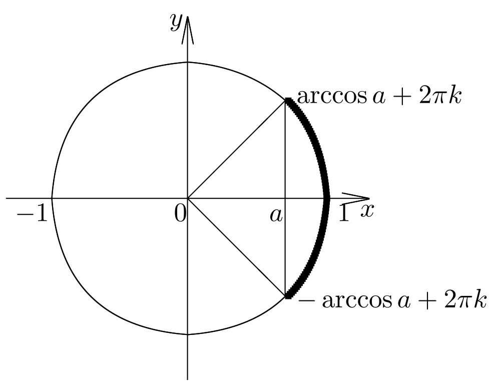


Afirmatia 3. Multimea solutiilor inecuatiei

$$\cos x > a \tag{4}$$

este

1. **R**, daca $a < -1$;
2. $\bigcup\limits_{k \in \mathbf{Z}} (2\pi k - \arccos a; 2\pi k + \arccos a)$, daca $-1 \leq a < 1$;
3. Multimea vida, daca $a \geq 1$.

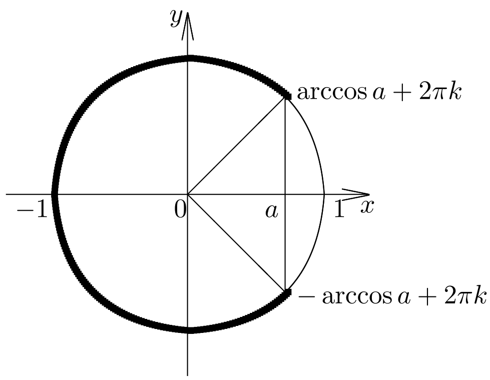


Afirmatia 4. Multimea solutiilor inecuatiei

$$\cos x < a \tag{5}$$

este

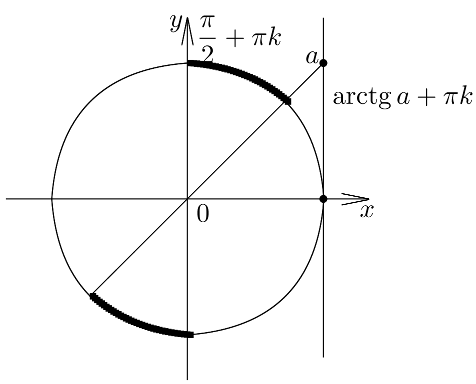
1. **R**, daca $a > 1$;
2. $\bigcup\limits_{k \in \mathbf{Z}} (2\pi k + \arccos a; 2\pi(k+1) - \arccos a)$, daca $-1 < a \leq 1$;
3. Multimea vida, daca $a \leq -1$.



Afirmatia 5. Multimea solutiilor inecuatiei

$$\operatorname{tg} x > a \tag{6}$$

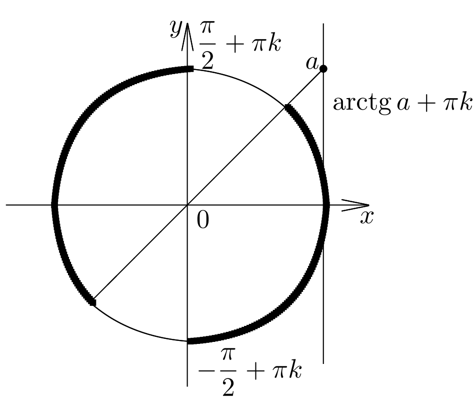
este $\bigcup_{k\in\mathbf{Z}} (\operatorname{arctg} a + \pi k; \frac{\pi}{2} + \pi k)$.



Afirmatia 6. Multimea solutiilor inecuatiei

$$\operatorname{tg} x < a \tag{7}$$

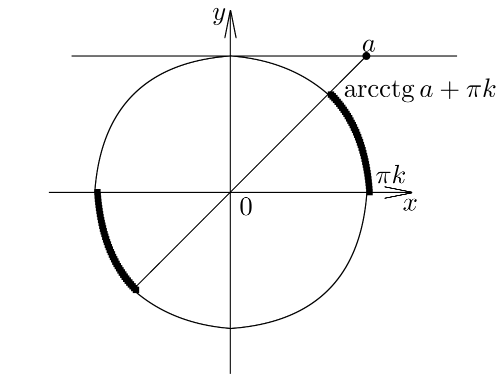
este $\bigcup_{k\in\mathbf{Z}} (-\frac{\pi}{2} + \pi k; \operatorname{arctg} a + \pi k)$.



Afirmatia 7. Multimea solutiilor inecuatiei

$$\operatorname{ctg} x > a \tag{8}$$

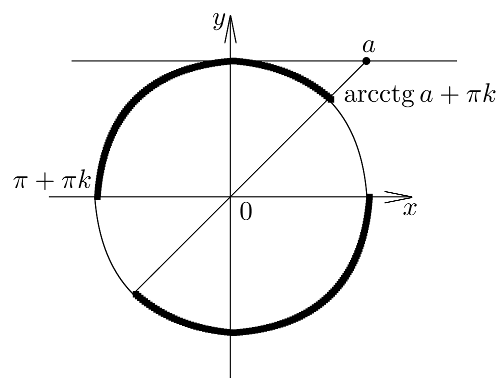
este $\bigcup_{k\in\mathbf{Z}}(\pi k; \operatorname{arcctg} a + \pi k)$.



Afirmatia 8. Multimea solutiilor inecuatiei

$$\operatorname{ctg} x < a \tag{9}$$

este $\bigcup_{k\in\mathbf{Z}}(\operatorname{arcctg} a + \pi k; \pi(k+1))$



Nota. 1. Daca semnul inegalitatii in (2)-(9) nu este strict, in multimea solutiilor inecuatiilor se includ si solutiile ecuatiei respective.

2. Afirmatiile 1-8 se obtin nemijlocit analizand graficul functiilor trigonometrice respective.

Exemplul 1. Sa se rezolve inecuatiile

1) $\sin 2x < \frac{1}{2}$;

2) $2\sin\left(\frac{\pi}{4} - x\right) \leq \sqrt{2}$;

3) $\cos^2 x \geq \frac{1}{4}$;

4) $-2 \leq \operatorname{tg} x < 1$;

5) $2\sin^2 x - 5\sin x + 2 > 0$;

6) $\sin^4 x + \cos^4 x \geq \frac{\sqrt{3}}{2}$;

7) $\operatorname{ctg}^2 x - \operatorname{ctg} x - 2 \leq 0$;

8) $\sin 2x - \sqrt{3}\cos 2x > \sqrt{2}$;

9) $\frac{2\operatorname{tg} x}{1+\operatorname{tg} x} + \frac{1}{\operatorname{tg} x} \geq 2$;

10) $4\sin x\cos x(\cos^2 x - \sin^2 x) < \sin 6x$;

11) $\sin x \sin 3x \geq \sin 5x \sin 7x$;

12) $\sin x + \sin 2x + \sin 3x > 0$.

Rezolvare. 1) Se noteaza $2x = t$ si se obtine inecuatia $\sin t < \frac{1}{2}$ care, conform afirmatiei 2 are solutiile

$$2\pi k - \pi - \arcsin\frac{1}{2} < t < \arcsin\frac{1}{2} + 2\pi k, \quad k \in \mathbf{Z}.$$

Se revine la variabila initiala si, tinand seama ca $\arcsin\frac{1}{2} = \frac{\pi}{6}$, se obtine

$$2\pi k - \pi - \frac{\pi}{6} < 2x < \frac{\pi}{6} + 2\pi k, \quad k \in \mathbf{Z},$$

de unde

$$2\pi k - \frac{7\pi}{6} < 2x < \frac{\pi}{6} + 2\pi k, \quad k \in \mathbf{Z},$$

sau

$$\pi k - \frac{7\pi}{12} < x < \frac{\pi}{12} + \pi k, \quad k \in \mathbf{Z}.$$

Asadar, solutiile inecuatiei enuntate formeaza multimea

$$\bigcup_{k\in\mathbf{Z}} \left(\pi k - \frac{7\pi}{12}; \frac{\pi}{12} + \pi k\right).$$

2) Cum functia sinus este impara,

$$2\sin\left(\frac{\pi}{4} - x\right) \leq \sqrt{2} \Leftrightarrow -2\sin\left(x - \frac{\pi}{4}\right) \leq \sqrt{2} \Leftrightarrow \sin\left(x - \frac{\pi}{4}\right) \geq -\frac{1}{\sqrt{2}}.$$

Se noteaza $t = x - \frac{\pi}{4}$ si se obtine inecuatia

$$\sin t \geq -\frac{1}{\sqrt{2}}$$

cu solutiile (a se vedea afirmatia 1 si nota 1)

$$2\pi k + \arcsin\left(-\frac{1}{\sqrt{2}}\right) \le t \le \pi - \arcsin\left(-\frac{1}{\sqrt{2}}\right) + 2\pi k, \quad k \in \mathbf{Z},$$

de unde, tinand seama ca $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$, se obtine

$$2\pi k - \frac{\pi}{4} \le x - \frac{\pi}{4} \le \pi + \frac{\pi}{4} + 2\pi k, \quad k \in \mathbf{Z},$$

sau

$$2\pi k \le x \le \frac{3\pi}{2} + 2\pi k, \quad k \in \mathbf{Z}.$$

3) Cum $\cos^2 x = \frac{1+\cos 2x}{2}$ inecuatia devine $\frac{1+\cos 2x}{2} \ge \frac{1}{4}$ sau $\cos 2x \ge -\frac{1}{2}$. Se aplica afirmatia 3 si se obtine

$$2\pi k - \arccos\left(-\frac{1}{2}\right) \le 2x \le \arccos\left(-\frac{1}{2}\right) + 2\pi k.$$

Cum $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$, rezulta

$$2\pi k - \frac{2\pi}{3} \le 2x \le \frac{2\pi}{3} + 2\pi k, \quad k \in \mathbf{Z},$$

de unde

$$\pi k - \frac{\pi}{3} \le x \le \frac{\pi}{3} + \pi k, \quad k \in \mathbf{Z}.$$

Altfel,

$$\cos^2 x \ge \frac{1}{4} \;\Leftrightarrow\; |\cos x| \ge \frac{1}{2} \;\Leftrightarrow\; \left[\begin{array}{l} \cos x \ge \frac{1}{2}, \\ \cos x \le -\frac{1}{2}, \end{array}\right. \quad \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} 2\pi n - \frac{\pi}{3} \le x \le \frac{\pi}{3} + 2\pi n, \quad n \in \mathbf{Z}, \\ 2\pi m + \frac{2\pi}{3} \le x \le \frac{4\pi}{3} + 2\pi m, \quad m \in \mathbf{Z} \end{array}\right. \;\Leftrightarrow\; \pi k - \frac{\pi}{3} \le x \le \frac{\pi}{3} + \pi k, \quad k \in \mathbf{Z}.$$

4) Se aplica afirmatiile 5 si 6 si se obtine

$$-2 \le \operatorname{tg} x < 1 \;\Leftrightarrow\; \left\{\begin{array}{l} \operatorname{tg} x < 1, \\ \operatorname{tg} x \ge -2, \end{array}\right. \;\Leftrightarrow\; \left\{\begin{array}{l} \pi n - \frac{\pi}{2} < x < \frac{\pi}{4} + \pi n, \quad n \in \mathbf{Z}, \\ \pi m - \operatorname{arctg} 2 < x < \frac{\pi}{2} + \pi m, \quad m \in \mathbf{Z}, \end{array}\right. \;\Leftrightarrow$$

$$\Leftrightarrow\; \pi k - \operatorname{arctg} 2 \le x < \frac{\pi}{4} + \pi k, \quad k \in \mathbf{Z}.$$

5) Se noteaza $t = \sin x$ si se obtine inecuatia de gradul al doilea

$$2t^2 - 5t + 2 > 0$$

cu solutiile

$$\left[\begin{array}{l} t < \dfrac{1}{2}, \\ t > 2, \end{array}\right.$$

de unde rezulta totalitatea de inecuatii

$$\left[\begin{array}{l} \sin x > 2, \\ \sin x < \dfrac{1}{2}. \end{array}\right.$$

Prima inecuatie a totalitatii solutii nu are, iar din cea secunda se obtine

$$2\pi k - \frac{7\pi}{6} < x < \frac{\pi}{6} + 2\pi k, \quad k \in \mathbf{Z}.$$

6) Cum

$$\sin^4 x + \cos^4 x = (\sin^2 x)^2 + (\cos^2 x)^2 = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x =$$

$$= 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{2} \cdot \frac{1 - \cos 4x}{2} = 1 - \frac{1 - \cos 4x}{4},$$

inecuatia devine

$$1 - \frac{1 - \cos 4x}{4} \geq \frac{\sqrt{3}}{2}$$

sau $\cos 4x \geq 2\sqrt{3} - 3$. Cum $|2\sqrt{3} - 3| \leq 1$, se aplica afirmatia 3 si se obtine

$$2\pi k - \arccos(2\sqrt{3} - 3) < 4x < \arccos(2\sqrt{3} - 3) + 2\pi k, \quad k \in \mathbf{Z},$$

sau

$$\frac{\pi k}{2} - \frac{1}{4}\arccos(2\sqrt{3} - 3) < x < \frac{1}{4}\arccos(2\sqrt{3} - 3) + \frac{\pi k}{2}, \quad k \in \mathbf{Z}.$$

7) Se noteaza $t = \operatorname{ctg} x$ si se obtine inecuatia patrata

$$t^2 - t - 2 \leq 0$$

cu solutiile $-1 \leq t \leq 2$, de unde $-1 \leq \operatorname{ctg} x \leq 2$. Ultima inecuatie se rezolva utilizand afirmatiile 7 si 8:

$$-1 \leq \operatorname{ctg} x \leq 2 \;\Leftrightarrow\; \left\{\begin{array}{l} \operatorname{ctg} x \leq 2, \\ \operatorname{ctg} x \geq -1, \end{array}\right. \;\Leftrightarrow\; \left\{\begin{array}{ll} \pi k + \operatorname{arcctg} 2 \leq x < \pi + \pi n, & n \in \mathbf{Z} \\ \pi m < x \leq \dfrac{3\pi}{4} + \pi m, & m \in \mathbf{Z} \end{array}\right. \;\Leftrightarrow$$

$$\Leftrightarrow\; \pi k + \operatorname{arcctg} 2 \leq x \leq \frac{3\pi}{4} + \pi k, \quad k \in \mathbf{Z}.$$

8) Se utilizeaza metoda unghiului auxiliar si se obtine

$$\sin 2x - \sqrt{3}\cos 2x > \sqrt{2} \;\Leftrightarrow\; \frac{1}{2}\sin 2x - \frac{\sqrt{3}}{2}\cos 2x > \frac{\sqrt{2}}{2} \;\Leftrightarrow$$

$$\Leftrightarrow\; \sin 2x \cos\frac{\pi}{3} - \cos 2x \sin\frac{\pi}{3} > \frac{\sqrt{2}}{2} \;\Leftrightarrow\; \sin\left(2x - \frac{\pi}{3}\right) > \frac{\sqrt{2}}{2} \;\Leftrightarrow$$

$$\Leftrightarrow\; 2\pi k + \frac{\pi}{4} < 2x - \frac{\pi}{3} < \pi - \frac{\pi}{4} + 2\pi k, \quad k \in \mathbf{Z} \;\Leftrightarrow$$

$$\Leftrightarrow\; 2\pi k + \frac{\pi}{4} + \frac{\pi}{3} < 2x < \frac{3\pi}{4} + \frac{\pi}{3} + 2\pi k, \quad k \in \mathbf{Z} \;\Leftrightarrow$$

$$\Leftrightarrow \pi k + \frac{7\pi}{24} < x < \frac{13\pi}{24} + \pi k, \quad k \in \mathbf{Z}.$$

9) Se noteaza $\operatorname{tg} x = t$ si se rezolva inecuatia in t utilizand metoda intervalelor:

$$\frac{2t}{1+t} + \frac{1}{t} \geq 2 \;\Leftrightarrow\; \frac{2t^2 + 1 + t - 2t(1+t)}{t(t+1)} \geq 0 \;\Leftrightarrow\; \frac{1-t}{t(t+1)} \geq 0 \;\Leftrightarrow\; \left[\begin{array}{l} 0 < t \leq 1, \\ t < -1. \end{array}\right.$$

Asadar, se obtine totalitatea de inecuatii

$$\left[\begin{array}{l} 0 < \operatorname{tg} x \leq 1, \\ \operatorname{tg} x < -1, \end{array}\right.$$

ce se rezolva, utilizand afirmatiile 5 si 6:

$$\left[\begin{array}{l} 0 < \operatorname{tg} x \leq 1, \\ \operatorname{tg} x < -1, \end{array}\right. \;\Leftrightarrow\; \left[\begin{array}{l} \left\{\begin{array}{l} \operatorname{tg} x \leq 1, \\ \operatorname{tg} x > 0, \end{array}\right. \\ -\frac{\pi}{2} + \pi m < x < -\frac{\pi}{4} + \pi m, \quad m \in \mathbf{Z}, \end{array}\right. \;\Leftrightarrow$$

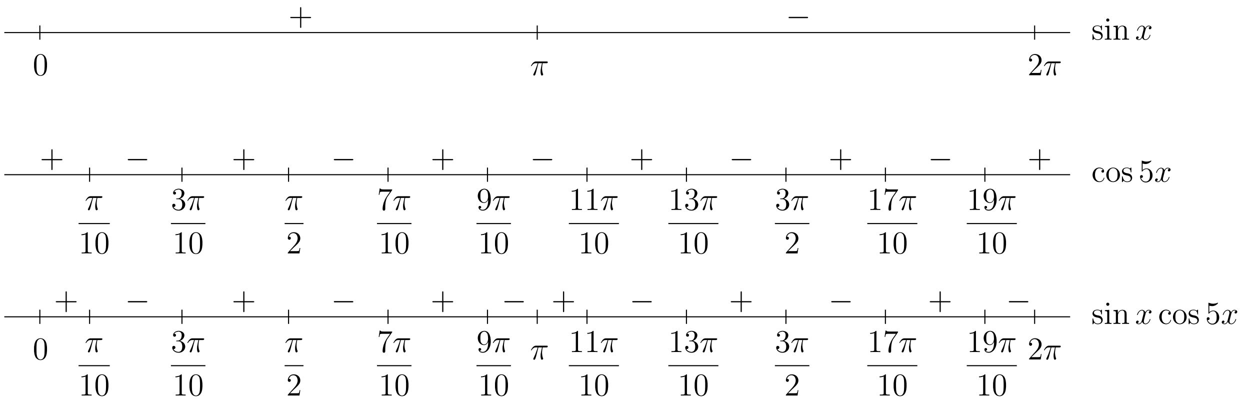
$$\Leftrightarrow\; \left[\begin{array}{l} \pi n < x \leq \frac{\pi}{4} + \pi n, \quad n \in \mathbf{Z}, \\ -\frac{\pi}{2} + \pi m < x < -\frac{\pi}{4} + \pi m, \quad m \in \mathbf{Z}. \end{array}\right.$$

10) Se utilizeaza formulele sinusului si cosinusului argumentului dublu si se obtine

$$4\sin x \cos x(\cos^2 x - \sin^2 x) < \sin 10x \;\Leftrightarrow\; 2\sin 2x \cdot \cos 2x < \sin 6x \;\Leftrightarrow$$

$$\Leftrightarrow\; \sin 4x < \sin 6x \;\Leftrightarrow\; \sin 6x - \sin 4x > 0 \;\Leftrightarrow\; 2\sin x \cos 5x > 0.$$

Se tine seama ca 2π este o perioada a functiei $f(x) = \sin x \cos 5x$ si se utilizeaza metoda generalizata a intervalelor pentru un interval de lungime 2π:



Astfel multimea solutiilor inecuatiei date este reuniunea multimilor

$$\left(2\pi k; \frac{\pi}{10}+2\pi k\right) \cup \left(\frac{3\pi}{10}+2\pi k; \frac{\pi}{2}+2\pi k\right) \cup \left(\frac{7\pi}{10}+2\pi k; \frac{9\pi}{10}+2\pi k\right) \cup$$

$$\cup \left(2\pi k+\pi; \frac{11\pi}{10}+2\pi k\right) \cup \left(2\pi k+\frac{13\pi}{10}; \frac{3\pi}{2}+2\pi k\right) \cup \left(\frac{17\pi}{10}+2\pi k; \frac{19\pi}{10}+2\pi k\right).$$

11) $\sin x \sin 3x \geq \sin 2x \sin 4x \;\Leftrightarrow\; \frac{1}{2}(\cos 2x - \cos 4x) \geq \frac{1}{2}(\cos 2x - \cos 6x) \;\Leftrightarrow$
$\Leftrightarrow\; -\cos 4x \geq -\cos 6x \;\Leftrightarrow\; \cos 6x - \cos 4x \geq 0 \;\Leftrightarrow\; -2\sin x \sin 5x \geq 0 \;\Leftrightarrow\; \sin x \sin 5x \leq 0.$
Ultima inecuatie se rezolva similar exemplului precedent si se obtine multimea solutiilor

$$\bigcup_{k\in\mathbf{Z}} \left[\frac{2\pi}{5}n; \frac{\pi}{5}+\frac{2\pi}{5}n\right].$$

12) $\sin x+\sin 2x+\sin 3x > 0 \;\Leftrightarrow\; (\sin x+\sin 3x)+\sin 2x > 0 \;\Leftrightarrow\; 2\sin 2x\cos x+\sin 2x > 0 \;\Leftrightarrow$

$$\Leftrightarrow\; \sin 2x(2\cos x+1) > 0 \;\Leftrightarrow\; \left[\begin{array}{l} \begin{cases} \sin 2x > 0, \\ \cos x > -\dfrac{1}{2}, \end{cases} \\ \begin{cases} \sin 2x < 0, \\ \cos x < -\dfrac{1}{2}, \end{cases} \end{array}\right. \;\Leftrightarrow$$

$$\Leftrightarrow\; \left[\begin{array}{l} \begin{cases} \pi n < x < \dfrac{\pi}{2}+\pi n, \quad n \in \mathbf{Z}, \\ -\dfrac{2\pi}{3}+2\pi m < x < \dfrac{2\pi}{3}+2\pi m, \quad m \in \mathbf{Z}, \end{cases} \\ \begin{cases} \dfrac{\pi}{2}+\pi n < x < \pi+\pi n, \quad n \in \mathbf{Z}, \\ 2\pi m+\dfrac{2\pi}{3} < x < \dfrac{4\pi}{3}+2\pi m, \quad m \in \mathbf{Z}, \end{cases} \end{array}\right. \;\Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} 2\pi m < x < \dfrac{\pi}{2} + 2\pi m, \quad m \in \mathbf{Z}, \\ 2\pi m - \dfrac{2\pi}{3} < x < -\dfrac{\pi}{2} + 2\pi m, \quad m \in \mathbf{Z}, \\ 2\pi m + \dfrac{2\pi}{3} < x < \pi + 2\pi m, \quad m \in \mathbf{Z}. \end{array} \right.$$

Exercitii pentru autoevaluare

Sa se rezolve inecuatiile:

1. $\operatorname{tg}^3 x + \operatorname{tg}^2 x - \operatorname{tg} x - 1 > 0$;
2. $\operatorname{tg} x + \operatorname{ctg} x \leq 2$;
3. $\sin 2x < \cos x$;
4. $\cos x + \cos 2x + \cos 3x \geq 0$;
5. $6 \sin^2 x - 5 \sin x + 1 > 0$;
6. $\dfrac{2 \cos^2 x + \cos x - 1}{\sin x - 1} < 0$;
7. $2 \cos \left(2x + \dfrac{\pi}{4}\right) - \sqrt{3} \leq 0$;
8. $\operatorname{tg} \left(\dfrac{\pi}{4} - 2x\right) < -\sqrt{3}$;
9. $2 \sin^2 x + 9 \cos x - 6 \geq 0$;
10. $\dfrac{\sin x}{1 + \cos x} \geq 0$;
11. $4 \sin x \cos x - \sqrt{2} < 2(\sqrt{2} \cos x - \sin x)$;
12. $\cos 2x + \sin x \geq 0$.