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Sensitivity analysis of efficient solution in vector MINMAX boolean programming problem*

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Abstract

We consider a multiple criterion Boolean programming problem with MINMAX partial criteria. The extreme level of independent perturbations of partial criteria parameters such that efficient (Pareto optimal) solution preserves optimality was obtained.

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Let $C = (c_{ij}) \in \mathbf{R}^{n\times m}$, $n, m \in \mathbf{N}$, $m \geq 2$, $C_i = (c_{i1}, c_{i2}, ..., c_{im})$, $\mathbf{E}^m = \{0,1\}^m$, T be the non-empty subset of the permutations set S_m which is defined on the set $N_m = \{1, 2, ..., m\}$. On the set of non-zero solutions (i.e. Boolean non-zero vectors) $X \subseteq \mathbf{E}^m$, $|X| > 1$, we define the vector criterion

$$f(x, C) = (f_1(x, C_1), f_2(x, C_2), ..., f_n(x, C_n)) \longrightarrow \min_{x\in X}.$$

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The components (partial criteria) are functions

$$f_i(x, C_i) = \max_{t \in T} \sum_{j \in N(x)} c_{it(j)}, \ i \in N_n,$$

where

$$t = \begin{bmatrix} 1 & 2 & ... & m \\ t(1) & t(2) & ... & t(m) \end{bmatrix}, \ N(x) = \{j \in N_m : \ x_j = 1\}.$$

Suppose $C_i[t] = (c_{it(1)}, c_{it(2)}, ..., c_{it(m)})$. Then we can rewrite partial criteria in the following form

$$f_i(x, C_i) = \max_{t \in T} C_i[t]x, \ i \in N_n,$$

where

$$x = (x_1, x_2, ..., x_m)^T.$$

The problem of finding the set of *efficient solutions* (the Pareto set)

$$P^n(C) = \{x \in X \ \ \pi(x, C) = \emptyset\}$$

we call *a vector minimax Boolean programming problem* and write $Z^n(C)$, where

$$\pi(x, C) = \{x' \in X : \ q(x, x', C) \geq 0_{(n)}, \ q(x, x', C) \neq 0_{(n)}\}.$$

$$q(x, x', C) = (q_1(x, x', C_1), \ q_2(x, x', C_2), \ ... \ , q_n(x, x', C_n)),$$

$$q_i(x, x', C_i) = f_i(x, C_i) - f_i(x', C_i), \ i \in N_n, \ 0_{(n)} = (0, 0, ..., 0) \in \mathbf{R}^n.$$

By analogy with [1 – 4], where the stability radius of efficient solution in different optimization problems was studied, the number

$$\rho^n(x^0, C) = \begin{cases} \sup \Omega, & \text{if } \Omega \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}$$

is called *the stability radius of the efficient solution* $x^0 \in P^n(C)$. Here

$$\Omega = \{\varepsilon > 0 : \ \forall C' \in \Re(\varepsilon) \ (x^0 \in P^n(C + C')\},$$

$$\Re(\varepsilon) = \{C' \in \mathbf{R}^{n\times m} : \| C' \|_\infty < \varepsilon\},$$

$$\| C' \|_\infty = \max\{|c'_{ij}| : \ (i,j) \in N_n \times N_m\}, \ C' = (c'_{ij}) \in \mathbf{R}^{n\times m}.$$

We consider $\rho^n(x^0, C) = \infty$ if for any matrix $C' \in \mathbf{R}^{n\times m}$

$$x' \in P^n(C + C').$$

For any $x^0 \neq x$ and any permutation $t \in T$ we introduce the following notifications:

$$T(x^0, x) = \{t \in T : \ \forall t' \in T \ (N(x^0, t) \neq N(x, t'))\},$$

$$N(x, t) = \{t(j) : \ j \in N_m \ \& \ x_j = 1\},$$

$$\bar{T}(x^0, x) = T \backslash T(x^0, x).$$

Lemma 1 *Assume that* $x^0 \neq x, \ x^0, x \in X \ t^0 \in \bar{T}(x^0, x)$. *Then*

$$C_i[t^0]x^0 \leq f_i(x, C_i)$$

for any index $i \in N_n$ *and matrix* $C \in \mathbf{R}^{n\times m}$.

Proof. Let $t^0 \in \bar{T}(x^0, x)$. Then there exists $t' \in T$ such that $N(x^0, t^0) = N(x, t')$. So for any $i \in N_n$ we have

$$C_i[t^0]x^0 = C_i[t']x \leq \max_{t \in T} C_i[t]x = f_i(x, C_i).$$

Lemma 1 is proved.

The efficient solution x^0 is called *trivial* if the set $T(x^0, x)$ is empty for any $x \in X \backslash \{x^0\}$ and *non-trivial* otherwise.

Theorem 1 *The stability radius $\rho^n(x^0, C)$ of any trivial solution x^0 of the problem $Z^n(C)$ is infinite.*

Proof. Let $x^0 \in P^n(C)$. Since x trivial, the equality $T = \bar{T}(x^0, x)$ is true for any $x \in X \backslash \{x^0\}$. By lemma 1, the inequality

$$(C + C')_i[t^0]x^0 \leq f_i(x, C_i + C'_i)$$

holds for any $x \in X \backslash \{x^0\},\ t^0 \in T,\ i \in N_n,\ C' \in \mathbf{R}^{n \times m}$. Hence

$$q(x^0, x, C + C') \leq 0_{(n)}.$$

So the solution $x^0 \in P^n(C)$ preserves the efficiency for any independent perturbations of matrix C. Thus $\rho^n(x^o, C) = \infty$. Theorem 1 is proved.

By definition, put

$$X(x^0) = \{x \in X \backslash \{x^0\} :\ T(x^0, x) \neq \emptyset\}.$$

Lemma 2 *Let x^0 be non-trivial efficient solution of the problem $Z^n(C)$, $\varphi > 0$. Suppose for any matrix $C' \in \Re(\varphi)$ and $x \in X(x^0)$ there exists an index $i \in N_n$ such that*

$$q_i(x, x^0, C_i + C'_i) > 0.$$

Then

$$x^0 \in P^n(C + C')$$

for any matrix $C' \in \Re(\varphi)$.

Proof. Let $x \notin X(x^0)$. Then for any $t \in T$ there exists $t' \in T$ such that $N(x^0, t) = N(x, t')$. Hence we have for any index $i \in N_n$ and any matrix $C' \in \Re(\varphi)$

$$q_i(x, x^0, C_i + C'_i) = \max_{t \in T}(C_i + C'_i)[t]x - \max_{t \in T}(C_i + C'_i)[t]x^0 =$$

$$= \max_{t \in T}(C_i + C'_i)[t]x - (C_i + C'_i)[t^*]x^0 \geq$$

$$\geq (C_i + C'_i)[t']x - (C_i + C'_i)[t^*]x^0 = 0.$$

It means that

$$x^0 \in P^n(C + C')$$

for any matrix $C' \in \Re(\varphi)$. Lemma 2 is proved.

For any non-trivial solution x^0 put

$$\varphi^n(x^0, C) = \min_{x \in X(x^0)} \max_{i \in N_n} \min_{t^0 \in T(x^0, x)} \max_{t \in T} \frac{C_i[t]x - C_i[t^0]x^0}{\sigma(x^0, t^0, x, t)},$$

where

$$\sigma(x^0, t^0, x, t) = |(N(x^0, t^0) \cup N(x, t)) \backslash (N(x^0, t^0) \cap N(x, t))|.$$

The following statements are true

$$t^0 \in \bar{T}(x, x^0) \implies \forall t \in T \ (\sigma(x^0, t^0, x, t) = 0). \quad (1)$$

$$C_i[t]x - C_i[t^0]x^0 + ||C_i||_\infty \sigma(x^0, t^0, x, t) \geq 0, \ i \in N_n, \quad (2)$$

It is easy to see that $0 \leq \varphi^n(x^0, C) < \infty$.

Theorem 2 *The stability radius $\rho^n(x^0, C)$ of any non-trivial efficient solution x^0 of the problem $Z^n(C)$ is expressed by the formula*

$$\rho^n(x^0, C) = \varphi^n(x^0, C).$$

Proof. First let us prove that $\rho^n(x^0, C) \geq \varphi := \varphi^n(x^0, C)$. For $\varphi = 0$, it is nothing to prove. Let $\varphi > 0$. Then for any $x \in X(x^0)$ there exists an index $i \in N_n$ such that

$$\min_{t^0 \in T(x^0, x)} \max_{t \in T} \frac{C_i[t]x - C_i[t^0]x^0}{\sigma(x^0, t^0, x, t)} \geq \varphi.$$

We have the following statements for any $C' \in \Re(\varphi)$

$$q_i(x, x^0, C_i + C_i') = \max_{t^0 \in T}(C_i + C_i')[t]x - \max_{t^0 \in T}(C_i + C_i')[t^0]x^0 =$$

$$= \min_{t^0 \in T} \max_{t \in T}(C_i[t]x - C_i[t^0]x^0 + C_i'[t]x - C_i'[t^0]x^0) \geq$$

$$\geq \min_{t^0 \in T} \max_{t \in T}(C_i[t]x - C_i[t^0]x^0 - ||C_i'||\sigma(x^0, t^0, x, t)).$$

Using (1) we continue

$$= \min_{t^0 \in T(x^0,x)} \max_{t \in T} (C_i[t]x - C_i[t^0]x^0 - ||C_i'||\sigma(x^0, t^0, x, t))$$

Applying (2) we finally conclude

$$> \min_{t^0 \in T(x^0,x)} \max_{t \in T} (C_i[t]x - C_i[t^0]x^0 - \varphi\sigma(x^0, t^0, x, t)) \geq 0.$$

Thus, by lemma 2, we obtain that non-trivial solution x^0 preserves efficiency for any perturbing matrix $C' \in \Re(\varphi)$, i.e. $\rho^n(x^0, C) \geq \varphi$.

It remains to check that $\rho^n(x^0, C) \leq \varphi$. According to the definition of φ, there exists $x \in X(x^0)$ such that for any $i \in N_n$

$$\varphi \geq \min_{t^0 \in T(x^0,x)} \max_{t \in T} \frac{C_i[t]x - C_i[t^0]x^0}{\sigma(x^0, t^0, x, t)} = \max_{t \in T} \frac{C_i[t]x - C_i[\tilde{t}]x^0}{\sigma(x^0, \tilde{t}, x, t)}. \quad (3)$$

Let $\varepsilon > 0$. Consider the following perturbing matrix $C^* \in \mathbf{R}^{n \times m}$. Every string $C_i^*,\ i \in N_n$ of this matrix consists of the elements

$$c_{ij}^* = \begin{cases} \alpha, & \text{if } j \in N(x^0, \tilde{t}), \\ -\alpha, & \text{otherwise,} \end{cases}$$

where $\varphi < \alpha < \varepsilon$. Using (3) we get the following expressions:

$$q_i(x, x^0, C_i + C_i^*) = \max_{t \in T} (C_i + C_i^*)[t]x - \max_{t \in T} (C_i + C_i^*)[t]x^0 \leq$$

$$\max_{t \in T} (C_i + C_i^*)[t]x - (C_i + C_i^*)[\tilde{t}]x^0 = (C_i + C_i^*)[\hat{t}]x - (C_i + C_i^*)[\tilde{t}]x^0 =$$

$$= C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \alpha\sigma(x^0, \tilde{t}, x, \hat{t}) < C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \varphi\sigma(x^0, \tilde{t}, x, \hat{t}) \leq$$

$$\leq C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \sigma(x^0, \tilde{t}, x, \hat{t}) \max_{t \in T} \frac{C_i[t]x - C_i[\tilde{t}]x^0}{\sigma(x^0, \tilde{t}, x, t)} \leq$$

$$\leq C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \sigma(x^0, \tilde{t}, x, \hat{t}) \frac{C_i[\hat{t}]x - C_i[\tilde{t}]x^0}{\sigma(x^0, \tilde{t}, x, \hat{t})} = 0.$$

Hence x^0 is not efficient solution of the problem $Z^n(C + C^*)$, where $C^* \in \Re(\varphi)$. It means that $\rho^n(x^0, C) \leq \varphi$. This completes the proof of Theorem 2.

Assume that $T = \{t^0\}$. $t^0 = \begin{bmatrix} 1 & 2 & ... & \mathrm{m} \\ 1 & 2 & ... & \mathrm{m} \end{bmatrix}$. Then our problem transforms into vector linear Boolean programming problem

$$f_i(x, C_i) = C_i x \longrightarrow \min_{x \in X}, \ i \in N_n,$$

where $X \subseteq \mathbf{E}^m$.

In this case one can see that any efficient solution is non-trivial. The next corollary follows from theorem 2.

Corollary 1 *[1] The stability radius of any efficient solution* x^0 *of vector linear Boolean programming problem* $Z^n(C)$, $n \geq 1$, *equals to*

$$\min_{x \in X \backslash \{x^0\}} \max_{i \in N_n} \frac{C_i(x - x^0)}{||x - x^0||^*},$$

where $||z||^* = \sum\limits_{j \in N_n} |z_j|, \ z = (z_1, z_2, ..., z_m) \in \mathbf{R}^m$.

Any efficient solution x^0 of the problem $Z^n(C)$ is called *stable* if $\rho^n(x^0, C) > 0$, and *strongly efficient* if there does not exist $x \in X \backslash \{x^0\}$ such that $C_i x^0 \geq C_i x$. From corollary 2 we have

Corollary 2 *[1] Any efficient solution of vector linear Boolean programming problem is stable iff it is strongly efficient.*

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