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The mixed hypergraphs

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Abstract

We introduce the notion of an anti-edge of a hypergraph, which is a non-overall polychromatic subset of vertices. The maximal number of colors, for which there exists a coloring of a hypergraph using all colors, is called an upper chromatic number of a hypergraph H and denoted by $\bar{\chi}(H)$. The general algorithm for computing the numbers of all colorings of mixed (containing edge and anti-edge sets) hypergraphs is proposed. Some properties of mixed hypergraph colorings and its application are discussed.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, $\mathbf{S} = \{S_1, S_2, \ldots, S_k\}$ be a family of subsets of X, in particular, $\mathbf{S}$ may be empty. The couple $H = (X, \mathbf{S})$ is called a hypergraph on X if $\bigcup_{i=1}^{k} S_i \subseteq X$(cf. [1,3]) Now let hypergraph $H = (X, \mathbf{S}), \mid X \mid = n$ and $\mathbf{S} = \mathcal{A} \bigcup \mathcal{E}$; in particular, $\mathcal{E}$ and/or $\mathcal{A}$ may be empty. Let $\mathcal{A} = \{A_1, \ldots, A_k\}, I = \{1, \ldots, k\}, \mathcal{E} = \{E_1, \ldots, E_m\}, J = \{1, \ldots, m\}$. We arrange that indices i and j may run on different sets of indices and we shall indicate them each time in order to avoid confuse. Throughout this paper we consider the hypergraphs without loops, i.e. $\mid A_i \mid \geq 2, i \in I, \mid E_j \mid \geq 2, j \in J$.

Every $E_j, j \in J$, is called "an edge", and every $A_i, i \in I$, is called an "anti-edge" or sometimes "co-edge". We conditionally shall use the prefix "anti-" or "co-" always when a statement concerns the sets from $\mathcal{A}$. In particular, if $\mathcal{E} = \emptyset$, then $H = H_{\mathcal{A}}$ will be called a "co-hypergraph", and in order to emphasize that for hyperghraph H may be $\mathcal{A} \neq \emptyset$ and/or $\mathcal{E} \neq \emptyset$ we call H a *mixed hypergraph*. Other terminology not explained here is taken from [1,3]. Let us have $\lambda \geq 0$ colors.

Definition 1 *The free colouring of a mixed hypergraph H, where H $= (X, \mathcal{A} \bigcup \mathcal{E})$ with λ colours is the coloring of its vertices X in such a way that the following four conditions hold:*

1. *any anti-edge $A_i, i \in I$, has at least two vertices of the same color;*
2. *any edge $E_j, j \in J$, has at least two vertices colored differently;*
3. *the number of used colors is not greater than λ;*
4. *all the vertices are colored.*

Note that this definition of coloring generalizes all those contained in [3], that correspond to the case $\mathcal{A} = \emptyset$. Now we can say in other words that hypergraphs edges present indeed the non-monochromatic subsets, and the anti-edges present the non-overall polychromatic subsets of vertices. The colorings, when not all the vertices of a hypergraph must be colored, will be investigated separately.

Two free colorings of a hypergraph H are said to be different, if there exists at least one vertex that changes color when passing from one coloring to another. Let $P(H, \lambda)$ be the chromatic polynomial of a hypergraph H, that expresses for any $\lambda \geq 0$ the number of different free colorings of H with λ colors [cf.1].

Definition 2 *A free coloring of a hypergraph H with $i \geq 0$ colors is said to be a strict coloring, if exactly i colors are used.*

So, the strict colorings exist only for such i, that $1 \leq i \leq n$. Let us consider that the two strict colorings of H are called different if there exist a pair of vertices in H which have the same color for one of these colorings and different colors for the other (cf.[2]).

Definition 3 *The maximal i for which there exists a strict coloring of a mixed hypergraph H with i colors is called an upper chromatic number of H and denoted by $\bar{\chi}(H)$.*

Let $r_i(H)$ be the number of strict colorings of a hypergraph H with $i \geq 1$ colors (cf.[2]), $\chi(H)$ is a usual chromatic number of H. We associate the vector $R(H) = (r_1, r_2, \ldots, r_n) \in \mathbf{R}^n$ with hypergraph H and call it the *chromatic spectrum* of H; hence $R(H) = (0, \ldots, 0, r_\chi, \ldots, r_{\bar{\chi}}, 0, \ldots 0)$.

Definition 4 *The value $\chi_m(H) = (\chi(H) + \bar{\chi}(H))/2$ is called the middle chromatic number of a hypergraph H.*

Definition 5 *The value $b(H) = \bar{\chi} - \chi + 1$ is called the breadth of chromatic spectrum of H.*

Definition 6 *The mixed hypergraph H, in which at least one pair of vertices cannot be colored because of costraints collision is called uncolorable; we put for such hypergraph $\chi(H) = \bar{\chi}(H) = 0$.*

Consequently, if $\chi_m(H)$ is not integer it means that $b(H)$ is even. If $\mathcal{A} = \emptyset$, then $\bar{\chi}(H) = n$, and we have thus a usual hypergraph colorings. If $\mathcal{E} = \emptyset$, then $\chi(H) = 1$, and we have unusual colorings. Moreover, if $\mathcal{A} \neq \emptyset$ and $\mathcal{E} \neq \emptyset$, then one can easy construct for every n an uncolorable hypergraph H for which we supposed $\chi(H) = \bar{\chi}(H) = 0$; for example any complete graph $K_n, n \geq 2$ with at least one added anti-edge of cardinality ≥ 2 cannot be colored. Such cases may by treated also as the hypergraphs with $b(H) \leq 0$ and will be investigated also separately.

Now in order to calculate $P(H, \lambda)$ and $R(H)$ for any mixed hypergraph $H = (X, \mathcal{A} \bigcup \mathcal{E})$ we formulate the following 5 true rules:

1. if some subset $\mathcal{K} \subseteq \mathcal{E}$ induces a usual complete subgraph $F = (X_F, \mathcal{K})$ and $X_F = A_j$ for some $j \in J$, then H is declared uncolorable (*elimination*);
2. if $E_i \subseteq E_j, i, j \in J$, then $P(H, \lambda) = P(H - E_j, \lambda)$, $R(H) = R(H - E_j)$(*clearing*);
3. if $A_i \subseteq A_j, i, j \in I$, then $P(H, \lambda) = P(H - A_j, \lambda)$, $R(H) = R(H - A_j)$(*co-clearing*);

4. if $A_i = \{x_k, x_l\}$, for some $i \in I$ and $x_k, x_l \in X$, then

$$P(H,\lambda) = P(H_1,\lambda), \quad R(H) = R(H_1), where$$

$$H_1 = (X_1, \mathcal{A}^1 \bigcup \mathcal{E}^1), X_1 = (X \backslash \{x_k, x_l\}) \bigcup \{y\}, y$$

is a new vertex, and
if $x_k \in E_j$, or $x_l \in E_j$, then $E_j^1 = (E_j \backslash \{x_k, x_l\}) \bigcup \{y\}$, otherwise

$$E_j^1 = E_{j,} j \in J;$$

if $x_k \in A_i$, or $x_l \in A_i$, then $A_i^1 = (A_i \backslash \{x_k, x_l\}) \bigcup \{y\}$, otherwise

$$A_i^1 = A_i, i \in I(contraction);$$

5. if $\{x_k, x_l\} \notin \mathcal{E}$ and $\{x_k, x_l\} \notin \mathcal{A}$, then

$$P(H,\lambda) = P(H_1,\lambda) + P(H_2,\lambda), R(H) = R(H_1) + R(H_2), \text{ where}$$

$$H_1 = (X, \mathcal{A} \bigcup \mathcal{E}_1), \mathcal{E}_1 = \mathcal{E} \bigcup \{x_k, x_l\},$$

$$H_2 = (X, \mathcal{A}_1 \bigcup \mathcal{E}), \mathcal{A}_1 = \mathcal{A} \bigcup \{x_k, x_l\} (splitting).$$

We propose a general algorithm that gives the possibility to compute $P(H,\lambda)$ and $R(H)$ and extends the Zykov's connection-contraction algorithm [2,4]. The idea is to find any pair of vertices, that does not belong to the edge and co-edge sets, after that to split all colorings of H onto two classes relatively these vertices, and, implementing elimination, clearing, co-clearing and contraction (in such order), to reduce the initial problem recurrently to the same one for the new pair of hypergraphs. Finally we obtain the list of complete graphs. We call this algorithm the 'splitting-contraction algorithm' and present in the following form:

ALGORITHM 1 (splitting-contraction).
INPUT: an arbitrary mixed hypergraph $H = (X, \mathcal{A} \bigcup \mathcal{E})$;
OUTPUT: list Z of complete graphs;

STEP 0. Add the hypergraph H to the empty list Y, suppose $Z = \{\emptyset\}$.

STEP 1. Verify the condition of elimination for each hypergraph from Y; delete uncolorable hypergraphs from Y.

STEP 2. Implement clearing, co-clearing and contraction (in such order) for all hypergraphs from Y.

STEP 3. Implement one splitting in each hypergraph from Y, where possible; delete complete graphs from the list Y and include them in the list Z; if a splitting is implemented for at least one hypergraph, then go to step 1, else go to step 4.

STEP 4. Output list $Z = \{K_{n_1}, K_{n_2}, \ldots, K_{n_t}\}$ of complete graphs. End.[]

Although Algorithm 1 is exponential, it is possible for some classes of hypergraphs to find the polynomial and effective modifications.

Theorem 1 *For any mixed hypergraph* $H = (X, \mathcal{A} \bigcup \mathcal{E})$ *Algorithm 1 gives the possibility to find chromatic polynomial* $P(H, \lambda)$ *and chromatic spectrum* $R(H)$*, and the following equality holds:*

$$P(H, \lambda) = \sum_{i=\chi(H)}^{\bar{\chi}(H)} r_i(H) \lambda^{(i)}.$$

Proof. If α_i is the number of all complete i-vertex graphs in list Z, then it follows from the algorithm and rules 1)-5) that

$$P(H, \lambda) = \sum_{i=\chi(H)}^{\bar{\chi}(H)} \alpha_i P(K_i, \lambda).$$

Since rules 1) - 5) and a whole algorithm are equivalent for $P(H, \lambda)$ as well as for $R(H)$, we have also

$$r_j(H) = \sum_{i=1}^{n} \alpha_i r_j(K_i), j = 1, \ldots n.$$

Hence from

$$r_j(K_i) = \begin{cases} 1, & \text{i=j} \\ 0, & otherwise \end{cases}$$

conclude that $\alpha_i = r_i, i = 1, \ldots, n$. Thus theorem follows from $P(K_i, \lambda) = \lambda^{(i)} = \lambda(\lambda - 1) \ldots (\lambda - i + 1)$.[]

Theorem 1 shows that in the most general case the vector $R(H)$ uniquely determines the chromatic polynomial $P(H, \lambda)$, and vice versa. A common criterion is unknown for an arbitrary polynomial to be chromatic one for a graph or a hypergraph. It is seen that this problem presents a partial case of the more general one. One can state now that the class of polynomials that may be chromatic, is essentially larger especially because of interactions between edge and anti-edge sets. Such interactions are not simple and bring many new of principle properties of hypergraph colorings.

For example, for $H = (X, \mathcal{A} \bigcup \mathcal{E})$, where $X = \{1, 2, 3\}, \mathcal{A} = A_1 = \{1, 2, 3\}, \mathcal{E} = E_1 = \{1, 2, 3\}$, we have $Z = \{K_2, K_2, K_2\} = \{3K_2\}$, $P(H, \lambda) = 3\lambda^{(2)} = 3\lambda^2 - 3\lambda, R(H) = (0, 3, 0), \chi = \bar{\chi} = \chi_m = 2, b(H) = 1$ and the corresponding three colorings are the following: $(\alpha\alpha\beta), (\alpha\beta\alpha)$ and $(\beta\alpha\alpha)$.

Another example; it is evident that adding of one anti-edge to a hypergraph H can increase $\chi(H)$. It is less evident that adding of one edge to mixed hypergraph can decrease $\bar{\chi}(H)$. Let for example $H = (X, \mathcal{A} \bigcup \mathcal{E})$, where $X = \{1, 2, 3, 4, 5\}$, $\mathcal{A} = \{(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 2)\}$, $\mathcal{E} = \{(3, 5)\}$; we have $\bar{\chi}(H) = 3$, and after adding of the edge (2,4) receive the new hypergraph H_1, for which $\bar{\chi}(H_1) = 2$.

One more example: an unusual property of co-hypergraph colorings, that is impossible for hypergraphs with $\mathcal{A} = \emptyset$.

We say that an anti-edge A_i of a mixed hypergraph $H = (X, \mathcal{A} \bigcup \mathcal{E})$ is *dead* if A_i does not contain any other anti-edge, and $R(H) = R(H - A_i)$. One can see, for example, that any co-edge of a co-hypergraph $H = (X, \mathcal{A})$, where $X = (1, 2, 3, 4)$, $\mathcal{A} = \{(1, 2, 3), (1, 3, 4), (1, 2, 4), (2, 3, 4)\}$, is dead, because $R(H) = R(H - A_j) = (1, 7, 0, 0), j = 1, 2, 3, 4$.

We omit the evident bounds on $\bar{\chi}(H)$ following from definition and suppose that the important and perspective directions of research in this area would be the following:

1. Investigate the upper chromatic number of unimodular, balanced,

arboreal, co-arboreal, normal, mengerian, paranormal co- and mixed hypergraphs [3].

2. Find and investigate the antipodes of perfect graphs relatively to the $\bar{\chi}(H)$ [5].

3. Characterize the mixed hypergraphs with $\chi = \bar{\chi}$. []

CONJECTURE. For any sequence of positive numbers $N = (n_1, n_2, \ldots, n_t)$ such that $n_i \geq (n_{i-1} + n_{i+1})/2, \quad i = 2, \ldots, n-1$, and $\max\{n_{[t/2]}, n_{[(t+2)/2]}\} = max\{n_i\}$ there exists such a mixed hypergraph H that $n_1 = r_\chi, n_2 = r_{\chi+1}, \ldots, n_t = r_{\bar{\chi}}$.

APPLICATION. Let X be a set of sources of power supply for some discrete system $H = (X, \mathcal{E} \bigcup \mathcal{A})$, the acting time of any source be 1, at least two sources of every $E_i \in \mathcal{E}, i \in I$ must work at different time, and at least two sources of every $A_j \in \mathcal{A}, j \in J$ must work at the same time. How can we *schedule* entirely system H in such a way that the time of work (it may be treated as a time of life) of a system would be the longest one? The latter will be equal to $\bar{\chi}(H)$.

References

[1] C.Berge. Graphs and Hypergraphs. North Holland, 1973.

[2] A.A.Zykov. Hypergraphs (Russian), Uspekhi Mat. Nauk 29, 1974, 89-154.

[3] C.Berge. Hypergraphs: combinatorics of finite sets. North Holland, 1989.

[4] A.A.Zykov. The space of Graphs and its Factorizations. Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity, J.Nesetril and M.Fiedler (Editors), Elsevier Science Publishers B.V., 1992, 371-374.

[5] V.I.Voloshin. On the upper chromatic number of a hypergraph. Scientific research Conference of the Moldova State University. Theses of reports. Kishinev, 1992, V.1, p.42.

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