Ecuatii si inecuatii trigonometrice

I. Ecuatii trigonometrice

Ecuatiile ce contin necunoscute sub semnul functiilor trigonometrice se numesc ecuatii trigonometrice.

Cele mai simple ecuatii trigonometrice sunt ecuatiile de tipul

$$\sin x = a,\ \cos x = a,\ \operatorname{tg} x = a,\ \operatorname{ctg} x = a,\ a \in \mathbf{R}. \tag{1}$$

Cum rezolvarea ecuatiilor trigonometrice se reduce la rezolvarea ecuatiilor de tipul (1) (utilizand diferite transformari), vom aminti afirmatiile de baza referitor solutiile ecuatiilor (1).

Afirmatia 1. Ecuatia

$$\sin x = a, \quad a \in \mathbf{R}, \tag{2}$$

pentru $|a| > 1$ solutii nu are, iar pentru $|a| \leq 1$ multimea solutiilor ei se contine in formula

$$x = (-1)^n \arcsin a + \pi n, \quad n \in \mathbf{Z}, \tag{3}$$

unde $\arcsin a \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ este unghiul, sinusul caruia este egal cu a, iar $\mathbf{Z}$ desemneaza multimea numerelor intregi, sau, echivalent (tinand seama de paritatea lui n), in totaliatea

$$\left[\begin{array}{l} x = \arcsin a + 2\pi k, \\ x = \pi - \arcsin a + 2\pi k, \end{array}\right. \qquad k \in \mathbf{Z}. \tag{4}$$

Nota 1. Daca in ecuatia (2) $a \in \{0; -1; 1\}$ solutiile ei (3) se scriu mai simplu, si anume

$$\sin x = 0 \ \Leftrightarrow\ x = \pi n,\ n \in \mathbf{Z},$$

$$\sin x = 1 \ \Leftrightarrow\ x = \frac{\pi}{2} + 2\pi n,\ n \in \mathbf{Z},$$

$$\sin x = -1 \ \Leftrightarrow\ x = -\frac{\pi}{2} + 2\pi n,\ n \in \mathbf{Z}.$$

Exemplul 1. Sa se rezolve ecuatiile

$$\text{a)}\ \sin x = \frac{\sqrt{3}}{2}; \quad \text{b)}\ \sin x = -\frac{1}{3}; \quad \text{c)}\ \sin x = \sqrt{11} - 2.$$

Rezolvare. a) Cum $\left|\frac{\sqrt{3}}{2}\right| \leq 1$, conform (3) solutiile ecuatiei date sunt

$$x = (-1)^n \arcsin \frac{\sqrt{3}}{2} + \pi n,\ n \in \mathbf{Z},$$

sau tinand seama ca $\arcsin\frac{\sqrt{3}}{2}=\frac{\pi}{3}$, se obtine

$$x=(-1)^n\frac{\pi}{6}+\pi n,\ n\in\mathbf{Z}.$$

b) Similar exemplului a) se obtine $x=(-1)^n\arcsin\left(-\frac{1}{3}\right)+\pi n,\ n\in\mathbf{Z}$ sau, tinand seama arcsinus ca functia este o functie impara,

$$x=(-1)^{n+1}\arcsin\frac{1}{3}+\pi n,\ n\in\mathbf{Z}.$$

c) Cum $\sqrt{11}-2>1$, rezulta ca ecuatia data nu are solutii.

Afirmatia 2. Ecuatia

$$\cos x=a \tag{5}$$

pentru $|a|>1$ nu are solutii, iar pentru $|a|\le 1$ multimea solutiilor ei se contine in formula

$$x=\pm\arccos a+2\pi n,\ n\in\mathbf{Z}, \tag{6}$$

unde $\arccos a\in[0;\pi]$ este unghiul, cosinusul caruia este egal cu a.

Nota 2. Daca in ecuatia (5) $a\in\{0;1;-1\}$ solutiile ei (6) se scriu mai simplu, si anume

$$\cos x=0\ \Leftrightarrow\ x=\frac{\pi}{2}+\pi n,\ n\in\mathbf{Z},$$
$$\cos x=1\ \Leftrightarrow\ x=2\pi n,\ n\in\mathbf{Z},$$
$$\cos x=-1\ \Leftrightarrow\ x=\pi+2\pi n,\ n\in\mathbf{Z}.$$

Exemplul 2. Sa se rezolve ecuatiile:

$$\text{a)}\ \cos x=-\frac{1}{2};\quad \text{b)}\ \cos x=\frac{2}{3};\quad \text{c)}\ \cos x=\frac{\sqrt{3}+1}{2}.$$

Rezolvare. a) Cum $\left|-\frac{1}{2}\right|\le 1$, conform (6) solutiile ecuatiei date sunt $x=\pm\arccos\left(-\frac{1}{2}\right)+2\pi n,\ n\in\mathbf{N}$, sau tinand seama ca $\arccos\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$, se obtine $x=\pm\frac{2\pi}{3}+2\pi n,\ n\in\mathbf{Z}$.

b) Similar exemplului a) se obtine $x=\pm\arccos\frac{2}{3}+2\pi n,\ n\in\mathbf{Z}$.

c) Cum $\frac{\sqrt{3}+1}{2}>1$, ecuatia data nu are solutii.

Afirmatia 3. Ecuatia

$$\operatorname{tg} x=a,\ a\in\mathbf{R} \tag{7}$$

are solutiile

$$x=\operatorname{arctg} a+\pi n,\ n\in\mathbf{Z}, \tag{8}$$

unde $\operatorname{arctg} a \in (-\frac{\pi}{2}; \frac{\pi}{2})$ este unghiul, tangenta caruia este egala cu a.

Afirmatia 4. Ecuatia

$$\operatorname{ctg} x = a,\ a \in \mathbf{R} \tag{9}$$

are solutiile

$$x = \operatorname{arcctg} a + \pi n,\ n \in \mathbf{Z}, \tag{10}$$

unde $\operatorname{arcctg} a \in (0; \pi)$ este unghiul, cotangenta caruia este egala cu a.

Exemplul 3. Sa se rezolve ecuatiile

$$\text{a) } \operatorname{tg} x = 1; \quad \text{b) } \operatorname{tg} x = -2; \quad \text{c) } \operatorname{ctg} x = -1; \quad \text{d) } \operatorname{ctg} x = 3.$$

Rezolvare. a) Conform (8) solutiile ecuatiei date sunt $x = \operatorname{arctg} 1 + \pi n,\ n \in \mathbf{Z}$, sau tinand seama ca $\operatorname{arctg} 1 = \dfrac{\pi}{4}$, se obtine $x = \dfrac{\pi}{4} + \pi n,\ n \in \mathbf{Z}$.

b) Similar exemplului precedent se obtine $x = \operatorname{arctg}(-2) + \pi n,\ n \in \mathbf{Z}$, sau tinand seama ca arctangenta este o functie impara, $x = -\operatorname{arctg} 2 + \pi n,\ n \in \mathbf{Z}$.

c) Se tine seama de (10) si se obtine

$$x = \operatorname{arcctg}(-1) + \pi n,\ n \in \mathbf{Z},$$

sau, cum $\operatorname{arcctg}(-1) = \dfrac{3\pi}{4}$, $x = \dfrac{3\pi}{4} + \pi n,\ n \in \mathbf{Z}$.

d) Similar exemplului c) se obtine $x = \operatorname{arcctg} 3 + \pi n,\ n \in \mathbf{Z}$.

Observatie. Ecuatiile

$$\sin f(x) = a, \quad \cos f(x) = a, \quad \operatorname{tg} f(x) = a, \quad \operatorname{ctg} f(x) = a \tag{11}$$

prin intermediul substitutiei $f(x) = t$ se reduc la rezolvarea ecuatiilor (1).

Exemplul 4. Sa se rezolve ecuatiile

$$\text{a) } \sin(2x - 1) = 1; \quad \text{b) } \cos(x^2 + 4) = -1; \quad \text{c) } \operatorname{tg} 2x = \sqrt{3}; \quad \text{d) } \operatorname{ctg} x^3 = -2.$$

Rezolvare. a) $\sin(2x - 1) = 1 \Leftrightarrow \begin{cases} \sin t = 1; \\ t = 2x - 1, \end{cases} \Leftrightarrow 2x - 1 = \dfrac{\pi}{2} + 2\pi n,\ n \in \mathbf{Z} \Leftrightarrow$ $\Leftrightarrow 2x = \dfrac{\pi}{2} + 2\pi n + 1,\ n \in \mathbf{Z} \Leftrightarrow x = \dfrac{\pi}{4} + \pi n + \dfrac{1}{2},\ n \in \mathbf{Z}$.

b) $\cos(x^2 + 4) = -1 \Leftrightarrow \begin{cases} \cos t = -1, \\ t = x^2 + 4, \end{cases} \Leftrightarrow \begin{cases} x^2 + 4 = \pi + 2\pi n,\ n \in \mathbf{Z}, \\ \pi + 2\pi n \geq 4, \end{cases} \Leftrightarrow$ $\Leftrightarrow x^2 = \pi + 2\pi n - 4, n = 1, 2, 3, \ldots \Leftrightarrow x = \pm\sqrt{\pi + 2\pi n - 4},\ n = 1, 2, 3, \ldots$ (se tine seama ca radicalul de ordin par exista doar din valori nenegative).

c) $\operatorname{tg} 2x = \sqrt{3} \Leftrightarrow 2x = \operatorname{arctg}\sqrt{3} + \pi n,\ n \in \mathbf{Z} \Leftrightarrow 2x = \dfrac{\pi}{3} + \pi n,\ n \in \mathbf{Z} \Leftrightarrow$ $x = \dfrac{\pi}{6} + \dfrac{\pi}{2} n,\ n \in \mathbf{Z}$.

d) $\operatorname{ctg} x^3 = -2 \Leftrightarrow x^3 = \operatorname{arcctg}(-2) + \pi n,\ n \in \mathbf{Z} \Leftrightarrow x = \sqrt[3]{\operatorname{arcctg}(-2) + \pi n},\ n \in \mathbf{Z}$.

Ecuatii trigonometrice reductibile la ecuatii de gradul al doilea

Ecuatia

$$a\sin^2 x + b\sin x + c = 0,\ \ a, b, c \in \mathbf{R},\ \ a \neq 0 \qquad (12)$$

prin intermediul substitutiei $t = \sin x$, $(|t| \leq 1)$ se reduce la ecuatia patrata $at^2 + bt + c = 0$.

Exemplul 5. Sa se rezolve ecuatiile

a) $2\sin^2 x - 5\sin x + 2 = 0$; b) $\sin^2 2x - \sin 2x = 0$; c) $\sin^2 x - \sin x + 6 = 0$.

Rezolvare. a) Se noteaza $\sin x = t$ si ecuatia devine

$$2t^2 - 5t + 2 = 0,$$

de unde $t_1 = \dfrac{1}{2}$ si $t_2 = 2$. Cum $|t| \leq 1$, ramane $t = \dfrac{1}{2}$ si prin urmare ecuatia initiala este echivalenta cu ecuatia

$$\sin x = \frac{1}{2},$$

solutiile careia sunt (a se vedea (3)) $x = (-1)^n \dfrac{\pi}{6} + \pi n,\ n \in \mathbf{Z}$.

b) Se noteaza $\sin x = t$ si se obtine ecuatia patrata $t^2 - t = 0$ cu solutiile $t_1 = 0$ si $t_2 = 1$. Astfel ecuatia initiala este echivalenta cu totalitatea de ecuatii

$$\left[\begin{array}{l} \sin 2x = 0, \\ \sin 2x = 1, \end{array}\right.$$

de unde

$$\left[\begin{array}{ll} x = \dfrac{\pi}{2}n, & n \in \mathbf{Z}, \\ x = \dfrac{\pi}{4} + \pi k, & k \in \mathbf{Z}. \end{array}\right.$$

c) Similar exemplelor precedente se obtine ecuatia patrata $t^2 - t + 6 = 0$, care nu are solutii. Rezulta ca si ecuatia trigonometrica nu are solutii.

Ecuatiile

$$a\cos^2 x + b\cos x + c = 0, \qquad (13)$$

$$a\,\mathrm{tg}^2\, x + b\,\mathrm{tg}\, x + c = 0, \qquad (14)$$

$$a\,\mathrm{ctg}^2\, x + b\,\mathrm{ctg}\, x + c = 0, \qquad (15)$$

unde $a, b, c \in \mathbf{R},\ a \neq 0$ se rezolva similar ecuatiei (12).

In cazul ecuatiei (13) se tine seama ca $t = \cos x$ in modul urmeaza sa nu intreaca unu, iar pentru $t = \mathrm{tg}\, x$ $(t = \mathrm{ctg}\, x)$ in ecuatia (14) (respectiv (15)) restrictii nu sunt.

Exemplul 6. Sa se rezolve ecuatiile

a) $6\cos^2 x - 5\cos x + 1 = 0$; b) $\operatorname{tg}^2 2x - 4\operatorname{tg} 2x + 3 = 0$; c) $\operatorname{ctg}^2 \frac{x}{2} - \operatorname{ctg}\frac{x}{2} - 2 = 0$.

Rezolvare. a) Se noteaza $\cos x = t$ si se obtine ecuatia patrata

$$6t^2 - 5t + 1 = 0$$

cu solutiile $t = \frac{1}{3}$ si $t_2 = \frac{1}{2}$. Cum ambele solutii verifica conditia $|t| \leq 1$ se obtine totalitatea

$$\left[\begin{array}{l} \cos x = \frac{1}{3}, \\ \cos x = \frac{1}{2}, \end{array}\right.$$

de unde $x = \pm \arccos\frac{1}{3} + 2\pi n,\ n \in \mathbf{Z}, \quad x = \pm\frac{\pi}{3} + 2\pi k,\ k \in \mathbf{Z}$.

b) Se noteaza $\operatorname{tg} 2x = t$ si se obtine ecuatia patrata

$$t^2 - 4t + 3 = 0$$

cu solutiile $t_1 = 1$ si $t_2 = 3$. Prin urmare

$$\left[\begin{array}{l} \operatorname{tg} 2x = 1, \\ \operatorname{tg} 2x = 3, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 2x = \frac{\pi}{4} + \pi n,\ n \in \mathbf{Z}, \\ 2x = \operatorname{arctg} 3 + \pi k,\ k \in \mathbf{Z}, \end{array}\right.$$

de unde $x = \frac{\pi}{8} + \frac{\pi}{2}n, \quad x = \frac{1}{2}\operatorname{arctg} 3 + \frac{\pi}{2}k,\ n, k \in \mathbf{Z}$.

c) Se rezolva similar exemplului precedent si se obtine $x = \frac{3\pi}{2} + 2\pi n, \quad x = 2\operatorname{arcctg} 2 + 2\pi k,\ n, k \in \mathbf{Z}$.

Ecuatia

$$a\cos^2 x + b\sin x + c = 0, \tag{16}$$

utilizand identitatea trigonometrica de baza $\sin^2 x + \cos^2 x = 1$, se reduce la rezolvarea unei ecuatii de tipul (12):

$$a(1 - \sin^2 x) + b\sin x + c = 0.$$

Similar, ecuatia

$$a\sin^2 x + b\cos x + c = 0 \tag{17}$$

se reduce la rezolvarea unei ecuatii de tipul (13):

$$a(1 - \cos^2 x) + b\cos x + c = 0.$$

Utilizand formulele

$$\cos 2x = 1 - 2\sin^2 x, \quad \cos 2x = 2\cos^2 x - 1$$

ecuatiile

$$a\cos 2x + b\sin x + c = 0, \tag{18}$$

$$a\cos 2x + b\cos x + c = 0, \tag{19}$$

se reduc la rezolvarea ecuatiilor de tipul (12) si respectiv (13).

Exemplul 7. Sa se rezolve ecuatiile:

$$\text{a) } 2\sin^2 x + 5\cos x - 5 = 0; \qquad \text{b) } \cos 4x + \sqrt{2}\sin 2x - 1 = 0.$$

Rezolvare. a) Cum $\sin^2 x = 1 - \cos^2 x$, ecuatia devine

$$2(1 - \cos^2 x) + 5\cos x - 5 = 0$$

sau

$$2\cos^2 x - 5\cos x + 3 = 0,$$

de unde $\cos x = \dfrac{3}{2}$ (aceasta ecuatie nu are solutii) sau $\cos x = 1$, cu solutiile $x = 2\pi k,\ k \in \mathbf{Z}$.

b) Cum $\cos 4x = 1 - 2\sin^2 2x$, ecuatia devine

$$-2\sin^2 2x + \sqrt{2}\sin 2x = 0,$$

sau

$$\sin 2x(\sqrt{2}\sin 2x - 1) = 0,$$

de unde

$$\left[\begin{array}{l} \sin 2x = 0, \\ \sin 2x = \dfrac{1}{\sqrt{2}}, \end{array}\right.$$

si $x = \dfrac{\pi}{2}k,\ x = (-1)^n\dfrac{\pi}{8} + \dfrac{\pi n}{2},\ k, n \in \mathbf{Z}$.

Ecuatia

$$a\,\mathrm{tg}\, x + b\,\mathrm{ctg}\, x + c = 0 \tag{20}$$

tinand seama ca $\mathrm{tg}\, x \cdot \mathrm{ctg}\, x = 1$ ($x \neq \dfrac{\pi}{2} \cdot k,\ k \in \mathbf{Z}$) prin intermediul substitutiei $t = \mathrm{tg}\, x$ (atunci $\mathrm{ctg}\, x = \dfrac{1}{t}$) se reduce la o ecuatie trigonometrica de tipul (14).

Exemplul 8. Sa se rezolve ecuatia:

$$\mathrm{tg}\, x - 5\,\mathrm{tg}\left(x - \frac{3\pi}{2}\right) = 6\sin\frac{7\pi}{2}.$$

Rezolvare. Cum $\sin\dfrac{7\pi}{2} = 1$ si $\mathrm{tg}\left(x - \dfrac{3\pi}{2}\right) = -\,\mathrm{tg}\left(\dfrac{3\pi}{2} - x\right) = -\,\mathrm{ctg}\, x$, ecuatia devine

$$\mathrm{tg}\, x + 5\,\mathrm{ctg}\, x - 6 = 0.$$

Se noteaza $\operatorname{tg} x = t$, atunci $\operatorname{ctg} x = \frac{1}{t}$ $(x \neq \frac{\pi}{2}k)$ si se obtine ecuatia patrata

$$t^2 - 6t + 5 = 0$$

cu solutiile $t_1 = 1$ si $t_2 = 5$. Asadar

$$\left[\begin{array}{l} \operatorname{tg} x = 1, \\ \operatorname{tg} x = 5, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \frac{\pi}{4} + \pi k, \ k \in \mathbf{Z}, \\ x = \operatorname{arctg} 5 + \pi n, \ n \in \mathbf{Z}. \end{array}\right.$$

Ecuatii omogene.

Ecuatia

$$a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + \ldots + a_{k-1} \sin x \cos^{n-1} x + a_n \cos^n x = 0, \quad (21)$$

unde $a_0 \cdot a_n \neq 0$, se numeste ecuatie omogena de gradul n in raport cu $\sin x$ si $\cos x$.

Cum $x = \frac{\pi}{2} + \pi k, \ k \in \mathbf{Z}$ nu verifica ecuatia (21) (toti termenii, incepand cu al doilea sunt nuli, iar primul este diferit de zero) multiplicand ecuatia cu $\frac{1}{\cos^n x}$ $(\neq 0)$ se obtine ecuatia echivalenta

$$a_0 \operatorname{tg}^n x + a_1 \operatorname{tg}^{n-1} x + \ldots + a_{n-1} \operatorname{tg} x + a_n = 0$$

care prin substitutia $\operatorname{tg} x = t$, se reduce la rezolvarea unei ecuatii algebrice de gradul n.

Exemplul 9. Sa se rezolve ecuatiile

a) $\sin 2x - \cos 2x = 0$;

b) $\sin^2 x + \sin 2x - 3\cos^2 x = 0$;

c) $5\sin^2 x + 5 \sin x \cos x = 3$;

d) $\cos 2x + \sin 2x = \sqrt{2}$.

Rezolvare. a) Ecuatia a) reprezinta o ecuatie trigonometrica omogena de gradul intai. Se multiplica cu $\frac{1}{\cos 2x}$ si se obtine ecuatia liniara in raport cu $\operatorname{tg} 2x$

$$\operatorname{tg} 2x - 1 = 0$$

de unde $\operatorname{tg} 2x = 1$ si $x = \frac{\pi}{8} + \frac{\pi}{2}n, \ n \in \mathbf{Z}$.

b) Cum $\sin 2x = 2 \sin x \cos x$ ecuatia b) se scrie $\sin^2 x + 2 \sin x \cos x - 3\cos^2 x = 0$ si reprezinta o ecuatie trigonometrica omogena de gradul al doilea. Se multiplica cu $\frac{1}{\cos^2 x}$ si se obtine ecuatia patrata

$$\operatorname{tg}^2 x + 2 \operatorname{tg} x - 3 = 0$$

cu solutiile $\operatorname{tg} x = -3$ si $\operatorname{tg} x = 1$. Prin urmare

$$\left[\begin{array}{l} x = -\operatorname{arctg} 3 + \pi n, \ n \in \mathbf{Z}, \\ x = \dfrac{\pi}{4} + \pi k, \ k \in \mathbf{Z}. \end{array}\right.$$

c) Se scrie $3 = 3 \cdot 1 = 3 \cdot (\sin^2 x + \cos^2 x)$ si ecuatia devine

$$5\sin^2 x + 5\sin x \cdot \cos x = 3\sin^2 x + 3\cos^2 x$$

sau

$$2\sin^2 x + 5\sin x \cdot \cos x - 3\cos^2 x = 0$$

adica o ecuatie trigonometrica omogena de gradul al doilea. Se rezolva similar exemplelor precedente si se obtin solutiile $x = -\operatorname{arctg} 3 + \pi k, \ k \in \mathbf{Z}$ si $x = \operatorname{arctg} \dfrac{1}{2} + \pi n, \ n \in \mathbf{Z}$.

d) Cum $\cos 2x = \cos^2 x - \sin^2 x$, $\sin 2x = 2\sin x \cos x$, $\sqrt{2} = \sqrt{2}(\sin^2 x + \cos^2 x)$, ecuatia devine

$$\cos^2 x - \sin^2 x + 2\sin x \cos x = \sqrt{2}\sin^2 x + \sqrt{2}\cos^2 x$$

sau

$$(\sqrt{2}+1)\sin^2 x - 2\sin x \cos x + (\sqrt{2}-1)\cos^2 x = 0,$$

adica este o ecuatie trigonometrica omogena de gradul al doilea. Se multiplica cu $\dfrac{1}{\cos^2 x}$ si se obtine ecuatia patrata

$$(\sqrt{2}+1)\operatorname{tg}^2 x - 2\operatorname{tg} x + \sqrt{2} - 1 = 0$$

cu solutia $\operatorname{tg} x = \dfrac{1}{\sqrt{2}+1}$ sau, rationalizand numitorul, $\operatorname{tg} x = \sqrt{2} - 1$.
Asadar, $x = \operatorname{arctg}(\sqrt{2}-1) + \pi n, \ n \in \mathbf{Z}$.

Metoda transformarii sumei functiilor trigonometrice in produs.

Ecuatiile de forma

$$\sin \alpha(x) \pm \sin \beta(x) = 0 \tag{22}$$

$$\cos \alpha(x) \pm \cos \beta(x) = 0 \tag{23}$$

cu ajutorul formulelor transformarii sumei in produs

$$\sin \alpha(x) \pm \sin \beta(x) = 2\sin\frac{\alpha(x) \pm \beta(x)}{2}\cos\frac{\alpha(x) \mp \beta(x)}{2} \tag{24}$$

$$\cos \alpha(x) + \cos \beta(x) = 2\cos\frac{\alpha(x) + \beta(x)}{2}\cos\frac{\alpha(x) - \beta(x)}{2} \tag{25}$$

$$\cos \alpha(x) - \cos \beta(x) = -2\sin\frac{\alpha(x) - \beta(x)}{2}\sin\frac{\alpha(x) + \beta(x)}{2} \tag{26}$$

se reduc la ecuatii trigonometrice simple.

Exemplul 10. Sa se rezolve ecuatiile

$$\text{a) } \sin 3x + \sin x = 0; \qquad \text{c) } \cos 5x = \sin 3x;$$
$$\text{b) } \cos x + \cos 3x = 0; \qquad \text{d) } \sin x + \cos 2x + \sin 3x + \cos 4x = 0.$$

Rezolvare. a) $\sin 3x + \sin x = 0 \Leftrightarrow 2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2} = 0 \Leftrightarrow \left[\begin{array}{l}\sin 2x = 0,\\ \cos x = 0,\end{array}\right. \Leftrightarrow$
$\Leftrightarrow \left[\begin{array}{l}x = \frac{\pi n}{2},\ n \in \mathbf{Z},\\ x = \frac{\pi}{2} + \pi k,\ k \in \mathbf{Z}\end{array}\right. \Leftrightarrow x = \frac{\pi n}{2},\ n \in \mathbf{Z}$ (se observa ca solutiile $x = \frac{\pi}{2} + \pi k,\ k \in \mathbf{Z}$ se contin in solutiile $x = \frac{\pi n}{2},\ n \in \mathbf{Z}$ - a se desena cercul trigonometric si a se depune pe el solutiile obtinute).

b) $\cos x + \cos 3x = 0 \Leftrightarrow 2\cos 2x \cos(-x) = 0$. Cum functia cosinus este o functie para, se obtine totalitatea

$$\left[\begin{array}{l}\cos 2x = 0,\\ \cos x = 0,\end{array}\right.$$

de unde $x = \frac{\pi}{4} + \frac{\pi}{2}k,\ k \in \mathbf{Z}$, $x = \frac{\pi}{2} + \pi n,\ n \in \mathbf{Z}$.

c) Cum $\cos 5x = \sin\left(\frac{\pi}{2} - 5x\right)$ (formulele de reducere) se obtine ecuatia

$$\sin\left(\frac{\pi}{2} - 5x\right) - \sin 3x = 0$$

sau

$$2\sin\left(\frac{\pi}{4} - 4x\right)\cos\left(\frac{\pi}{4} - x\right) = 0,$$

de unde, tinand seama ca functia sinus este impara, iar functia cosinus este para, se obtine totalitatea

$$\left[\begin{array}{l}\sin\left(4x - \frac{\pi}{4}\right) = 0,\\ \cos\left(x - \frac{\pi}{4}\right) = 0,\end{array}\right.$$

sau

$$\left[\begin{array}{l}4x - \frac{\pi}{4} = \pi k,\\ x - \frac{\pi}{4} = \frac{\pi}{2} + \pi n,\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x = \frac{\pi}{16} + \frac{\pi}{4}k,\ k \in \mathbf{Z},\\ x = \frac{3\pi}{4} + \pi n,\ n \in \mathbf{Z}.\end{array}\right.$$

d) Se grupeaza convenabil: $(\sin x + \sin 3x) + (\cos 2x + \cos 4x) = 0$, se aplica formulele (24) si (25) si se obtine ecuatia

$$2\sin 2x \cos x + 2\cos 3x \cos x = 0$$

sau

$$2\cos x(\sin 2x + \cos 3x) = 0,$$

de unde rezulta totalitatea de ecuatii

$$\left[\begin{array}{l} \cos x = 0, \\ \sin 2x + \cos 3x = 0. \end{array}\right.$$

Din prima ecuatie se obtine $x = \frac{\pi}{2} + \pi n,\ n \in \mathbf{Z}$. Ecuatia secunda a totalitatii se rezolva similar exemplului c) si se obtine $x = \frac{\pi}{2} + 2\pi m,\ m \in \mathbf{Z}$ (se contine in solutia deja obtinuta) si $x = \frac{3\pi}{10} + \frac{2\pi k}{5},\ k \in \mathbf{Z}$. Asadar solutiile ecuatiei initiale sunt $x = \frac{\pi}{2} + \pi n$, $x = \frac{3\pi}{10} + \frac{2\pi k}{5}$, $n, k \in \mathbf{Z}$.

Metoda transformarii produsului in suma
(utilizarea formulelor $\sin(\alpha \pm \beta),\ \cos(\alpha \pm \beta)$).

Exemplul 11. Sa se rezolve ecuatiile

$$a)\ \cos x \cos 2x - \sin x \sin 2x = 1; \qquad b)\ \cos x \cos 3x = \cos 4x.$$

Rezolvare. a) $\cos x \cos 2x - \sin x \sin 2x = 1 \ \Leftrightarrow\ \cos(x + 2x) = 1 \ \Leftrightarrow\ \cos 3x = 1 \ \Leftrightarrow$ $\Leftrightarrow\ 3x = 2\pi k,\ k \in \mathbf{Z} \ \Leftrightarrow\ x = \frac{2\pi}{3}k,\ k \in \mathbf{Z}$.

b) Cum $\cos x \cos 3x = \frac{1}{2}[\cos(x + 3x) + \cos(x - 3x)] = \frac{1}{2}(\cos 4x + \cos 2x)$ se obtine

$$\frac{1}{2}\cos 4x + \frac{1}{2}\cos 2x = \cos 4x,$$

sau $\cos 2x - \cos 4x = 0$, de unde rezulta

$$2\sin(-x)\sin 3x = 0.$$

Ultima ecuatie este echivalenta cu totalitatea

$$\left[\begin{array}{l} \sin x = 0, \\ \sin 3x = 0, \end{array}\right.$$

de unde $x = \frac{\pi k}{3},\ k \in \mathbf{Z}$ (solutiile primei ecuatii se contin in solutiile ecuatiei secunde).

Metoda micsorarii puterii

Aceasta metoda utilizeaza formulele

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \tag{27}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}, \tag{28}$$

$$\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x, \tag{29}$$

$$\sin^6 x + \cos^6 x = 1 - \frac{3}{4}\sin^2 2x, \tag{30}$$

$$\sin^8 x + \cos^8 x = \cos^2 2x + \frac{1}{8}\sin^4 2x, \tag{31}$$

in scopul micsorarii gradului ecuatiei ce urmeaza a fi rezolvate. Formulele (27) si (28) se utilizeaza si la rezolvarea ecuatiilor

$$\sin^2 ax + \sin^2 bx = \sin^2 cx + \sin^2 dx, \tag{32}$$

$$\cos^2 ax + \cos^2 bx = \cos^2 cx + \cos^2 dx, \tag{33}$$

daca numerele a, b, c si d verifica una din conditiile $a+b=c+d$ sau $a-b=c-d$.

Exemplul 12. Sa se rezolve ecuatiile

a) $\cos^2 x + \cos^2 2x + \cos^2 3x = \frac{3}{2}$;

b) $\sin^4 2x + \cos^4 2x = \sin 2x \cos 2x$;

c) $\cos^6 x + \sin^6 x = \cos 2x$.

Rezolvare. a) Se utilizeaza formula (27) si se obtine ecuatia echivalenta

$$\frac{1+\cos 2x}{2} + \frac{1+\cos 4x}{2} + \frac{1+\cos 6x}{2} = \frac{3}{2}$$

sau

$$\cos 2x + \cos 4x + \cos 6x = 0.$$

Se grupeaza convenabil si se obtine

$$(\cos 2x + \cos 6x) + \cos 4x = 0 \;\Leftrightarrow\; 2\cos 4x \cos 2x + \cos 4x = 0 \;\Leftrightarrow$$

$$\Leftrightarrow\; \cos 4x(2\cos 2x + 1) = 0 \;\Leftrightarrow\; \left[\begin{array}{l} \cos 4x = 0, \\ \cos 2x = -\frac{1}{2}, \end{array}\right. \;\Leftrightarrow\; \left[\begin{array}{ll} x = \frac{\pi}{8} + \frac{\pi}{4}n, & n \in \mathbf{Z}, \\ x = \pm\frac{\pi}{3} + \pi k, & k \in \mathbf{Z}. \end{array}\right.$$

b) Cum (a se vedea (29)) $\sin^4 2x + \cos^4 2x = 1 - \frac{1}{2}\sin^2 4x$, iar $\sin 2x \cos 2x = \frac{1}{2}\sin 4x$, ecuatia devine

$$1 - \frac{1}{2}\sin^2 4x = \frac{1}{2}\sin 4x$$

sau $\sin^4 2x + \sin 4x - 2 = 0$, de unde rezulta $\sin 4x = 1$ si $x = \frac{\pi}{8} + \frac{\pi}{2}n,\ n \in \mathbf{Z}$.

c) Cum $\cos^6 x + \sin^6 x = 1 - \frac{3}{4}\sin^2 2x = 1 - \frac{3}{4}(1-\cos^2 2x) = \frac{1}{4} + \frac{3}{4}\cos^2 2x$, ecuatia devine

$$\frac{1}{4} + \frac{3}{4}\cos^2 2x - \cos 2x = 0 \quad \text{sau} \quad 3\cos^2 2x - 4\cos 2x + 1 = 0,$$

de unde rezulta totalitatea

$$\left[\begin{array}{l} \cos 2x = 1, \\ \cos 2x = \frac{1}{3}, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \pi n,\ n \in \mathbf{Z}, \\ x = \pm\frac{1}{2}\arccos\frac{1}{3} + \pi k,\ k \in \mathbf{Z}. \end{array}\right.$$

Ecuatii de tipul

$$a\sin x + b\cos x = c, \quad a\cdot b\cdot c \neq 0. \tag{34}$$

Se propun urmatoarele metode de rezolvare a ecuatiilor de forma (34):

a) Reducerea la o ecuatie omogena de gradul al doilea in raport cu $\sin\frac{x}{2}$ si $\cos\frac{x}{2}$.

Se scrie

$$\begin{aligned} \sin x &= \sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}, \\ \cos x &= \cos 2\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}, \\ c &= c\cdot 1 = c\cdot\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) \end{aligned}$$

si ecuatia (34) devine

$$(b+c)\sin^2\frac{x}{2} - 2a\sin\frac{x}{2}\cos\frac{x}{2} + (c-b)\cos^2\frac{x}{2} = 0,$$

- omogena de gradul 2 daca $(c-b)(b+c) \neq 0$, sau, in caz contrar, se reduce la rezolvarea unei ecuatii omogene de gradul 1 si a unei ecuatii de tipul (2) sau (5).

Exemplul 13. Sa se rezolve ecuatiile

$$\text{a) } \sin 2x + \cos 2x = 1; \qquad \text{b) } \sin x + \cos x = \sqrt{2}.$$

Rezolvare. a) $\sin 2x + \cos 2x = 1 \Leftrightarrow 2\sin x\cos x + \cos^2 x - \sin^2 x = \sin^2 x + \cos^2 x \Leftrightarrow$ $2\sin x\cos x - 2\sin^2 x = 0 \Leftrightarrow 2\sin x(\cos x - \sin x) = 0 \Leftrightarrow \left[\begin{array}{l} \sin x = 0, \\ \cos x - \sin x = 0, \end{array}\right. \Leftrightarrow$

$$\Leftrightarrow \left[\begin{array}{l} \sin x = 0, \\ \operatorname{tg} x = 1, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \pi k,\ k \in \mathbf{Z}, \\ x = \frac{\pi}{4} + \pi n,\ n \in \mathbf{Z}. \end{array}\right.$$

b) $\sin x + \cos x = \sqrt{2} \;\Leftrightarrow\; 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \sqrt{2}\sin^2\frac{x}{2} + \sqrt{2}\cos^2\frac{x}{2} \;\Leftrightarrow$ $\Leftrightarrow\; (\sqrt{2}+1)\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} + (\sqrt{2}-1)\cos^2\frac{x}{2} = 0 \;\Leftrightarrow\; (\sqrt{2}+1)\operatorname{tg}^2 x - 2\operatorname{tg} x + \sqrt{2} - 1 = 0 \;\Leftrightarrow$ $\Leftrightarrow\; \operatorname{tg} x = \sqrt{2} - 1 \;\Leftrightarrow\; x = \operatorname{arctg}(\sqrt{2}-1) + \pi n,\; n \in \mathbf{Z}$.

b) Utilizarea formulelor

$$\sin\alpha = \frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}, \quad \cos\alpha = \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} \quad (\alpha \neq \pi + 2\pi k, \quad k \in \mathbf{Z})\,. \tag{35}$$

Cu ajutorul formulelor indicate, ecuatia (34) se reduce al o ecuatie patrata in raport cu tg $\frac{x}{2}$. Se tine seama ca aplicarea acestor formule aduce la pierderea solutiilor $\alpha = \pi + 2\pi k,\; k \in \mathbf{Z}$, din ce cauza se verifica (prin substituirea directa in ecuatia initiala), daca ele sunt sau ba solutii ale ecuatiei (34).

Exemplul 14. Sa se rezolve ecuatiile

a) $\sin 2x + \cos 2x = 1$; b) $\sqrt{3}\sin x + \cos x = -1$.

Rezolvare. a) Cum $\sin 2x = \dfrac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x},\; \cos 2x = \dfrac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x},\; \left(x \neq \dfrac{\pi}{2} + \pi n,\; n \in \mathbf{Z}\right)$ si cum $x = \dfrac{\pi}{2} + \pi n,\; n \in \mathbf{Z}$ nu verifica ecuatia data, ecuatia este echivalenta cu ecuatia

$$\frac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x} + \frac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = 1 \;\text{ sau }\; 1 + \operatorname{tg}^2 x = 2\operatorname{tg} x + 1 - \operatorname{tg}^2 x,$$

de unde rezulta

$$\left[\begin{array}{l} \operatorname{tg} x = 0, \\ \operatorname{tg} x = 1, \end{array}\right. \;\Leftrightarrow\; \left[\begin{array}{l} x = \pi k,\; k \in \mathbf{Z}, \\ x = \dfrac{\pi}{4} + \pi n,\; n \in \mathbf{Z}. \end{array}\right.$$

b) Se aplica formulele (35) si se obtine

$$\begin{cases} \dfrac{2\sqrt{3}\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} + \dfrac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = -1, \\ x \neq \pi + 2\pi k,\; k \in \mathbf{Z}, \end{cases}$$

sau

$$\begin{cases} 2\sqrt{3}\operatorname{tg}\dfrac{x}{2} + 1 - \operatorname{tg}^2\dfrac{x}{2} = 1 - \operatorname{tg}^2\dfrac{x}{2}, \\ x \neq \pi + 2\pi k,\; k \in \mathbf{Z}, \end{cases}$$

de unde $\begin{cases} \operatorname{tg}\dfrac{x}{2} = -\dfrac{1}{\sqrt{3}}, \\ x \neq \pi + 2\pi k, \end{cases}$ si $x = -\dfrac{\pi}{3} + 2\pi n,\; n \in \mathbf{Z}$. Verificarea directa arata ca si $x = \pi + 2\pi k,\; k \in \mathbf{Z}$ sunt solutii ale ecuatiei date. Asadar solutiile ecuatiei date sunt $x = -\dfrac{\pi}{3} + 2\pi k,\; x = \pi + 2\pi n,\; k, n \in \mathbf{Z}$.

c) Metoda unghiului auxiliar.

Cum $a \cdot b \cdot c \neq 0$ ecuatia (34) se scrie

$$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}} \tag{36}$$

si cum $\left|\frac{a}{\sqrt{a^2+b^2}}\right| \leq 1$, $\left|\frac{b}{\sqrt{a^2+b^2}}\right| \leq 1$ si $\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$ rezulta ca exista un unghi α, astfel incat

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}} \quad \text{si} \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}} \tag{37}$$

sau un unghi β, astfel incat

$$\sin\beta = \frac{a}{\sqrt{a^2+b^2}} \quad \text{si} \quad \cos\beta = \frac{b}{\sqrt{a^2+b^2}}. \tag{38}$$

Atunci ecuatia (36) se scrie

$$\sin(x+\alpha) = \frac{c}{\sqrt{a^2+b^2}},$$

sau

$$\cos(x-\beta) = \frac{c}{\sqrt{a^2+b^2}}.$$

Ultimile ecuatii nu prezinta greutati in rezolvare.

Nota. Se observa ca ecuatia (34) are solutii daca si numai daca $\left|\frac{c}{\sqrt{a^2+b^2}}\right| \leq 1$, iar valoarea maxima a functiei $f(x) = a\sin x + b\cos x$ este $\sqrt{a^2+b^2}$ si valoarea minima este $-\sqrt{a^2+b^2}$.

Exemplul 15. Sa se rezolve ecuatiile

a) $\sin 2x + \cos 2x = 1$; b) $3\sin x + 4\cos x = 5$; c) $\sin 2x + \cos 2x = \sqrt{3}$.

Rezolvare. a) $\sin 2x + \cos 2x = 1 \;\Leftrightarrow\; \frac{1}{\sqrt{2}}\sin 2x + \frac{1}{\sqrt{2}}\cos 2x = \frac{1}{\sqrt{2}} \;\Leftrightarrow$

$\Leftrightarrow\; \cos 2x \cos\frac{\pi}{4} + \sin 2x \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \;\Leftrightarrow\; \cos\left(2x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \;\Leftrightarrow\; 2x - \frac{\pi}{4} = \pm\frac{\pi}{4} + 2\pi k,\ k \in \mathbf{Z} \;\Leftrightarrow$

$$\Leftrightarrow\; 2x = \frac{\pi}{4} \pm \frac{\pi}{4} + 2\pi k,\ k \in \mathbf{Z} \;\Leftrightarrow\; \left[\begin{array}{l} x = \pi n,\ n \in \mathbf{Z}, \\ x = \dfrac{\pi}{4} + \pi k,\ k \in \mathbf{Z}. \end{array}\right.$$

b) $3\sin x + 4\cos x = 5 \;\Leftrightarrow\; \frac{3}{5}\sin x + \frac{4}{5}\cos x = 1 \;\Leftrightarrow\; \begin{cases} \sin x\cos\alpha + \cos x \sin\alpha = 1, \\ \sin\alpha = \dfrac{4}{5};\ \cos\alpha = \dfrac{3}{5}, \end{cases} \;\Leftrightarrow$

$$\Leftrightarrow\; \begin{cases} \sin(x+\alpha) = 1, \\ \operatorname{tg}\alpha = \dfrac{4}{3}, \end{cases} \;\Leftrightarrow\; x = \frac{\pi}{2} + 2\pi k - \operatorname{arctg}\frac{4}{3}, \quad k \in \mathbf{Z}.$$

c) Cum valoarea maxima a membrului din stanga ecuatiei este $\sqrt{1+1}=\sqrt{2}$ si $\sqrt{2}<\sqrt{3}$ rezulta ca ecuatia nu are solutii.

Ecuatii de tipul $F(\sin x \pm \cos x, \sin x \cos x) = 0$.

Ecuatiile de asa tip se rezolva cu ajutorul substitutiei $t = \sin x \pm \cos x,\ |t| \le \sqrt{2}$.

Exemplul 16. Sa se rezolve ecuatiile:

$$\text{a) } 2(\sin x + \cos x) + \sin 2x + 1 = 0;$$

$$\text{b) } 1 - \sin 2x = \cos x - \sin x;$$

$$\text{c) } \frac{1}{\cos x} + \frac{1}{\sin x} + \frac{1}{\sin x \cos x} = 5.$$

Rezolvare. a) Se noteaza $t = \sin x + \cos x$, atunci $t^2 = (\sin x + \cos x)^2 = 1 + \sin 2x$, si ecuatia devine $2t + t^2 = 0$, de unde $t = 0$ sau $t = -2$. Cum ecuatia $\sin x + \cos x = -2$ nu are solutii, ramane $\sin x + \cos x = 0$ - ecuatie omogena de gradul intai cu solutiile $x = -\frac{\pi}{4} + \pi n,\ n \in \mathbf{Z}$.

b) Se noteaza $\cos x - \sin x = t$, atunci $\sin 2x = 1 - t^2$ si ecuatia devine $t^2 = t$ cu solutiile $t = 0,\ t = 1$. Asadar

$$\left[\begin{array}{l}\cos x - \sin x = 0,\\ \cos x - \sin x = 1,\end{array}\right. \Leftrightarrow \left[\begin{array}{l}1 - \operatorname{tg} x = 0,\\ \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l}x = \frac{\pi}{4} + \pi k,\ k \in \mathbf{Z},\\ x = -\frac{\pi}{4} \pm \frac{\pi}{4} + 2\pi n,\ n \in \mathbf{Z}\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x = \frac{\pi}{4} + \pi k,\ k \in \mathbf{Z},\\ x = 2\pi n,\ n \in \mathbf{Z},\\ x = -\frac{\pi}{2} + 2\pi m,\ m \in \mathbf{Z}.\end{array}\right.$$

c) *DVA* al ecuatiei este $\mathbf{R} \setminus \left\{\frac{\pi}{2} \cdot n,\ n \in \mathbf{Z}\right\}$. In *DVA* ecuatia se scrie

$$\sin x + \cos x - 5 \sin x \cos x + 1 = 0.$$

Se noteaza $t = \sin x + \cos x$ si se obtine ecuatia patrata

$$5t^2 - 2t - 7 = 0,$$

cu solutiile $t = -1$ si $t = \frac{7}{5}$. Prin urmare $\sin x + \cos x = -1$, de unde $x = \frac{\pi}{4} \pm \frac{3\pi}{4} + 2\pi m,\ m \in \mathbf{Z}$ (nu verifica *DVA* al ecuatiei) $\sin x + \cos x = \frac{7}{5}$, de unde $x = \frac{\pi}{4} \pm \arccos \frac{7}{5\sqrt{2}} + 2\pi k,\ k \in \mathbf{Z}$.

Metoda descompunerii in factori

Aceasta metoda este una din cele mai frecvente si presupune o cunoastere satisfacatoare a *formulelor trigonometrice.*

Exemplul 17. Sa se rezolve ecuatiile

$$\text{a) } \sin^3 x - \cos^3 x = \cos 2x;$$
$$\text{b) } \sin 3x - \sin 2x + 2\cos x = 2\cos^2 x - \sin x;$$
$$\text{c) } 4\sin x + 2\cos x = 2 + 3\,\mathrm{tg}\, x.$$

Rezolvare. a) $\sin^3 x - \cos^3 x = \cos 2x \Leftrightarrow (\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x) = = \cos^2 x - \sin^2 x \Leftrightarrow (\sin x - \cos x)(1 + \sin x\cos x + (\cos x + \sin x)) = 0 \Leftrightarrow$

$$\Leftrightarrow \left[\begin{array}{l} \sin x - \cos x = 0, \\ 1 + \sin x\cos x + (\cos x + \sin x) = 0, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \mathrm{tg}\, x = 1, \\ \left\{\begin{array}{l} 1 + \dfrac{t^2-1}{2} + t = 0, \\ t = \sin x + \cos x, \end{array}\right. \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} x = \dfrac{\pi}{4} + \pi n,\ n \in \mathbf{Z}, \\ \left\{\begin{array}{l} t^2 + 2t + 1 = 0, \\ t = \sin x + \cos x, \end{array}\right. \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \dfrac{\pi}{4} + \pi k,\ k \in \mathbf{Z}, \\ \sin x + \cos x = -1, \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} x = \dfrac{\pi}{4} + \pi k,\ k \in \mathbf{Z}, \\ x = -\dfrac{\pi}{2} + 2\pi n,\ n \in \mathbf{Z}, \\ x = \pi + 2\pi m,\ m \in \mathbf{Z}. \end{array}\right.$$

b) Se trec toti termenii in stanga ecuatiei si se grupeaza convenabil:

$$(\sin 3x + \sin x) + 2\cos x - (\sin 2x + 2\cos^2 x) = 0.$$

Se utilizeaza formulele sumei sinusurilor si sinusului unghiului dublu si se obtine

$$(2\sin 2x\cos x + 2\cos x) - (2\sin x\cos x + 2\cos^2 x) = 0$$

sau

$$2\cos x \cdot [(\sin 2x + 1) - (\sin x + \cos x)] = 0.$$

Se tine seama ca $\sin 2x + 1 = 2\sin x\cos x + \sin^2 x + \cos^2 x = (\sin x + \cos x)^2$ si ecuatia devine

$$2\cos x[(\sin x + \cos x)^2 - (\sin x + \cos x)] = 0$$

sau

$$2\cos x(\sin x + \cos x)(\sin x + \cos x - 1) = 0,$$

de unde se obtine totalitatea

$$\left[\begin{array}{l}\cos x = 0,\\ \sin x + \cos x = 0,\\ \sin x + \cos x - 1 = 0.\end{array}\right.$$

Din prima ecuatie a totalitatii se obtine $x = \frac{\pi}{2} + \pi k,\ k \in \mathbf{Z}$. Cea secunda reprezinta o ecuatie trigonometrica omogena de gradul intai cu solutiile $x = -\frac{\pi}{4} + \pi m,\ m \in \mathbf{Z}$. Ecuatia a treia se rezolva, de exemplu, prin metoda introducerii unghiului auxiliar si are solutiile $x = 2\pi n,\ n \in \mathbf{Z}$ si $x = \frac{\pi}{2} + 2\pi l,\ l \in \mathbf{Z}$. Ultimul set de solutii se contine in multimea solutiilor primei ecuatii si prin urmare multimea solutiilor ecuatiei initiale este

$$x = \frac{\pi}{2} + \pi k, \quad x = \frac{\pi}{4} + \pi m, \quad x = 2\pi n, \quad k, m, n \in \mathbf{Z}.$$

c) *DVA* al ecuatiei este $\mathbf{R} \setminus \left\{\frac{\pi}{2} + \pi k,\ k \in \mathbf{Z}\right\}$ $(\cos x \neq 0)$. Ecuatia se scrie

$$4\sin x + 2\cos x = 2 + 3\frac{\sin x}{\cos x}$$

sau

$$4\sin x \cos x + 2\cos^2 x - 2\cos x - 3\sin x = 0.$$

Se grupeaza convenabil:

$$2\cos x(2\sin x - 1) + (2\cos^2 x - 3\sin x) = 0,$$

sau, cum $2\cos^2 x = 2(1 - \sin^2 x) = 2 - 2\sin^2 x$,

$$2\cos x(2\sin x - 1) + (2 - 3\sin x - 2\sin^2 x) = 0.$$

Cum $2 - 3\sin x - 2\sin^2 x = 2 - 4\sin x + \sin x - 2\sin^2 x = 2(1 - 2\sin x) + \sin x(1 - 2\sin x) =$
$= (1 - 2\sin x)(2 + \sin x)$, ecuatia devine

$$2\cos x(2\sin x - 1) + (1 - 2\sin x)(2 + \sin x) = 0,$$

sau

$$(2\sin x - 1)(2\cos x - \sin x - 2) = 0.$$

Cum $2\cos x - \sin x - 2 = 2(\cos x - 1) - \sin x = 2 \cdot (-2\sin^2 \frac{x}{2}) - 2\sin\frac{x}{2}\cos\frac{x}{2} =$
$= -2\sin\frac{x}{2}\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)$, ecuatia se scrie

$$-2(2\sin x - 1)\sin\frac{x}{2}\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right) = 0.$$

de unde rezulta

$$\sin x = \frac{1}{2}, \quad \text{cu solutiile} \quad x = (-1)^n \frac{\pi}{6} + \pi n, \; n \in \mathbf{Z},$$

$$\sin\frac{x}{2} = 0, \quad \text{cu solutiile} \quad x = 2\pi m, \; m \in \mathbf{Z},$$

$$2\sin\frac{x}{2} + \cos\frac{x}{2} = 0, \quad \text{cu solutiile} \quad x = -2\,\mathrm{arctg}\,\frac{1}{2} + 2\pi k, \; k \in \mathbf{Z}.$$

Toate solutiile obtinute verifica DVA al ecuatiei.

In incheiere vom prezenta unele metode utile de rezolvare a ecuatiilor trigonometrice.

Exemplul 18. Sa se rezolve ecuatiile:

a) $\cos x + \cos 2x + \cos 3x + \ldots + \cos nx = n, \quad n \in \mathbf{N}, \; n \geq 1;$

b) $\sin x + \sin 2x + \sin 3x + \ldots + \sin nx = n, \quad n \in \mathbf{N}, \; n \geq 2;$

c) $\sin^{11} x + \cos^{11} x = 1;$

d) $\sin^{10} x - \cos^7 x = 1;$

e) $\sin\frac{x}{2}\cos 2x = -1;$

f) $3\sin 2x + 4\cos 6x \cos 2x + 2\sin 10x = 7;$

g) $\sin 2x\left(\cos\frac{x}{2} - 2\sin 2x\right) + \cos 2x\left(1 + \sin\frac{x}{2} - 2\cos 2x\right) = 0;$

h) $4\sin^2 x - 4\sin^2 3x \sin x + \sin^2 3x = 0;$

i) $\sqrt{\frac{1}{16} + \cos^4 x - \frac{1}{2}\cos^2 x} + \sqrt{\frac{9}{16} + \cos^4 x - \frac{3}{2}\cos^2 x} = \frac{1}{2};$

j) $\cos x \cos 2x \cos 4x \cos 8x = \frac{1}{16}.$

Rezolvare. a) Cum pentru orice m natural $|\cos mx| \leq 1$, membrul din stanga ecuatiei va fi egal cu n daca si numai daca fiecare termen va fi egal cu unu. Asadar rezulta sistemul

$$\begin{cases} \cos x = 1, \\ \cos 2x = 1, \\ \ldots \\ \cos nx = 1 \end{cases}$$

cu solutiile $x = 2\pi k, \; k \in \mathbf{Z}$.

b) Se rezolva similar exemplului a) si se obtine sistemul

$$\begin{cases} \sin x = 1, \\ \sin 2x = 1, \\ \ldots \\ \sin nx = 1, \end{cases}$$

care este incompatibil. Intr-adevar, solutiile primei ecuatii: $x = \frac{\pi}{2} + 2\pi n,\ n \in \mathbf{Z}$ nu verifica a doua ecuatie a sistemului: $\sin 2\left(\frac{\pi}{2} + 2\pi n\right) = \sin(\pi + 4\pi n) = 0 \neq 1$. Prin urmare ecuatia nu are solutii.

c) Cum $\sin^{11} x \leq \sin^2 x,\ \cos^{11} x \leq \cos^2 x$ implica $\sin^{11} x + \cos^{11} x \leq \sin^2 x + \cos^2 x$, sau $\sin^{11} x + \cos^{11} x \leq 1$, iar in ultima inegalitate semnul egalitatii se atinge daca si numai daca

$$\left[\begin{array}{l} \begin{cases} \sin x = 0, \\ \cos x = 1, \end{cases} \\ \begin{cases} \sin x = 1, \\ \cos x = 0. \end{cases} \end{array}\right.$$

rezulta ca ecuatia are solutiile $x = 2\pi m,\ m \in \mathbf{Z}$ (din primul sistem al totalitatii) si $x = \frac{\pi}{2} + 2\pi n,\ n \in \mathbf{Z}$ (din sistemul secund).

d) Se utilizeaza acelasi procedeu ca si in exemplul precedent: $\sin^{10} x \leq \sin^2 x,\ -\cos^7 x \leq \cos^2 x$, de unde $\sin^{10} x - \cos^7 x \leq 1$ si, prin urmare, semnul egalitatii se atinge cand

$$\begin{cases} \sin^{10} x = \sin^2 x, \\ -\cos^7 x = \cos^2 x, \end{cases}$$

adica $\sin x \in \{0; -1; 1\}$, iar $\cos x \in \{0; -1\}$. Asadar se obtine $x = \frac{\pi}{2} + \pi n;\ x = \pi + 2\pi m$, $n, m \in \mathbf{Z}$.

e) Cum $\left|\sin \frac{x}{2}\right| \leq 1$, $|\cos 2x| \leq 1$, membrul din stanga ecuatiei va fi egal cu minus unu, daca si numai daca

$$\left[\begin{array}{l} \begin{cases} \sin \frac{x}{2} = 1, \\ \cos 2x = -1, \end{cases} \\ \begin{cases} \sin \frac{x}{2} = -1, \\ \cos 2x = 1. \end{cases} \end{array}\right.$$

Din $\sin \frac{x}{2} = 1$, rezulta $x = \pi + 4\pi n$ si atunci $\cos 2x = \cos(2\pi + 8\pi n) = 1 \neq -1$, adica primul sistem al totalitatii este incompatibil. Din $\sin \frac{x}{2} = -1$ rezulta $x = -\pi + 4\pi k$ si atunci $\cos 2(-\pi + 4\pi k) = \cos 2\pi = 1$, deci $x = -\pi + 4\pi k,\ k \in \mathbf{Z}$ sunt solutiile sistemului (si ecuatiei enuntate).

f) Cum $3\sin 2x + 4\cos 6x \cos 2x \leq 3\sin 2x + 4\cos 2x \leq 5$ (a se vedea nota la Metoda unghiului auxiliar), $2\sin 10x \leq 2$ se obtine $3\sin 2x + 4\cos 6x \cos 2x + 2\sin 10x \leq 7$, si semnul egalitatii se atinge doar pentru

$$\begin{cases} |\cos 6x| = 1, \\ \sin 10x = 1, \end{cases} \quad \text{sau} \quad \begin{cases} \sin 6x = 0, \\ \sin 10x = 1, \end{cases}$$

de unde

$$\begin{cases} x = \frac{\pi n}{6},\ n \in \mathbf{Z}, \\ x = \frac{\pi}{20} + \frac{\pi m}{10},\ m \in \mathbf{Z}. \end{cases}$$

Ultimul sistem este incompatibil. In adevar

$$\frac{\pi n}{6} = \frac{\pi}{20} + \frac{\pi m}{10}, \quad n, m \in \mathbf{Z}$$

conduce la ecuatia in numere intregi

$$10n = 3 + 6m \;\text{ sau }\; 10n - 6m = 3$$

care nu are solutii: diferenta a doua numere pare nu este un numar impar. Prin urmare ecuatia enuntata nu are solutii.

g) Ecuatia se scrie

$$\sin\left(2x + \frac{x}{2}\right) - 2(\sin^2 2x + \cos^2 2x) + \cos 2x = 0$$

sau

$$\sin\frac{5x}{2} + \cos 2x = 2.$$

Membrul din stanga nu intrece doi ($\sin\frac{5x}{2} \le 1,\ \cos 2x \le 1$), prin urmare ecuatia are solutii daca si numai daca

$$\begin{cases} \sin\frac{5x}{2} = 1, \\ \cos 2x = 1, \end{cases} \text{sau} \begin{cases} x = \frac{\pi}{5} + \frac{4\pi k}{5},\ k \in \mathbf{Z} \\ x = \pi n,\ n \in \mathbf{Z}. \end{cases}$$

Sistemul obtinut (si deci si ecuatia initiala) are solutii daca vor exista asa $n, k \in \mathbf{Z}$ astfel incat

$$\frac{\pi}{5} + \frac{4\pi k}{5} = \pi n,$$

sau

$$1 + 4k = 5n$$

de unde $4k = 5n - 1$ sau $4k = 4n + (n - 1)$. Asadar, $n - 1$ urmeaza a fi divizibil prin 4, adica

$$n - 1 = 4s,\ s \in \mathbf{Z}$$

de unde $n = 4s + 1$ si cum $1 + 4k = 5n$, adica $4k = 5(4s + 1) - 1$ se obtine $k = 5s + 1$, si

$$x = \pi + 4\pi s,\ s \in \mathbf{Z}.$$

h) Membrul din stanga ecuatiei se considera trinom patrat in raport cu $\sin x$. Discriminantul acestui trinom este

$$D = 16\sin^4 3x - 16\sin^2 3x,$$

de unde rezulta ca ecuatia enuntata va avea solutii doar pentru $\sin^2 3x \le 0$ sau $\sin^2 3x \ge 1$. Prin urmare (cum $\sin^2\alpha \ge 0$ si $\sin^2\beta \le 1$) ecuatia poate avea solutii doar daca $\sin^2 3x = 0$ sau $\sin^2 3x = 1$ adica $x = \frac{\pi n}{3}$ respectiv $x = \frac{\pi}{6} + \frac{\pi}{3}m,\ n, m \in \mathbf{Z}$.

Se substituie in ecuatie si se obtine

1. $4\sin^2\frac{\pi}{3}\cdot n - 4\sin^2\pi n\cdot\sin\frac{\pi}{3}n + \sin^2\pi n = 0$. Cum $\sin^2\pi n = 0$, ramane $4\sin^2\frac{\pi}{3}n = 0$, de unde $n = 3m,\ m\in\mathbf{Z}$, adica din primul set se obtine solutiile $x = \pi m,\ m\in\mathbf{Z}$.

2. $4\sin^2\left(\frac{\pi}{6}+\frac{\pi}{3}m\right) - 4\sin^2\left(\frac{\pi}{2}+\pi m\right)\sin\left(\frac{\pi}{6}+\frac{\pi}{3}m\right) + \sin^2\left(\frac{\pi}{2}+\pi n\right) = 0.$
   Cum $\sin^2\left(\frac{\pi}{2}+\pi m\right) = \cos^2\pi m = 1$, se obtine

$$4\sin^2\left(\frac{\pi}{6}+\frac{\pi}{3}m\right) - 4\sin\left(\frac{\pi}{6}+\frac{\pi}{3}m\right) + 1 = 0$$

   adica

$$\left(2\sin\left(\frac{\pi}{6}+\frac{\pi}{3}m\right) - 1\right)^2 = 0,$$

   de unde rezulta $x = \frac{\pi}{6}+\pi k$ sau $x = \frac{5\pi}{6}+\pi k,\ k\in\mathbf{Z}$ adica $x = (-1)^k\frac{\pi}{6}+\pi k,\ k\in\mathbf{Z}$.

Asadar solutiile ecuatiei date sunt

$$x = \pi n,\ n\in\mathbf{Z},\quad x = (-1)^n\frac{\pi}{6}+\pi k,\ k\in\mathbf{Z}.$$

i) Se noteaza $\cos^2 x = t$ si ecuatia devine

$$\frac{\sqrt{16t^2-8t+1}}{4} + \frac{\sqrt{16t^2-24t+9}}{4} = \frac{1}{2}$$

sau

$$\sqrt{(4t-1)^2} + \sqrt{(4t-3)^3} = 2,$$

de unde

$$|4t-1| + |4t-3| = 2.$$

Se tine seama ca $|4t-3| = |3-4t|$ si $2 = |2| = |4t-1+3-4t|$ si utilizand proprietatile modulului se obtine inecuatia

$$(4t-1)(3-4t)\geq 0,$$

de unde

$$\frac{1}{4}\leq t\leq\frac{3}{4},$$

adica $\frac{1}{4}\leq\cos^2 x\leq\frac{3}{4}$ sau $\frac{1}{2}\leq|\cos x|\leq\frac{\sqrt{3}}{2}$. Din ultima inecuatie se obtine (a se vedea tema Inecuatii trigonometrice) solutiile ecuatiei enuntate

$$x\in\left\{\frac{\pi}{6}+\pi k;\ \frac{\pi}{3}+\pi k\right\}\cup\left\{\pi k-\frac{\pi}{3};\ \pi k-\frac{\pi}{6}\right\},\ k\in\mathbf{Z}.$$

j) Cum $x = \pi k,\ k\in\mathbf{Z}$ nu sunt solutii ale ecuatiei date ($\cos\pi k = \pm 1,\ \cos 2\pi k = \cos 4\pi k = \cos 8\pi k = 1$) se multiplica ambii membri ai ecuatiei cu $16\sin x$ si se utilizeaza formula sinusului unghiului dublu

$$16\sin x\cos x\cos 2x\cos 4x\cos 8x = \sin x,$$

$$8\sin 2x\cos 2x\cos 4x\cos 8x = \sin x,$$
$$4\sin 4x\cos 4x\cos 8x = \sin x,$$
$$2\sin 8x\cos 8x = \sin x,$$
$$\sin 16x = \sin x,$$
sau $\sin 16x - \sin x = 0$, $2\sin\frac{15x}{2}\cos\frac{17x}{2} = 0$, de unde $\sin\frac{15x}{2} = 0$, $x = \frac{2\pi k}{15}$, $k \in \mathbf{Z}$, $k \neq 15s$, $s \in \mathbf{Z}$ (deoarece $x \neq \pi m$) si $\cos\frac{17x}{2} = 0$, $x = \frac{\pi}{17} + \frac{2\pi m}{17}$, $m \in \mathbf{Z}$, $m \neq 17s+8$, $s \in \mathbf{Z}$.

Exercitii pentru autoevaluare

Sa se rezolve ecuatiile

1. $2\sin^2 x - 1 = \cos x$;
2. $7\operatorname{tg} x - 4\operatorname{ctg} x = 12$;
3. $\operatorname{tg}^2 x - 3\operatorname{tg} x + 2 = 0$;
4. $6\cos^2 x + 5\cos x + 1 = 0$;
5. $\sin^2 x - \cos^2 x = \cos x$;
6. $3\cos^2 x + 4\sin x\cos x + 5\sin^2 x = 2$;
7. $3\cos^2 x - \sin^2 x - 2\sin x\cos x = 0$;
8. $\cos 2x\cos x = \sin 7x\sin 6x + 8\cos\frac{3\pi}{2}$;
9. $\cos 3x\cos 6x = \cos 5x\cos 8x$;
10. $\sin^2 x + \sin^2 2x = \sin^2 3x + \sin^2 4x$;
11. $\frac{1}{2}(\sin^4 x + \cos^4 x) = \sin^2 x\cos^2 x + \sin x\cos x - \frac{1}{2}$;
12. $\cos 3x = \cos x$;
13. $\sin 2x = \sin x$;
14. $\sin 5x = \cos 13x$;
15. $\cos^2 x + 3|\cos x| - 4 = 0$;
16. $8\sin^2 x\cos^2 x + 4\sin 2x - 1 = (\sin x + \cos x)^2$;
17. $\sin 3x + \sin x = \sqrt{2}\cos x$;
18. $8\cos^4 x = 3 + 5\cos^4 x$;

19. $2\sqrt{3}\sin 2x(3+\cos 4x) = 7\sin 4x$;

20. $2\sin 4x - 3\sin^2 2x = 1$;

21. $\cos^2 x + \cos^2 \frac{3x}{4} + \cos^2 \frac{x}{2} + \cos^2 \frac{x}{4} = 2$;

22. $6\cos^2 x + \cos 3x = \cos x$;

23. $\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$;

24. $\operatorname{tg} 2x = 4\cos^2 x - \operatorname{ctg} x$;

25. $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$.