Transformari identice ale expresiilor algebrice.

Definitie. Vom numi **expresie algebrica** expresia, ce se obtine din constante si variabile prin intermediul operatiilor de adunare, scadere, inmultire, impartire, ridicare la o putere intreaga si extragerea radacinii.

Exemple de expresii algebrice:

$$E = \sqrt[3]{2+\sqrt{5}}; \qquad E(x,y) = \left(\frac{x+\sqrt[3]{2xy^2}-1}{3\sqrt{z-x}-\sqrt{y}}\right)^{\frac{5}{4}};$$

$$E(x,y,z) = \frac{x+y}{xy} - z; \qquad E(x,y) = (x+y)^3 - 3xy(x+y).$$

Definitie. Domeniu al valorilor admisibile (concis DVA) al expresiei algebrice $E(x_1, x_2, \ldots, x_n)$ $(D(E))$ se numeste multimea tuturor cortegiilor $(x_1, x_2, \ldots, x_n)$ pentru care expresia $E(x_1, x_2, \ldots, x_n)$ are sens.

De exemplu, DVA al expresiei $E(x,y) = \dfrac{x+y}{xy} - z$ este multimea $D(E) = \{(x,y) \mid x \in \mathbf{R},\ y \in \mathbf{R},\ xy \neq 0\}$, iar DVA al expresiei $E(x,y) = \sqrt{xy} - 2z$, este multimea tripletelor $\{(x,y,z) \mid x,y,z \in \mathbf{R},\ xy \geq 0\}$.

Definitie. Expresiile algebrice E_1 si E_2 se numesc identic egale pe multimea $M \subset D(E_1) \cap D(E_2)$, daca valorile numerice ale acestor expresii sunt egale pentru orice valori ale variabilelor din M.

De exemplu $\sqrt{x^2} = x$ pe multimea $[0; +\infty)$, $\sqrt{a^2} = -a$, pe multimea $(-\infty; 0]$, $\dfrac{x^2-1}{x+1} = x-1$ pe multimea $\mathbf{R} \setminus \{-1\}$, $(x+y)^2 = x^2 + 2xy + y^2$ pe multimea $\{(x,y) \mid x \in \mathbf{R},\ y \in \mathbf{R}\}$.

Definitie. Vom numi transformare identica a expresiei algebrice pe multimea $M \subseteq D(E)$ inlocuirea acestei expresii cu o expresie identic egala cu ea.

Nota. Tinem sa mentionam, ca uneori multimea M pe care expresiile algebrice sunt identic egale nu se evidentiaza, avandu-se in vedere egalitatea identica a acestor expresii pe intersectia domeniilor lor de definitie.

De exemplu

$$\sqrt{a^2} = |a|,\ (M = \mathbf{R}), \qquad \frac{a(a+1)}{a} = \frac{a^2-1}{a-1},\ (M = \mathbf{R} \setminus \{0; 1\}).$$

In procesul transformarilor identice sunt utile urmatoarele formule:

I. Formulele inmultirii prescurtate

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$,

2. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$,

3. $a^2 - b^2 = (a - b)(a + b)$,

4. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

Aceste formule se obtin ca consecinte ale urmatoarelor formule generale:

5. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}) \quad (n \in \mathbf{N})$,

6. $a^{2n+1} + b^{2n+1} = (a + b)(a^{2n} - a^{2n-1}b + \ldots - ab^{2n-1} + b^{2n}) \quad (n \in \mathbf{N})$,

7. (binomul lui Newton)

$$(a+b)^n = C_n^0 a^0 b^n + C_n^1 a^1 b^{n-1} + \ldots + C_n^k a^k b^{n-k} + \ldots + C_n^n a^n b^0,$$

$$\text{unde } n \in \mathbf{N}, \quad C_n^k = \frac{n!}{k!(n-k)!}, \quad n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n, \quad 0! = 1.$$

II. Proprietatile puterilor

Egalitatile ce urmeaza sunt juste pentru orice numere pozitive a si b si orice numere reale α si β.

1. $a^0 = 1$;

2. $a^{\alpha+\beta} = a^\alpha \cdot a^\beta$;

3. $a^{\alpha-\beta} = \dfrac{a^\alpha}{a^\beta}$;

4. $(a^\alpha)^\beta = a^{\alpha\beta}$;

5. $(ab)^\alpha = a^\alpha \cdot b^\alpha$;

6. $\left(\dfrac{a}{b}\right)^\alpha = \dfrac{a^\alpha}{b^\alpha}$;

7. $a^{-\alpha} = \dfrac{1}{a^{\alpha}}$.

Nota 1. Tinem sa mentionam, ca numerele negative la fel pot fi ridicate la anumite puteri (intregi, sau mai general, rationale de tipul $\dfrac{m}{2n-1}$), unde m - intreg,

Nota 2. $0^{\alpha} = 0$, pentru orice $\alpha > 0$.

III. Proprietatile radicalilor

1. $\sqrt[n]{a^n} = \begin{cases} a, & \text{daca } n - \text{ impar,} \\ |a|, & \text{daca } n - \text{ par,} \end{cases}$

2. $\sqrt[2k]{ab} = \sqrt[2k]{a} \cdot \sqrt[2k]{b}$, daca $a \geq 0,\ b \geq 0,\ k \in \mathbf{N}$,

3. $\sqrt[2k]{ab} = \sqrt[2k]{|a|}\ \sqrt[2k]{|b|}$, daca $ab \geq 0,\ k \in \mathbf{N}$.

4. $\sqrt[2k+1]{ab} = \sqrt[2k+1]{a}\ \sqrt[2k+1]{b}, \quad k \in \mathbf{N}$.

5. $\left(\sqrt[m]{a}\right)^k = \sqrt[m]{a^k}, \quad a \geq 0$ daca m este par, $a \in \mathbf{R}$ daca m este impar.

6. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, daca $a \geq 0,\ b > 0,\ n$ - par sau $b \neq 0,\ a \in \mathbf{R}$, daca n - impar.

7. $\sqrt[n]{a^k} = a^{\frac{k}{n}}, \quad a \geq 0$.

8. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[m \cdot n]{a},\ a \geq 0$ daca m par sau n par, $a \in \mathbf{R}$ daca $m \cdot n$ - impar.

9. $\sqrt[2k+1]{-a} = -\sqrt[2k+1]{a}, \quad a \in \mathbf{R}$.

10. $\sqrt{a \pm b\sqrt{c}} = \sqrt{\dfrac{a + \sqrt{a^2 - cb^2}}{2}} \pm \sqrt{\dfrac{a - \sqrt{a^2 - cb^2}}{2}}$, unde $a > 0,\ b > 0,\ c > 0$ si $a^2 \geq b^2 c$.

Exemplul 1. Sa se determine *DVA* al expresiilor algebrice:

$$a)\ E(x) = \sqrt[6]{x + x^2 - 2x^3};$$

$$b)\ E(x,y) = \frac{x^2 + xy}{x^2 + y^2}\left(\frac{x}{|x-y|} + \frac{y}{x+y}\right);$$

$$c)\ E(a,b,c,d) = \frac{a}{b+c} + \frac{\sqrt{d}}{b^2 c + c^2 b}.$$

Rezolvare. a) *DVA* al expresiei enuntate se determina rezolvand inecuatia $x+x^2-2x^3 \geq 0$ cu ajutorul metodei intervalelor:

$$x+x^2-2x^3 \geq 0 \;\Leftrightarrow\; x(1+x-2x^2) \geq 0 \;\Leftrightarrow\; x(2x+1)(1-x) \geq 0 \;\Leftrightarrow\; x \in (-\infty; -\frac{1}{2}] \cup [0;1].$$

Asadar $D(E) = (-\infty; -\frac{1}{2}] \cup [0;1]$.

b) Se tine seama ca expresia are sens, daca

$$\begin{cases} x^2+y^2 \neq 0, \\ |x-y| \neq 0, \\ x+y \neq 0, \end{cases}$$

de unde rezulta $D(E) = \{(x,y) \mid x \neq y,\ x \neq -y\}$.

c) Cum numitorii fractiilor rationale urmeaza a fi diferiti de zero, iar radacina de ordinul doi exista numai din expresii nenegative, se obtine sistemul

$$\begin{cases} b+c \neq 0, \\ b^2c+c^2b \neq 0, \\ d \geq 0, \end{cases} \Leftrightarrow \begin{cases} b+c \neq 0, \\ bc(b+c) \neq 0, \\ d \geq 0, \end{cases} \Leftrightarrow \begin{cases} b+c \neq 0, \\ b \neq 0, \\ c \neq 0, \\ d \geq 0. \end{cases}$$

Asadar *DVA* al expresiei date este multimea $\{(a,b,c,d) \mid b+c \neq 0,\ b \neq 0,\ c \neq 0,\ d \geq 0\}$.

Exemplul 2. Sa se determine daca expresiile A si B sunt identic egale pe multimea M.

$$a)\ A = \left(\frac{a\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}} - \sqrt{ab}\right)^{\frac{1}{2}} \frac{\sqrt{a}+\sqrt{b}}{ab(a-b)}, \quad B = \frac{1}{ab}, \quad M = \{(a,b) \mid a > b > 0\};$$

$$b) A = \frac{a-\sqrt{3}}{\sqrt{\left(\frac{a^2+3}{2a}\right)^2 - 3}}, \quad B = \frac{2a}{a+\sqrt{3}}, \quad M = \{a \mid a > \sqrt{3}\}.$$

Rezolvare. a) Cum $a\sqrt{a} = (\sqrt{a})^3$, $b\sqrt{b} = (\sqrt{b})^3$, $a = (\sqrt{a})^2$, $b = (\sqrt{b})^2$ pe mulitmea M, se aplica formulele inmultirii prescurtate si se obtine:

$$A = \left(\frac{(\sqrt{a})^3 + (\sqrt{b})^3}{\sqrt{a} + \sqrt{b}} - \sqrt{ab} \right)^{\frac{1}{2}} \cdot \frac{\sqrt{a} + \sqrt{b}}{ab((\sqrt{a})^2 - (\sqrt{b})^2)} =$$

$$= \left(\frac{(\sqrt{a} + \sqrt{b})((\sqrt{a})^2 - \sqrt{ab} + (\sqrt{b})^2)}{\sqrt{a} + \sqrt{b}} - \sqrt{ab} \right)^{\frac{1}{2}} \cdot \frac{(\sqrt{a} + \sqrt{b})}{ab(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})} =$$

$$= \left[(\sqrt{a})^2 - 2\sqrt{ab} + (\sqrt{b})^2 \right]^{\frac{1}{2}} \cdot \frac{1}{ab(\sqrt{a} - \sqrt{b})} = \sqrt{(\sqrt{a} - \sqrt{b})^2} \frac{1}{ab(\sqrt{a} - \sqrt{b})} =$$

$$= |\sqrt{a} - \sqrt{b}| \cdot \frac{1}{ab(\sqrt{a} - \sqrt{b})},$$

deoarece $a > b > 0$ implica $\sqrt{a} > \sqrt{b}$, si prin urmare $|\sqrt{a} - \sqrt{b}| = \sqrt{a} - \sqrt{b}$. Asadar $A = (\sqrt{a} - \sqrt{b}) \frac{1}{ab(\sqrt{a} - \sqrt{b})} = \frac{1}{ab} = B$. Astfel pe multimea M expresiile A si B sunt egale.

b) Similar exemplului precedent

$$A = \frac{a - \sqrt{3}}{\sqrt{\left(\frac{a^2 + 3}{2a} \right)^2 - 3}} = \frac{a - \sqrt{3}}{\sqrt{\frac{(a^2 + 3)^2 - 4a^2 \cdot 3}{4a^2}}} = \frac{(a - \sqrt{3}) \cdot 2|a|}{\sqrt{(a^2 - 3)^2}} = \frac{(a - \sqrt{3}) \cdot 2|a|}{|a^2 - 3|} = \frac{2a}{a + \sqrt{3}} = B.$$

S-a tinut seama ca daca $a > \sqrt{3}$, atunci $\sqrt{a^2} = a$ si $\sqrt{(a^2 - 3)^2} = |a^2 - 3| = a^2 - 3$.

Exemplul 3. Sa se aduca la o forma mai simpla expresiile:

$$a)\ \frac{\sqrt{2b+2\sqrt{b^2-4}}}{\sqrt{b^2-4}+b+2};$$

$$b)\ \left(\frac{\sqrt[3]{mn^2}+\sqrt[3]{m^2n}}{\sqrt[3]{m^2}+2\sqrt[3]{mn}+\sqrt[3]{n^2}}-2\sqrt[3]{n}+\frac{m-n}{\sqrt[3]{m^2}-\sqrt[3]{n^2}}\right)\frac{1}{\sqrt[6]{m}+\sqrt[6]{n}};$$

$$c)\ \frac{\left(\frac{\sqrt[4]{bc^3}+\sqrt[4]{a^2bc}}{\sqrt{a}+\sqrt{c}}+\sqrt[4]{bc}\right)^2+bc+3}{\sqrt{bc}+3};$$

$$d)\ \frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)};$$

$$e)\ \frac{a^2b^2}{(a-c)(b-c)}+\frac{a^2c^2}{(a-b)(c-b)}+\frac{b^2c^2}{(b-a)(c-a)};$$

$$f)\ \frac{y-z}{(x-y)(x-z)}+\frac{z-x}{(y-x)(y-z)}+\frac{x-y}{(z-x)(z-y)};$$

$$g)\ \frac{m|m-3|}{(m^2-m-6)|m|}.$$

Rezolvare. *DVA* al expresiei se determina din sistemul $\begin{cases} b^2-4\geq 0, \\ 2b+2\sqrt{b^2-4}\geq 0, \\ \sqrt{b^2-4}+b+2>0, \end{cases}$ de unde rezulta $b \geq 2$.

In *DVA* expresia este identic egala cu:

$$\frac{\sqrt{2b+2\sqrt{b^2-4}}}{\sqrt{b^2-4}+b+2}=\frac{\sqrt{(\sqrt{b-2})^2+(\sqrt{b+2})^2+2\sqrt{b-2}\sqrt{b+2}}}{\sqrt{b-2}\sqrt{b+2}+(\sqrt{b+2})^2}=$$

$$=\frac{\sqrt{(\sqrt{b-2}+\sqrt{b+2})^2}}{\sqrt{b+2}(\sqrt{b-2}+\sqrt{b+2})}=\frac{|\sqrt{b-2}+\sqrt{b+2}|}{\sqrt{b+2}(\sqrt{b-2}+\sqrt{b+2})}=\frac{1}{\sqrt{b+2}},$$

deoarece in *DVA* $\sqrt{b-2}+\sqrt{b+2}\geq 2$ si prin urmare $|\sqrt{b-2}+\sqrt{b+2}|=\sqrt{b-2}+\sqrt{b+2}$. Asadar, pentru $b \geq 2$ expresia enuntata este egala cu $\frac{1}{\sqrt{b+2}}$.

b) *DVA* al expresiei este multimea $\{(m,n)\mid m\geq 0,\ n\geq 0,\ m\neq n\}$. Se noteaza $\sqrt[6]{m}=a$,

$\sqrt[6]{n} = b$, atunci $\sqrt[3]{m} = a^2$, $\sqrt[3]{m^2} = a^4$, $m = a^6$ si $\sqrt[3]{n} = b^2$, $\sqrt[3]{n^2} = b^4$, $n = b^6$ si expresia devine

$$\left(\frac{a^2b^4 + a^4b^2}{a^4 + 2a^2b^2 + b^4} - 2b^2 + \frac{a^6 - b^6}{a^4 - b^4}\right)\frac{1}{a+b} =$$

$$= \left(\frac{a^2b^2(a^2+b^2)}{(a^2+b^2)^2} - 2b^2 + \frac{(a^2-b^2)(a^4+a^2b^2+b^4)}{(a^2-b^2)(a^2+b^2)}\right)\frac{1}{(a+b)} =$$

$$= \frac{a^2b^2 - 2b^2(a^2+b^2) + a^4 + a^2b^2 + b^4}{a^2+b^2} \cdot \frac{1}{a+b} =$$

$$= \frac{a^4 - b^4}{a^2+b^2} \cdot \frac{1}{a+b} = \frac{(a^2+b^2)(a-b)(a+b)}{(a^2+b^2)(a+b)} = a - b.$$

Asadar expresia initiala pe DVA este identic egala cu $\sqrt[6]{m} - \sqrt[6]{n}$.

c) In $DVA:\ \{(a,b,c) \mid a \geq 0,\ b \geq 0,\ c \geq 0,\ a^2 + c^2 \neq 0\}$ expresia se tranforma astfel:

$$\frac{\left(\dfrac{\sqrt[4]{bc^3} + \sqrt[4]{a^2bc}}{\sqrt{a} + \sqrt{c}} + \sqrt[4]{bc}\right)^2 + bc + 3}{\sqrt{bc} + 3} = \frac{\left(\dfrac{\sqrt[4]{bc}(\sqrt{c} + \sqrt{a})}{\sqrt{a} + \sqrt{c}} + \sqrt[4]{bc}\right)^2 + bc + 3}{\sqrt{bc} + 3} =$$

$$= \frac{4\sqrt{bc} + bc + 3}{\sqrt{bc} + 3} = \frac{3\sqrt{bc} + 3 + \sqrt{bc} + bc}{\sqrt{bc} + 3} = \frac{3(\sqrt{bc} + 1) + (1 + \sqrt{bc})\sqrt{bc}}{\sqrt{bc} + 3} =$$

$$= \frac{(\sqrt{bc} + 1)(\sqrt{bc} + 3)}{\sqrt{bc} + 3} = \sqrt{bc} + 1.$$

d) DVA al expresiei este multimea $\{(a,b,c) \mid a \neq b,\ a \neq c,\ b \neq c\}$. Se aduce expresia la numitor comun si se obtine

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} = \frac{a^3(c-b) + b^3(a-c) + c^3(b-a)}{(a-b)(a-c)(c-b)}.$$

Se tine seama de expresia de la numitor si se descompune in factori numaratorul

$$a^3(c-b) + b^3(a-c) + c^3(b-a) = c(a^3 - b^3) + ab(b^2 - a^2) + c^3(b-a) =$$

$$= (a-b)(c(a^2 + ab + b^2) - ab(a+b) - c^3) = (a-b)(c(a^2 - c^2) + ab(c-a) + b^2(c-a)) =$$

$$= (b-c)(a-b)(-a^2b - a^2c + c^2(a+b)) = (a-b)(b-c)(b(c^2 - a^2) + ac(c-a)) =$$

$$= (a-b)(b-c)(c-a)(ab + bc + ca).$$

Asadar pe DVA expresia enuntata este identic egala cu $ab + bc + ca$.

f) DVA al expresiei este multimea $\{(x,y,z) \mid x\neq y,\ y\neq z,\ z\neq x\}$. Primul termen al expresiei se descompune in modul urmator:

$$\frac{y-z}{(x-y)(x-z)}=\frac{x-z-x+y}{(x-y)(x-z)}=\frac{x-z}{(x-y)(x-z)}-\frac{x-y}{(x-y)(x-z)}=\frac{1}{x-y}-\frac{1}{x-z}.$$

Similar se descompun si ceilalti termeni

$$\frac{z-x}{(y-z)(y-x)}=\frac{1}{y-z}-\frac{1}{y-x};$$
$$\frac{x-y}{(z-x)(z-y)}=\frac{1}{z-x}-\frac{1}{z-y}.$$

Prin urmare

$$\frac{y-z}{(x-y)(x-z)}+\frac{z-x}{(y-z)(y-x)}+\frac{x-y}{(z-x)(z-y)}=$$
$$=\frac{1}{x-y}-\frac{1}{x-z}+\frac{1}{y-z}-\frac{1}{y-x}+\frac{1}{z-x}-\frac{1}{z-y}=$$
$$=\frac{2}{x-y}+\frac{2}{y-z}+\frac{2}{z-x}.$$

g) DVA al expresiei este $\mathbf{R}\setminus\{-2;0;3\}$. Se tine seama ca expresia contine $|m|$ si $|m-3|$ si se considera urmatoarele trei cazuri:

1. fie $m\in(-\infty;-2)\cup(-2;0)$. Atunci $|m|=-m$, $|m-3|=-(m-3)$ si expresia devine

$$\frac{m|m-3|}{(m^2-m-6)|m|}=\frac{-m(m-3)}{-(m+2)(m-3)m}=\frac{1}{m+2};$$

2. fie $m\in(0;3)$. Atunci $|m|=m$, $|m-3|=-(m-3)$ si expresia devine

$$\frac{m|m-3|}{(m^2-m-6)|m|}=\frac{-m(m-3)}{(m+2)(m-3)m}=-\frac{1}{m+2};$$

3. fie $m\in(3;+\infty)$. Atunci $|m|=m$, $|m-3|=m-3$ si expresia devine

$$\frac{m|m-3|}{(m^2-m-6)|m|}=\frac{1}{m+2}.$$

Asadar

$$\frac{m|m-3|}{(m^2-m-6)|m|}=\begin{cases}\dfrac{1}{m+2}, & \text{daca } m\in(-\infty;-2)\cup(-2;0)\cup(3;+\infty),\\[2mm] -\dfrac{1}{m+2}, & \text{daca } m\in(0;3).\end{cases}$$

Exemplul 4. Sa se descompuna in factori

$$a)\ (x+y)(y+z)(z+x)-xyz;$$

$$b)\ x^3+y^3+z^3-3xyz;$$

$$c)\ x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1;$$

$$d)\ x^5+x+1.$$

Rezolvare. a) Se aduna si se scade $z(y+z)(z+x)$ si se grupeaza convenabil

$$(x+y)(y+z)(z+x)+z(y+z)(z+x)-z(y+z)(z+x)-xyz=$$

$$=(y+z)(z+x)(x+y+z)-z((y+z)(z+x)-xy)=$$

$$=(y+z)(z+x)(x+y+z)-z(z^2+yz+zx)=$$

$$=(y+z)(z+x)(x+y+z)-z^2(x+y+z)=$$

$$=(x+y+z)((y+z)(z+x)-z^2)=(x+y+z)(xy+yz+zx).$$

b) Se aplica formula pentru suma a doua cuburi si se rezolva similar exemplului precedent:

$$x^3+y^3+z^3-3xyz=(x+y)(x^2-xy+y^2)+z(z^2-3xy)=$$

$$=(x+y+z)(x^2-xy+y^2)+z(z^2-3xy-x^2+xy-y^2)=$$

$$=(x+y+z)(x^2-xy+y^2)+z(z^2-(x+y)^2)=$$

$$=(x+y+z)(x^2-xy+y^2+z(z-x-y))=$$

$$=(x+y+z)(x^2+y^2+z^2-xy-xz-yz).$$

c) Se aplica formulele inmultirii prescurtate si se obtine

$$x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1=\frac{x^9-1}{x-1}=\frac{(x^3)^3-1^3}{x-1}=$$

$$=\frac{(x^3-1)(x^6+x^3+1)}{x-1}=(x^2+x+1)(x^6+x^3+1).$$

d) $x^5+x+1=1+x+x^2-x^2+x^5=1+x+x^2-x^2(1-x^3)=(1+x+x^2)-x^2(1-x)(1+x+x^2)=$
$=(1+x+x^2)(1-x^2(1-x))=(1+x+x^2)(1-x^2+x^3).$

Exemplul 5. Sa se rationalizeze numitorii urmatoarelor expresii irationale:

$$a)\ \frac{1}{1+\sqrt{2}-\sqrt{3}};\quad b)\ \frac{1}{\sqrt{3}+\sqrt{5}+\sqrt{7}};\quad c)\ \frac{1}{1+\sqrt[3]{2}+2\sqrt[3]{4}}.$$

Rezolvare. Se multiplica cu conjugata numitorului si se obtine:
a) $\dfrac{1}{1+\sqrt{2}-\sqrt{3}} = \dfrac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})^2-3} = \dfrac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}} = \dfrac{(1+\sqrt{2}+\sqrt{3})\sqrt{2}}{2\sqrt{2}\cdot\sqrt{2}} = \dfrac{\sqrt{2}+2+\sqrt{6}}{4}.$
b) Similar exemplului a) se obtine

$$\frac{1}{\sqrt{3}+\sqrt{5}+\sqrt{7}} = \frac{\sqrt{3}+\sqrt{5}-\sqrt{7}}{(\sqrt{3}+\sqrt{5})^2-7} = \frac{\sqrt{3}+\sqrt{5}-\sqrt{7}}{1+2\sqrt{15}} =$$

$$= \frac{(\sqrt{3}+\sqrt{5}-\sqrt{7})(2\sqrt{15}-1)}{(2\sqrt{15})^2-1} = \frac{(\sqrt{3}+\sqrt{5}-\sqrt{7})(2\sqrt{15}-1)}{59}.$$

c) Se utilizeaza formula (a se vedea exemplul 4 b)):

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

de unde

$$x+y+z = \frac{(x^3+y^3+z^3-3xyz)}{(x^2+y^2+z^2-xy-yz-zx)}$$

si se obtine

$$\frac{1}{1+\sqrt[3]{2}+2\sqrt[3]{4}} = \frac{1^2+(\sqrt[3]{2})^2+(2\sqrt[3]{4})^2-\sqrt[3]{2}-2\sqrt[3]{4}-2\sqrt[3]{8}}{1^3+(\sqrt[3]{2})^3+(2\sqrt[3]{4})^3-3\cdot 1\cdot\sqrt[3]{2}\cdot 2\cdot\sqrt[3]{4}} = \frac{7\sqrt[4]{2}-\sqrt[3]{4}-3}{23};$$

Exemplul 6. Sa se arate ca numarul

$a)\ \sqrt{|20\sqrt{7}-53|}-\sqrt{20\sqrt{7}+53},$

$b)\ \sqrt[3]{26-15\sqrt{3}}+\sqrt{3},$

$c)\ \sqrt{26+6\sqrt{13-4\sqrt{8+2\sqrt{6-2\sqrt{5}}}}}+\sqrt{26-6\sqrt{13+4\sqrt{8-2\sqrt{6+2\sqrt{5}}}}},$

este numar intreg.

Rezolvare. a) Se evidentiaza sub radicali patrate complete si se obtine:

$$\sqrt{(5-2\sqrt{7})^2}-\sqrt{(5+2\sqrt{7})^2} = |5-2\sqrt{7}|-|5+2\sqrt{7}| = 2\sqrt{7}-5-5-2\sqrt{7} = -10.$$

b) Se evidentiaza sub semnul radacinii de ordinul trei un cub complet si se obtine

$$\sqrt[3]{26-15\sqrt{3}}+\sqrt{3}=\sqrt[3]{8-12\sqrt{3}+18-3\sqrt{3}}+\sqrt{3}=\sqrt[3]{(2-\sqrt{3})^3}+\sqrt{3}=2-\sqrt{3}+\sqrt{3}=2.$$

c) Se tine seama, ca

$$\begin{aligned}
&6\pm 2\sqrt{5}=5\pm 2\sqrt{5}+1=(\sqrt{5}\pm 1)^2,\\
&\sqrt{8+2\sqrt{6-2\sqrt{5}}}=\sqrt{8+2\sqrt{5}-2}=\sqrt{6+2\sqrt{5}}=\sqrt{5}+1,\\
&\sqrt{8-2\sqrt{6+2\sqrt{5}}}=\sqrt{8-2\sqrt{5}-2}=\sqrt{6-2\sqrt{5}}=\sqrt{5}-1,\\
&\sqrt{13-4(\sqrt{5}+1)}=\sqrt{9-4\sqrt{5}}=\sqrt{5-2\cdot 2\cdot\sqrt{5}+4}=\sqrt{5}-2,\\
&\sqrt{13+4(\sqrt{5}-1)}=\sqrt{5}+2,\\
&\sqrt{26+6(\sqrt{5}-2)}=\sqrt{14+6\sqrt{5}}=\sqrt{5+2\cdot 3\cdot\sqrt{5}+9}=\sqrt{5}+3,\\
&\sqrt{26-6\sqrt{\sqrt{5}+2}}=\sqrt{14-6\sqrt{5}}=\sqrt{(\sqrt{5}-3)^2}=|\sqrt{5}-3|=3-\sqrt{5}
\end{aligned}$$

si se obtine ca expresia initiala este egala cu

$$\sqrt{5}+3+3-\sqrt{5}=6.$$

Identitati conditionate.

Exemplul 7. a) Sa se calculeze $x^2+y^2+z^2$, daca $x+y+z=1$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$.

b) Sa se arate ca egalitatea $xyz=1$ implica

$$\frac{1}{1+x+xy}+\frac{1}{1+y+yz}+\frac{1}{1+z+zx}=1.$$

c) Sa se arate ca daca $x+y+z=0$, atunci $x^4+y^4+z^4=2(xy+yz+zx)^2$.

d) Sa se demonstreze, ca pentru orice trei termeni consecutivi ai unei progresii geometrice are loc egalitatea

$$a_1^2+a_2^2+a_3^2=(a_1+a_2+a_3)(a_1-a_2+a_3).$$

e) Sa se arate ca daca $x_1+x_2+x_3=y_1+y_2+y_3=x_1y_1+x_2y_2+x_3y_3=0$ si nu toate numerele $x_j,\ j=\overline{1,3}$ si $y_i,\ i=\overline{1,3}$ sunt egale cu zero, atunci

$$\frac{x_1^2}{x_1^2+x_2^2+x_3^2}+\frac{y_1^2}{y_1^2+y_2^2+y_3^2}=\frac{2}{3}.$$

Rezolvare. a) Cum egalitatea $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$ implica $xy+yz+zx=0$, rezulta

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)=1^2-2\cdot 0=1.$$

b) Se observa, ca in conditia $xyz=1$

$$\frac{1}{1+z+zx}=\frac{1}{xyz+z+zx}=\frac{1}{z(1+x+xy)}=\frac{xy}{1+x+xy},$$

si atunci

$$\frac{1}{1+y+yz}=\frac{1}{y(1+z+zx)}=\frac{1}{y\cdot z(1+x+xy)}=\frac{x}{1+x+xy}.$$

Prin urmare,

$$\frac{1}{1+x+xy}+\frac{1}{1+y+yz}+\frac{1}{1+z+zx}=\frac{1+x+xy}{1+x+xy}=1.$$

c) Se tine seama ca $x+y=-z$ si se obtine

$$2(xy+yz+zx)^2=2(xy+z(x+y))^2=2(xy-(x+y)^2)^2=$$
$$=2\left(x^2+xy+y^2\right)^2=2(x^4+x^2y^2+y^4+2x^3y+2x^2y^2+2xy^3)=$$
$$=x^4+y^4+(y^4+4xy^3+6x^2y^2+4yx^3+x^4)=x^4+y^4+(x+y)^4=$$
$$=x^4+y^4+(-z)^4=x^4+y^4+z^4.$$

d) Cum $a_1a_3=a_2^2$ (proprietatea caracteristica a progresiei geometrice) rezulta

$$(a_1+a_2+a_3)^2=a_1^2+a_2^2+a_3^2+2a_2(a_1+a_2+a_3).$$

Prin urmare

$$a_1^2+a_2^2+a_3^2=(a_1+a_2+a_3)^2-2a_2(a_1+a_2+a_3)=(a_1+a_2+a_3)(a_1-a_2+a_3).$$

e) Se considera vectorii (x_1, x_2, x_3), (y_1, y_2, y_3) si $(1, 1, 1)$. Cum produsurile scalare ale acestor vectori sunt egale cu zero, rezulta ca ei sunt doi cate doi ortogonali. Asadar vectorii

$$g_1 = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right),$$
$$g_2 = \left(\frac{y_1}{\sqrt{y_1^2 + y_2^2 + y_3^2}}, \frac{y_2}{\sqrt{y_1^2 + y_2^2 + y_3^2}}, \frac{y_3}{\sqrt{y_1^2 + y_2^2 + y_3^2}}\right),$$
$$g_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$$

formeaza o baza ortonormata in spatiul $\mathbf{R}^3$. Prin urmare

$$(x, g_1)^2 + (x, g_2)^2 + (x, g_3)^2 = 1,$$

pentru orice vector $x \in \mathbf{R}^3$.

In particular, pentru $x = e_1 = (1, 0, 0)$, se obtine

$$\frac{x_1^2}{x_1^2 + x_2^2 + x_3^2} + \frac{y_1^2}{y_1^2 + y_2^2 + y_3^2} + \frac{1}{3} = 1,$$

de unde rezulta egalitatea initiala.

Exercitii pentru autoevaluare

1. Sa se determine *DVA* al expresiei

a) $y = \sqrt{3x - x^2}$ *R:* $[0; 3]$.

b) $y = \sqrt{\dfrac{x^2 - 7x + 12}{x^2 - 2x - 3}}$ *R:* $(-\infty; -1) \cup [4; +\infty)$.

c) $y = \dfrac{(b + c)(a + c)}{ab + cd + cb + ad}$ *R:* $\{(a, b, c) \mid a \neq -c,\ b \neq -d\}$.

2. Sa se determine, daca expresiile A si B sunt identic egale pe multimea M, daca:

$$A = \frac{a - \sqrt{3}}{\sqrt{\left(\dfrac{a^2 + 3}{2a}^2 - 3\right)}}, \quad B = \frac{2a}{a + \sqrt{3}}, \quad M = \{a :\ a > \sqrt{3}\}.$$

R: Da.

3. Sa se aduca la forma mai simpla expresia

a) $\left(\frac{\sqrt{a+1}}{2}-\frac{1}{2\sqrt{a+1}}\right)\cdot\left(\frac{\sqrt{a+1}-1}{\sqrt{a+1}+1}-\frac{\sqrt{a+1}+1}{\sqrt{a+1}-1}\right)\cdot\frac{\sqrt{a+1}}{a}$; *R:* -1

b) $\left(a+\frac{ab}{a-b}\right)\left(\frac{ab}{a+b}-a\right):\frac{a^2+b^2}{a^2-b^2}$; *R:* $-\frac{a^4}{a^2+b^2}$

c) $\frac{\sqrt{x}+1}{1+\sqrt{x}+x}:\frac{1}{x^2-\sqrt{x}}$; *R:* $\sqrt{x}(x-1)$

d) $\frac{ab}{a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}}-\frac{a-b}{a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}}-\frac{a^{\frac{2}{3}}-b^{\frac{2}{3}}}{a^{\frac{1}{3}}-b^{\frac{1}{3}}}$; *R:* $b^{\frac{1}{3}}-a^{\frac{1}{3}}$

e) $\sqrt[4]{(1-2a+a^2)(a^2-1)(a-1)}:\frac{a^2+2a-3}{\sqrt[4]{a+1}}$;

R: $\frac{\sqrt{a+1}}{a+3}$ pentru $a>1$; $-\frac{\sqrt{a+1}}{a+3}$ pentru $-1<a<1$.

f) $\frac{a^2-4-|a-2|}{a^3+2a^2-5a-6}$;

R: $\frac{1}{a+3}$ pentru $a>2$; $\frac{1}{a+2}$ pentru $a<2,\ a\neq-3,\ a\neq-1$.

g) $\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{a-b}{a+b}+\frac{(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}$; *R:* 0.

4. Sa se descompuna in factori:

a) x^4+x^2+1; *R:* $(x^2+x+1)(x^2-x+1)$

b) $(x-y)^3+(y-z)^3+(z-x)^3$; *R:* $3(x-y)(y-z)(z-x)$.

5. Sa se rationalizeze

$$a)\ \frac{9}{\sqrt{10}+\sqrt{15}+\sqrt{14}+\sqrt{21}}; \qquad R\text{: } \frac{9(\sqrt{7}-\sqrt{5})(\sqrt{3}-\sqrt{2})}{2}.$$

$$b)\ \frac{\sqrt{7}}{\sqrt[12]{5}+\sqrt[12]{3}}; \qquad R\text{: } \frac{\sqrt{7}(\sqrt[12]{5}-\sqrt[12]{3})(\sqrt[6]{5}+\sqrt[6]{3})(\sqrt[3]{25}+\sqrt[3]{15}+\sqrt[3]{9})}{2}.$$

6. Sa se arate ca expresiile

$$a)\ \frac{\sqrt{2\sqrt[4]{8}-2\sqrt{\sqrt{2}+1}}}{\sqrt{\sqrt[4]{8}+\sqrt{\sqrt{2}-1}}-\sqrt{\sqrt[4]{8}-\sqrt{\sqrt{2}-1}}}; \qquad R\text{: } 1$$

$$b)\ \sqrt[3]{9+\sqrt{80}}+\sqrt[3]{9-\sqrt{80}}; \qquad R\text{: } 3$$

reprezinta numere intregi.