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On the optimal control of heat apportionment systems

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Abstract

The theory of optimal control for distributed parameters systems is adjusted for solving the problems of determining the controled temperature fields of hidrotehnical construction with the aim of appointment of the tehnical and technological operations, which assure their integrity. The problem is simulated with the help of the conduction equation with initial and boundary condition. The control function is introduced as the heat flowing, boundary influence or as the thermal diffusivity coefficient.

1 Introduction

An important place among the technical problems belongs to the continuos heat apportionment systems. The heat emition stipulates the appearance of complex temperature fields in the systems, that causes the appearance and concentration of temperature tensions, that further bring to the formation of cracks, stratifications and other defects, which lead to the distruction of the buildings and constructions. An example of sach systems can be the massive concrete constructions.

The cause of appearance of cracks in them is the temperature tensions, which appear at the internal heat apportionment, because of great differences between the temperatures in the internal and external parts of the concrete massive and deformation at its cooling to the normal exploitation temperature.

To present the formation of cracks in the concrete laying in the period of building of hydrotechnical constructions it's necessary to support certain temperature regimes, at which the tensions values do not exceld the permisible ones.

In this connection the problem of computing the controled temperature fields of such constructions is as quite importance. The computed temperature fields permit to take off temperature picks, and also to set corresponding technical and technological measures, which assume the achievement of given temperatures in a period of construction.

Mathematically this problem is given by the parabolic equation with initial and boundary conditions. The unstationary control action is found in the right part of the heat conduction equation as the densites of sources distribution, and in the boundary conditions as the heat flowing or is the thermal diffusivity coefficient.

The criterium of optimization is considered the square cost function. The one- two- and three-dimensional control problems are considered depending of the aims of investigation.

2 Control problem

First we make same assumptions concerning the erection of massive concrete constructions, than we'll state the optimal control problem of the temperature regime of the cooling concrete massive.

a) Concrete is considered a quasihomogeneous izotopic material.

b) The thermophysical characteristics of the concrete are constant values, that do not depend neither on time, nor on temperature, except the case when one of the thermophysical coefficients is a control influence.

c) After the concrete laying a heat emision process begins, which is determined by the expression:

$$q = q_0 e^{-mt},$$

where q — quantity of heat, emited in a unit time, q_0, m — constants, t — time.

d) We'll consider that the control $p(t,x) \in L_2$. In the case when $p(t,x)$ enters in the heat conduction equation or in the boundary condition as the heat flowing, this can be technically performed as pipe cooling system with the pipe situated enouqh close to one another. In the case, when the control is a thermal diffusivity coefficient, this can be one of the compound components with corresponding properties.

An unbounded concrete wall of thickness R, one side of which is thermo–isolated, and on the other the heat exchange with the environment takes place is considered. The controlled process is given by the parabolic equation:

$$\frac{\partial U(t,x)}{\partial t} = \frac{\partial (ap(x)\frac{\partial U(t,x)}{\partial x})}{\partial x} + q_0 e^{-mt} + bp(t,x), \tag{1}$$

$$t \in (0, t_1], \;\; x \in (0, R)$$

with the initial condition

$$U(0,x) = \varphi(x), \;\; x \in (0,R) \tag{2}$$

and boundary conditions

$$\frac{\partial U(t,0)}{\partial x} = 0, \frac{\partial U(t,R)}{\partial x} = \alpha[cp(t) - U(t,R)], \;\; 0 < t \leq t_1, \tag{3}$$

here a — the thermal diffusivity coefficient, α — the heat exchange coefficient.

Now let's formulate the optimal control problem. Let t_1 be fixed constant. It's necessary to find such a control function $p(x)$, that at the moment of time $t = t_1$ the corresponding solution of the boundary problem (1) - (3) satisfies the condition:

$$U(t_1, x) = y(x), \;\; y(x) \in L_2(0,R), \tag{4}$$

and the cost function

$$J(p) = \int_0^R \int_0^{t_1} [ap^2(x) + bp^2(t,x) + cp^2(t)] dt dx \tag{5}$$

achieves the minimal value. a, b, c — constants, b and c get the value 1 or 0, depending of the control function considered.

3 Distributed control ($p(t) = 1, \;\; c = 0, b = 1$)

Let's formulate the auxiliary problems. Let t_1 be fixed. It is necessary to find such a control function $p(t, x)$ and the corresponding solution of the problem (1)–(3), on which the cost function

$$J_1(p) = \int_0^R (U(t_1, x) - y(x))^2 dx + \beta \int_0^R \int_0^{t_1} p^2(t, x) dt dx \qquad (6)$$

achives the minimal value, where β is a given positive number.

Theorem 1 *(Necessary and sufficient optimality conditions) If the admisible control $p^0(t, x)$ and the corresponding solution $U^0(t, x)$ of the boundary problem (1)–(3) to be optimal, it is necessary and sufficient, the function*

$$H(\Psi^0, U^0, p^0) = p^0(t, x)\Psi^0(t, x) - \beta p^0(t, x)^2$$

to sutisfy the condition

$$H(\Psi^0, U^0, p^0)(=) \max H(\Psi^0, U^0, p), \qquad (7)$$

where Ψ^0 is the solution of the boundary problem

$$\frac{\partial \Psi(t, x)}{\partial t} + a \frac{\partial^2 \Psi(t, x)}{\partial x^2} = 0, \qquad (8)$$

$$t \in [0, t_1), \;\; x \in (0, R).$$

$$\Psi(t_1) = 2[U(t_1, x) - y(x)], \;\; x \in (0, R), \qquad (9)$$

$$\frac{\partial \Psi(t, 0)}{\partial x} = 0, \;\; \frac{\partial \Psi(t, R)}{\partial x} = -\alpha \Psi(t, R), \;\; t \in [0, t_1) \qquad (10)$$

Solving in common problems (1)–(3) and (8)–(10) with the condition

$$p(t, x) = \Psi(t, x)/2\beta, \qquad (11)$$

we build the optimal control for the auxiliary problem. And the solution of the initial problem we obtain by the limit passing at $\beta \to 0$. For this case it is possible to obtain the analityc optimal solution $U^0(t, x)$, $p^0(t, x)$ as a infinite series of decomposition by the proper functions which correspond to the Sturm – Liuvilly problem.

4 Boundary control ($a = const, \;\; b = 0$)

We'll consider the controlled process, given by the two dimensional heat conduction equation

$$\frac{\partial U(t,x,y)}{\partial t} = a[\frac{\partial^2 U(t,x,y)}{\partial x^2} + \frac{\partial^2 U(t,x,y)}{\partial y^2}] + q_0 e^{-mt}, \qquad (12)$$

$$t \in (0, t_1], \;\; x \in (0, R_0), \;\; y \in (0, R_1)$$

with the initial

$$U(0,x,y) = U^{la}, \;\; 0 < x < R_0, \;\; 0 < y < R_1 \qquad (13)$$

and boundary conditions

$$\frac{\partial U(t,0,y)}{\partial x} = 0, \;\; \frac{\partial U(t,R_0,y)}{\partial x} = \alpha_0[g_0(t,y) - U(t,R_0,y)],$$

$$\frac{\partial U(t,x,0)}{\partial y} = 0, \;\; \frac{\partial U(t,x.R_1)}{\partial y} = \alpha_1[g_1(t,x) - U(t,x,R_1)]. \qquad (14)$$

The functions $g_0(t,y)$ and $g_1(t,x)$ are of the form $g_0(t,y) = g_0(y)p(t)$, $g_1(t,x) = g_1(x)p(t)$, where $g_0(y)$, $g_1(x)$ — are given function $\in L_2$, and $p(t)$ — are arbitrary control function $\in L_2$.

It is necessary to find such an admisible control $p(t)$, that the corresponding solution of the problem (12)–(14) at the moment of time $t = t_1$ satisfies the condition

$$U(t_1,x,y) = UO, \;\; 0 < x < R_0, \;\; 0 < y < R_1 \qquad (15)$$

and the cost function

$$J(p) = \int_0^{t_1} p^2(t)dt$$

achieves the minimal value.

We'll search the approximate solution of the formulated problem by solving the following problem. It is necessary to find such control

$p(t) \in L_2(0,t_1)$ that together with the corresponding solution of the problem (12)–(14) it will minimize the cost function

$$J_\beta(p) = 1/\beta \int_0^{R_0} \int_0^{R_1} [U(t_1,x,y) - UO]^2 dxdy + \int_0^{t_1} p^2(t)dt. \qquad \text{(A)}$$

For each admisibile control $p(t)$ and corresponding solution $U(t,x,y)$ we'll confront the function $V(x,y)$, defined by the correlations:

$$V_t(t,x,y) = a[V_{xx}(t,x,y) + V_{yy}(t,x,y)],$$

$$0 \le t < t_1,\ \ 0 < x < R_0,\ \ 0 < y < R_1; \qquad (16)$$

$$V(t_1,x,y) = -2[U(t_1,x,y) - U_O],$$

$$0 < x < R_0, 0 < y < R_1; \qquad (17)$$

$$V_x(t,0,y) = 0,\ \ V_x(t,R_0,y) = -\alpha_0 V(t,R_0,y),$$

$$0 \le t < t_1,\ \ 0 < y < R_1;$$

$$V_y(t,x,0) = 0,\ \ V_y(t,x,R_1) = -\alpha_1 V(t,x,R_1),$$

$$0 < x < R_0,\ \ 0 \le t < T_1. \qquad (18)$$

The following theorem is true.

Theorem 2 *In order the admisibile control $p^0(t,\beta)$ in the boundary problem (12)–(14) to be optimal it is necessary and sufficient, that the function*

$$h = p(t,\beta)r(t) + \beta p^2(t,\beta),$$

$$r(t) = \int_0^{R_1} g_0(y)V(t,0,y)dy + \int_0^{R_0} g_1(x)V(t,x,0)dx,$$

where $V(t,x,y)$ is the solution of the problem (16)–(18), corresponding to $p^0(t,\beta)$, to satisfy the condition

$$h(V^0(t,x,y), p^0(t,\beta))\ \ (=)\ \ \sup h(V(t,x,y), p(t,\beta)).$$

Theorem 3 *The optimal control $p^0(t,\beta)$ is the solution of the integral equation*

$$\beta p(t) = F(t) - \int_0^{t_1} K(t,\tau)p(t,\tau)d\tau, \tag{19}$$

where

$$F(t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} g_{nk}\delta_{nk}e^{-\lambda_n k^2 a(t-\tau)},$$

$$K(t,\tau) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} g_{nk}^2 e^{-\lambda_{nk}^2 a(t+\tau-2t_1)},$$

$$\delta_{nk} = UO_{nk} - U_{nk}^{la}e^{-a\lambda_{nk}^2 t_1} + q_{nk}^0\int_0^{t_1} e^{-m\tau}e^{-\lambda_{nk}^2 a(t_1-\tau)}d\tau,$$

$$q_{nk}^0 = \int_0^{R_0}\int_0^{R_1} q_0 X_n(x)Y_k(y)w_{nk}dxdy,$$

$$U_{nk}^{la} = \int_0^{R0}\int_0^{R_1} U^{la}X_n(x)Y_k(y)w_{nk}dxdy,$$

$$UO_{nk} = \int_0^{R0}\int_0^{R_1} UOX_n(x)Y_k(y)w_{nk}dxdy,$$

$$g_{nk} = (\alpha_1 aX_n(R_1)Y(k)w_{nk} + \alpha_0 aY_k(r_0)X(n)w_{nk})/w_{nk}^2,$$

$$X(n) = \int_0^{R_0} X_n(x)dx,$$

$$Y(k) = \int_0^{R_1} Y_k(y)dy.$$

$X_n(x)$ and $Y_k(y)$ is a full in $L_2(0,R_0;0,R_1)$ orthonormal system of proper functions corresponding to the Sturm–Liuvilly problem. The solution of this equation is unique.

Now we'll substitute the equation (19) with an approximative one

$$\beta p(t) = F_n - \int_0^{t_1} K_{NN}(t,\tau)p(\tau)d\tau,$$

in wich

$$F_N = \sum_{n=1}^{N}\sum_{k=1}^{N} g_{nk}e^{-\lambda_{nk}^2 a(t_1-\tau)}\delta_{nk},$$

$$K_{NN} = \sum_{n=1}^{N} \sum_{k=1}^{N} g_{nk}^2 e^{-\lambda_{nk}^2 a(t+\tau-2t_1)}.$$

Its unique solution is determined by the function

$$p_N(t, t_1, \beta) = 1/\beta \sum_{n=1}^{N} \sum_{k=1}^{N} g_{nk}[\delta_{nk} - C_{nk}(\beta)] e^{-\lambda_{nk}^2 a(t-t_1)},$$

where $C_{nk}(\beta)$ is the unique solution of the system equations

$$C_{ij}^N + 1/\beta \sum_{n=1}^{N} \sum_{k=1}^{N} K_{ijnk} C_{nk}^N = b_{ij}^n,$$

$$b_{ij}^N = 1/\beta \sum_{n=1}^{N} \sum_{k=1}^{N} K_{ijkn} \delta_{nk}, \quad i, j = 1, 2, \ldots, N.$$

The three–dimensional problem is formulated and solved in the same way. The approximate solutions for the both of this problems and alsow the analysise of the given results are given.

5 Pipe cooling

It is supposed that a pipe cooling system was laied inside the concrete massive with a real step of pipes in the period of building. Around a pipe with radius r_0 a cylinder of radius R is cut out, which is situated far enough from the massive edge. The temperature field of this cylinder is given by the equation

$$\frac{\partial U(t,r)}{\partial t} = a(\frac{\partial^2 U(t,r)}{\partial r^2} + 1/r \frac{\partial U(t,r)}{\partial r}) + q_0 e^{-mt}, \qquad (20)$$

$$r < r < R, \ \ 0 < t \leq t_1,$$

with the initial

$$U(0,r) = U^0(r), \ \ r_0 < r < R \qquad (21)$$

and boundary conditions

$$\frac{\partial U(t,r_0)}{\partial r} = -\alpha[p(t) - U(t,r_0)], \ \ \frac{\partial U(t,R)}{\partial r} = 0, \ \ 0 < t \leq t_1. \qquad (22)$$

It is necessary to find such a control $p(t) \in P$, where $P = \{P|\bmod(p(t)) \le U_v\}$, for which the coresponding solution $U(t,r)$ of the boundary problem (20)–(22) at the moment of time $t = t_1$ achieves given values

$$U(t_1, r) = \Phi(r), \;\; r_0 < r < R \tag{23}$$

and the cost function

$$J(p) = \int_0^{t_1} p^2(t)dt \tag{24}$$

gets the minimal value.

The optimal control is presented in the following way

$$p(t) = \begin{cases} U_v, & \Phi(t) \ge U_v \\ \Psi, & \bmod(\Psi(t)) \le U_v \\ -U_v, & \Phi(t) \le -U_v \end{cases} \tag{25}$$

where

$$\Phi(t) = \alpha/2\beta\Psi(t,R), \tag{26}$$

$\beta = const > 0$ and $\Psi(t,R)$ is defined as unique solution of the initial–baundary problem

$$\frac{\partial\Psi(t,r)}{\partial t} = -a(\frac{\partial^2\Psi(t,r)}{\partial r^2} + 1/r\frac{\partial\Psi(t,r)}{\partial r}),$$

$$r_0 < r < R, \;\; 0 \le t < t_1;$$

$$\begin{aligned} \Psi(t_1, r) &= -2\sqrt{r}[U(t_1,r) - \varphi(r)], \;\; r_0 < r < R; \\ \frac{\partial\Psi(t,r_0)}{\partial r} &= \alpha\Psi(t,r_0), \;\; \frac{\partial\Psi(t,R)}{\partial r} = 0, \;\; 0 \le t < t_1. \end{aligned} \tag{27}$$

It means, that it is necessary to solve the problem (20)–(22) and (27) with condition (26) in order to find the optimal control.

The control $p(t)$ will be searched in the following form

$$p(t) = -1/\beta \sum_{k=1}^{\infty} \gamma_k(\beta)^{\lambda_k^2 a(t_1-t)}. \tag{28}$$

Substituting (28) into (26), we'll get the following system of linear nonhomogeneous algebraic equations relativly the coefficients $\gamma_k(\beta)$

$$\gamma_n(\beta) + \alpha^2/\beta b_n \sum_{n=1}^{\infty}(1 - e^{(-\lambda_k^2+\lambda_n^2)at_1})/a(\lambda_k^2 + \lambda_n^2)\gamma_k(\beta) = \alpha/\beta a_n,$$

$$n = 1, 2, \ldots$$

Solving this system relativly $\gamma_n(\beta)$ we define $p(t)$, $J(p)$ and $U(t, r)$. To find the optimal control we solved the problem of minimization of the cost function

$$J_1(p) = \int_{R_0}^{R_1} [\sqrt{r}(U(t_1, r) - \Phi(r))]^2 + \beta \int_0^{t_1} p^2(\tau) d\tau,$$

and we have got the optimality conditions for it. So we obtained the temperature of the cooling liquid, which is feeded into the pipe in order to obtain a given temperature distribution around the pipe.

Next, outcoming from the given cooling liquid temperature we'll find the minimal number of pipes necessary for the temperature tensions be not greater than the admissible ones at the given moment of time. Let $U(, x)$ be the temperature of the concrete massive in the point x at the moment of time t. Let $x_1, x_2, \ldots, x_n$ be the points of the pipes location. The quantity of heat from the concrete to the pipe is proportional to the value $[U(t, x) - U_v]$, where U_v is the temperature of the cooling liquid. Hence, the power of the heat flowing in the point x_i is equal to the value $C_0[U(t, x_i) - U_v]$.

The temperature distribution is given by the following initial–boundary problem

$$\frac{\partial U(t,x)}{\partial t} = \frac{\partial^2 U(t,x)}{\partial x^2} + f(t) + C_0 \sum_{k=1}^{n} \delta(x - x_k)[U(t,x) - U_v],$$

$$0 < t \le t_1, \ \ 0 < x < 1; \qquad (29)$$

$$U(0, x) = U^n, \ \ 0 < x < 1; \qquad (30)$$

$$\frac{\partial U(t,0)}{\partial x} = 0, \ \ \frac{\partial U(t,1)}{\partial x} = -\alpha U(t,1), \ \ 0 < t \le t_1. \qquad (31)$$

It is necessary to find the minimal number n of points, through which the cooling pipes pass, necessary for the temperature tension computed by the formula

$$\sigma(t,1,n) = \alpha E/(1-\nu)[\tilde{U}(t,1,n) - U(t,1,n)]$$

to sutisfy the condition $\sigma(t,1,n) \leq \sigma^*$. Here α is the linear coefficient of thermal expansion, E — elasticity module, ν — Poisson coefficient, $\tilde{U}(t,1,n) = \int_0^1 U(t,x,n)dx$ — middle section temperature. Both problems are solved approximately.

6 Thermal diffusivity coefficient control ($b = 0, a = 1, c = 0$)

The control process is given by the one dimentional equation of heat condaction

$$\frac{\partial U(t,x)}{\partial t} = \frac{\partial (p(x)\frac{\partial U(t,x)}{\partial x})}{\partial x} + f(t), \tag{32}$$
$$0 < t \leq t_1, \;\; 0 < x < R,$$

with initial

$$U(0,x) = \Phi(x), \;\; 0 < x < R \tag{33}$$

and boundary conditions

$$\frac{\partial U(t,0)}{\partial x} = 0, \;\; \frac{\partial U(t,R)}{\partial x} + \alpha U(t,R) = 0, \;\; 0 < t \leq t_1, \tag{34}$$

where $f(t,x)$, $\Phi(x)$ — given function from $L_2(0,t_1), L_2(0,R)$, control $p(x)$ — piece–wise constant function, that $p(x) \in P = \{p : p_1, p_2\}$.

The problem is to find such a control $p(x)$, that the cost function

$$J(p) = \int_0^R [U(t_1,x) - y(x)]^2 dx \tag{35}$$

achieves at the moment of time $t = t_1$ the minimal value, $y(x)$ — a known function from $L_2(0,R)$.

Since the problem (32)–(34) is incorrect, we'll consider the following cost function

$$J_\beta(p) = \int_0^R [U(t_1, x) - y(x)]^2 dx + \beta \int_0^R p^2(x) dx. \qquad (36)$$

Let's state the conjugate to (32)–(34) problem and let $\Psi(t, x)$ be it's solution

$$\frac{\partial \Psi(t,x)}{\partial t} = -\frac{\partial (p(x)\frac{\partial \Psi(t,x)}{\partial x})}{\partial x}, \ \ 0 \le t < t_1, \ \ 0 < x < R; \qquad (37)$$

$$\Psi(t_1, x) = -2[U(t_1, x, p) - y(x)], \ \ 0 < x < R; \qquad (38)$$

$$\frac{\partial \Psi(t,0)}{\partial x} = 0, \ \ \frac{\partial \Psi(t,R)}{\partial x} = -\alpha \Psi(t, R), \ \ 0 \le t < t_1. \qquad (39)$$

Let $H(t, x, U, \Psi, p) = p(x) \int_0^{t_1} \frac{\partial U(t,x)}{\partial x} \frac{\partial \Psi(t,x)}{\partial x} dt + \beta p^2(x)$, then the following theorem is true.

Theorem 4 *In order the function $p(x) \in P$ be the solution of the problem (32)–(34), (36) it is necessary the following conditions to fulfile*

$$H(t, x, U^*, \Psi^*, p^*) = \max_{p \in P} H(t, x, U, \Psi, p), \qquad (40)$$

where $U^*(t, x)$, $\Psi^*(t, x)$ are corresponding the solutions of the basic and conjugate problems at $p(x) = p^*(x)$.

The necessary condition of optimality and descent method on great variation were used for numerical solving of this problem .

7 Conclusion

A short characteristic of the problems, that appear when simulating the controlled temperature fields of massive concrete construction, is given in the article. Their analytic and numeric solutions are obtained by using the necessary and sufficient conditions of optimality in the maximum principle form. Pipe cooling is shown as an example of practical control application. Standard mathematic ensuring for this was elaborated.

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