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Methods of solving of the optimal stabilization problem for stationary smooth control systems Part I

G. Kondrat'ev A. Balabanov

Abstract

In this article some ideas of Hamilton mechanics and differential-algebraic Geometry are used to exact definition of the potential function (Bellman-Lyapunov function) in the optimal stabilization problem of smooth finite-dimensional systems

1 Statement of the problem

For the smooth finite-dimensional system

$$\dot{x} = f(x,u),\ x \in R^n, u \in R^m, \quad f(0,0) = 0 \tag{1.1}$$

and integral functional

$$J = \int_0^{+\infty} \omega(x,u)dt,\ \omega(x,u) \geq 0,\ \omega(0,0) = 0 \tag{1.2}$$

it is required to find the function $u_{opt}(x)$, ensuring global stability in some maximum neighbourhood of the origin of coordinates of the system (1.1) and realizing a minimal functional (1.2) along each its trajectory. All functions are assumed as smooth and partial ones, defined in some maximal in each current context neighbourhood of the origin of coordinates of appropriate space.

According to the Bellman principle [1,2] the problem (1.1), (1.2) is reduced to solving of the functional equation

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$$min_u\{f^i(x,u)V_i + \omega(x,u)\} = 0 \tag{1.3}$$

where $(f^1(x,u),\ldots,f^n(x,u))^T = f(x,u)$ – right side of the equation (1.1), $V_i dx^i = dV$ – differential of the Bellman-Lyapunov function.

The equation (1.3) is equivalent to the system

$$f^i(x,u)V_i + \omega(x,u) = 0 \tag{1.4.a}$$

$$f^i_{u^j}(x,u)V_i + \omega_{u^j}(x,u) = 0 \tag{1.4.b}$$

Here and further the following conventions are applied: a partial derivative of the value A with respect to x is designated by an appropriate subscript A_x; summation is produced on a repeating index located at different levels of the multiindex value; a function is called as of fixed sign in some vicinity of zero if it is equal to zero in the origin of coordinates and saving the sign in an open dense set of the vicinity (if the function is equal to zero only in zero, it. For nondegeneracy of the Bellman-Lyapunov function $V(x)$ it is sufficient nondegeneracy of $\omega(x,u)|_{u=0}$.

In favorable case by exception of variables u^j, $j = 1,\ldots,m$, the system (1.4) is reduced to the Hamilton-Jacoby equation

$$\varphi(x^i, V_j) = 0 \tag{1.5}$$

The problem (1.1), (1.2) and the equations (1.3)-(1.5) were studied from different points of view by different authors [3-11]. Along with theoretical proof of the method, research of optimal control existence conditions, definition of adequate space, in which the solution always exists [3,9,10], methods of solving of the particular problems originating in mechanics, biology, industry etc. [4,12], and also classes of the problems for systems of the defined sort [5,6,13,14] were developing.

By first completely investigated type of systems (1.1), (1.2), accepting exact solution, was linearly-square systems

$$f^i(x,u) = A^i_j x^j + B^i_k u^k,$$

$$\omega(x,u) = C_{ik}y^iy^k = D_{ij}x^ix^j + E_{ik}x^iu^k + F_{kl}u^ku^l,$$

where $(y^1,\dots,y^{n+m}) = (x^1,\dots,x^n,u^1,\dots,u^m)$,
et matrixes $(C_{ij})_{i,j=1,\dots,n+m}$, $(D_{ij})_{i,j=1,\dots,n}$, $(F_{kl})_{k,l=1,\dots,m}$ – are positive definite.

For linearly-square systems the Bellman-Lyapunov function, satisfying (1.5), is quadratic, positive definite function, which second partial derivatives can be found by solving of an algebraic system of the second order equations (Riccaty system).

$$\varphi_{x^ix^k}(0,0)+\varphi_{x^iV_l}(0,0)V_{lk}+\varphi_{V_lx^k}(0,0)V_{li}+\varphi_{V_jV_l}(0,0)V_{lk}V_{ji}=0 \quad (1.6)$$

The system (1.6) has two real solutions – required positive- definite matrix (V_{ij}) and negative definite matrix $(\tilde{V}_{ij})$. $u_{opt}(x)$ is a linear function on Lagrangian manifold of the function $V(x)$.

Therefore, linearly-square problem (1.1), (1.2) completely is solvable. The following steps in construction of a general theory of systems (1.1), (1.2) and appropriate equations (1.3)-(1.5), increasing polynomial dimensionality of functions f and ω and introduction of the special kind of nonlinearities reflected in [4,6,13,14,15,16], represent a stage of accumulation and transition to the following logical step – theory of analytical and smooth systems (1.1), (1.2) [11,12,14,17, 18,19].

However, main methods of synthesis of an optimal feed-back for nonlinear systems until now remain: a method of substitution of the source system by linearly-square approximation in a neighbourhood of singular point [6,7, 15,16,20,21], method of splitting of a neighbourhood of the origin of coordinates of the state space on rather small blocks with consequent pasting together of found on them with the Bellman principle of an optimality, $u_{opti}, i \in I$, in the uniform synthesis method of an optimal feed-back of a specific structure $u_{opt}(x,\alpha), \alpha$ – optimized parameters [14,20,21]. The work [19], in which the optimum control is restored on n to optimal control in the sense of a specially given criterion is selected from the class of stabilizing actions is interesting.

Surprisingly, that the theory of optimum control, evolved from an analytical mechanics, does not use at all methods of the last and also of differential geometry, traditionally connected to analytical mechanics. The given article to some degree completes it.

In section 2 the classical methods of Hamilton mechanics and possibility of calculation of coefficients of Taylor series of the Bellman-Lyapunov function in a neighbourhood of the singular point are considered, the way of construction of a evolutionary equation which allows to reduce algebraic solution of the Hamilton-Jacoby equation in holonomic one is given.

In section 3 some tools of differential-algebraic geometry are applied: the class of systems with invariant foliation of the Bellman-Lyapunov function, accepting exact solution of the problem of the optimal stabilization is described; the differential in variants of the Bellman-Lyapunov function of linearly-square problem are calculated; the definition of the nondegenerate potential function as equidistant function of Euclidean space is given; the way of obtaining of differential invariants with the help of suitable isomorphism of differential algebra is considered; the algebraic structure of the first integrals and separatrices of the Hamilton system is analyzed, because of that the way of calculation of the Lagrangian manifold of the potential function is formulated.

In section 4 the symmetries in the the optimal stabilization problem are considered.

In section 5 the heuristic algorithm of synthesis of suboptimal control of a given structure is offered and some additional facts able to be useful in synthesis problem are considered.

The obtained results represent a basis of the invariant theory of smooth optimal control systems.

2 Methods of mechanics and analysis

It is supposed that matrix $(f^i_{u^k u^l} V_i + \omega_{u^k u^l})|_{0 \in T^*R^n \times R^m}$, where T^*R^n – phase space, is nondegenerate, that is in some neighbourhood of the origin of coordinates T^*R^n there is uniquely defined function $u_{opt} = u_{opt}(x^i, V_j)$, satisfying (1.4.b) at which substitution in (1.4.a) the Hamilton-Jacoby equation (1.5) is obtained. In this case problem of the optimal stabilization is reduced to search of the equation $+ : V_i = V_i(x)$, $i = 1, \ldots, n$, of the Bellman-Lyapunov function $V(x)$. For the system (1.1), (1.2), accepting in the origin of coordinates non-degenerate linearly-square approximating (system of expression of the functional is square positive definite form), Lagrangian manifold L represents the separatrix of steady points of the Hamilton system

$$\begin{cases} \dot{x}^i = \varphi_{V_i}(x^k, V_l) \\ \dot{V}_i = -\varphi_{x^i}(x^k, V_l) \end{cases} \tag{2.1}$$

The system (2.1) has also separatrix L^- of unstable points, being Lagrangian manifold of accompanying solution (1.5) negative definite function $\tilde{V}$.

2.1 Solving of Cauchy problem for the Hamilton system

It is known [32,33,34], that association of trajectories of the Hamilton system (2.1), passing through $(n-1)$-dimensional Lagrangian manifold $L_0^{n-1} \subset \varphi^{-1}(0)$ transversally to trajectories of the Hamilton system, is the n-dimensional Lagrangian manifold $L^n \subset \varphi^{-1}(0)$. Unfortunately we can not directly take advantage of this fact for construction of the separatrix of steady points L^+ according to condition that only one point $(0, 0) \in L^+$ is known. Nevertheless we can use the following procedure:

- to calculate (linear) Lagrangian manifold L^+:

  $V_i = \alpha_{ij} x^j$, $\alpha_{ij} \in R$, of linear approximation in the origin of coordinates of the system (2.1) (α_{ij} is a solution of the Riccaty system (1.6));

- to count evolution of initial $(n-1)$-dimensional Lagrangian manifold $L_0^{n-1} = \{(x^i, V_j) \in T^* R^n \mid V_i = \alpha_{ij} x^j,\ (x^1)^2 + \ldots + (x^n)^2 = \varepsilon,\ \varepsilon > 0$ is small real number$\}$ along Hamilton vector field (2.1) in the opposite direction.

The approximate local parametric representation of the manifold L^+ for the analytical Hamilton system (2.1) is

$$x^i(\tau, c_2, \ldots, c_n) = (\exp(\tau Id\varphi)x^i)\Big|_{x^i=x^i(c_2,\ldots,c_n)} = ((1 + \frac{\tau}{1!}(Id\varphi) +$$

$$+\frac{\tau^2}{2!}(Id\varphi)^2 + \ldots)x^i)\Big|_{x^i=x^i(c_2,\ldots,c_n)};$$

$$V_j(\tau, c_2, \ldots, c_n) = (\exp(\tau Id\varphi)V_j)\Big|_{V_j=V_j(c_2,\ldots,c_n)} = ((1 + \frac{\tau}{1!}(Id\varphi) +$$

$$+\frac{\tau^2}{2!}(Id\varphi)^2 + \ldots)V_j)\Big|_{V_j=V_j(c_2,\ldots,c_n)};$$

where $Id\varphi \equiv \varphi_{V_i}\frac{\partial}{\partial x^i} - \varphi_{x^j}\frac{\partial}{\partial V_j}$ – Hamilton vector field;

$$\begin{cases} x^1(c_2,\ldots,c_n) = \frac{-2\sqrt{\varepsilon}}{(1+(c_2)^2+\ldots+(c_n)^2)} + \sqrt{\varepsilon} \\ x^2(c_2,\ldots,c_n) = \frac{-2\sqrt{\varepsilon}c_2}{(1+(c_2)^2+\ldots+(c_n)^2)} \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ x^n = (c_2,\ldots,c_n) = \frac{-2\sqrt{\varepsilon}c_n}{(1+(c_2)^2+\ldots+(c_n)^2)} \end{cases}$$

– rational parameterization of the sphere $S^{n-1}_{\varepsilon}((x^1)^2+\ldots+(x^n)^2 = \varepsilon)$ with the remote point $(\sqrt{\varepsilon}, 0, \ldots, 0)$;

$$V_i(c_2,\ldots,c_n) = \alpha_{ij}x^j(c_2,\ldots,c_n); \quad \tau, c_2, \ldots, c_n \in R;$$

for each analytical function $F(x^i, V_j)$ the series $(\exp(\tau Id\varphi)F(x^i, V_j))$ converges in some neighbourhood of the point $\tau = 0$.

2.2 Method of the first integrals

In a neighbourhood of a nonspecial point the vector field (2.1) has $(2n-1)$ functionally independent first integrals, commuting concerning a Poisson bracket with the Hamiltonian

$$(\varphi, \psi) = 0 \qquad (2.2.a)$$

that is satisfying to the linear homogeneous partial equation of the first order

$$\varphi_{V_i}\psi_{x^i} - \varphi_{x^j}\psi_{V_j} = 0 \qquad (2.2.b)$$

In a neighbourhood of the singular point of the system (2.1) exists no more than n functionally independent integrals (though in addition $L^+ \bigcup L^-$ the number of functionally independent integrals can be more than n). In any case, if $\{I_a\}$, $a \in A$, is some set of integrals of the system (2.1), at an approaching choice of constants c_a, $a \in A$, functions $(I_a - c_a)$ belong to an (reduced) ideal of the manifold $L^+ \bigcup L^-$. Actually, for the definition of Lagrangian manifold it is enough to have n functionally independent integrals, pairwise commuting concerning Poisson bracket.

By common algebraic solution (above a ring $C^\infty(T^*R^n)$ of smooth functions) of equation

$$\varphi_{V_i}a_i - \varphi_{x^j}b^j = 0 \qquad (2.2.c)$$

associated with the equation (2.2.c), is

$$\begin{aligned} a_i &= A_{ij}\varphi_{V_j} - B_i^k\varphi_{x^k} \\ b^j &= C_k^j\varphi_{V_k} - D^{jl}\varphi_{x^l} \end{aligned} \qquad (2.3)$$

where A_{ij}, $D^{jl} \in C^\infty(T^*R^n)$ are skew-symmetric matrixes, $B_i^k \in C^\infty(T^*R^n)$ – any matrix, $C_k^j = -B_j^k$.

The solutions of the equation (2.2.b) will be the closed 1-forms

$$\alpha = a_i dx^i + b^j dV_j = (A_{ij}\varphi_{V_j} - B_i^k\varphi_{x^k})dx^i + (C_k^j\varphi_{V_k} - D^{jl}\varphi_{x^l})dV_j,$$

$$d\alpha = 0 \qquad (2.4)$$

that is elements of matrixes A, B, C, D should satisfy following partial equations of the first order

$$A_{ijx^t}\varphi_{V_j} + A_{ij}\varphi_{V_jx^t} - B_{ix^t}^k\varphi_{x^k} - B_i^k\varphi_{x^kx^t} =$$

$$A_{tjx^i}\varphi_{V_j} + A_{tj}\varphi_{V_jx^i} - B_{tx^i}^k\varphi_{x^k} - B_t^k\varphi_{x^kx^i}$$

$$A_{ijV_s}\varphi_{V_j} + A_{ij}\varphi_{V_jV_s} - B^k_{iV_s}\varphi_{x^k} - B^k_i\varphi_{x^kV_s} =$$
$$C^s_{kx^i}\varphi_{V_k} + C^s_k\varphi_{V_kx^i} - D^{sl}_{x^i}\varphi_{x^l} - D^{sl}\varphi_{x^lx^i}$$

$$C^j_{kV_i}\varphi_{V_k} + C^j_k\varphi_{V_kV_i} - D^{jl}_{V_i}\varphi_{x^l} - D^{jl}\varphi_{x^lV_i} =$$
$$C^i_{kV_j}\varphi_{V_k} + C^i_k\varphi_{V_kV_j} - D^{il}_{V_j}\varphi_{x^l} - D^{il}\varphi_{x^lV_j} \tag{2.5}$$

The effective solving of the last system without any additional suppositions is undefined. Such suppositions can be some functional or differential dependences between elements of matrices A, B, C, D. For linear equation (2.1) it is natural to look for integrals of movement as having quadratic form. In this case A_{ij}, B^k_i, C^j_k, D^{jl} are constants and the solving of the equations (2.5) does not cause difficulties (they are reduced to the system of linear homogeneous algebraic equations).

Example 1.

$$\begin{cases} \dot{x}^1 = x^1 - x^2 \\ \dot{x}^2 = x^3 \\ \dot{x}^3 = u \end{cases} \qquad J = \int\limits_0^\infty ((x^1)^2 + u^2)dt$$

$u_{opt} = -\frac{1}{2}V_3$ – optimum control on the Lagrangian surface L^+ ;
$\varphi = (x^1)^2 + x^1V_1 - x^2V_1 + x^3V_2 - \frac{1}{4}(V_3)^2$ – Hamiltonian;
$Id\varphi = (x^1 - x^2)\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - \frac{1}{2}V_3\frac{\partial}{\partial x^3} - (2x^1 + V_1)\frac{\partial}{\partial V_1} + V_1\frac{\partial}{\partial V_2} - V_2\frac{\partial}{\partial V_3}$ is Hamilton vector field .

$$\begin{pmatrix} \psi_{x^1} \\ \psi_{x^2} \\ \psi_{x^3} \\ \psi_{V_1} \\ \psi_{V_2} \\ \psi_{V_3} \end{pmatrix} = \begin{pmatrix} 0 & a_2 & a_3 & b_1 & b_2 & b_3 \\ -a_2 & 0 & a_4 & b_4 & b_5 & b_6 \\ -a_3 & -a_4 & 0 & b_7 & b_8 & b_9 \\ -b_1 & -b_4 & -b_7 & 0 & c_2 & c_3 \\ -b_2 & -b_5 & -b_8 & -c_2 & 0 & c_4 \\ -b_3 & -b_6 & -b_9 & -c_3 & -c_4 & 0 \end{pmatrix} \begin{pmatrix} x^1 - x^2 \\ x^3 \\ -\frac{1}{2}V_3 \\ -(2x^1 + V_1) \\ V_1 \\ -V_2 \end{pmatrix}$$

where ψ – required first integral; a, b, c with indexes are constants.

Taking into account relations of equality of the second mixed derivative from ψ , we shall obtain the intermediate system of the linear algebraic equations concerning a, b, c, which solution is the 3-parametric family of matrices of coefficients of the noted above system:

$$\begin{pmatrix} 0 & -2b_7 & 0 & (-b_7+b_9) & 0 & 2c_3 \\ 2b_7 & 0 & -4c_3 & b_7 & b_9 & 0 \\ 0 & 4c_3 & 0 & b_7 & 0 & b_9 \\ (b_7-b_9) & -b_7 & -b_7 & 0 & -c_3 & c_3 \\ 0 & -b_9 & & c_3 & 0 & (c_3-\frac{1}{2}b_7) \\ -2c_3 & 0 & -b_9 & -c_3 & (\frac{1}{2}b_7-c_3) & 0 \end{pmatrix}$$

where b_7, b_9, c_3 – independent parameters.

Integrating the form

$$d\psi = \psi_{x^1}dx^1 + \psi_{x^2}dx^2 + \psi_{x^3}dx^3 + \psi_{V_1}dV_1 + \psi_{V_2}dV_2 + \psi_{V_3}dV_3 \ ,$$

we obtain a 3-parametric family of the first integrals:

$$\psi = b_7((x^1)^2 - (x^2)^2 + \frac{1}{4}(V_2)^2 - 2x^1x^3 + x^1V_1 - x^2V_1 - x^3V_1 + \frac{1}{2}V_1V_3) +$$
$$+ b_9(-(x^1)^2 + \frac{1}{4}(V_3)^2 - x^1V_1 + x^2V_1 - x^3V_2) +$$
$$+ c_3(2(x^3)^2 - \frac{1}{2}(V_1)^2 - \frac{1}{2}(V_2)^2 - 2x^1V_2 + 2x^2V_3 - V_1V_2) \equiv$$
$$\equiv b_7\psi_1 + b_9\psi_2 + c_3\psi_3,$$

where $\psi_1, \psi_2 \equiv -\varphi, \psi_3$ – independent integrals of the Hamilton vector field $Id\varphi$, defined on all, T^*R^3 .

Equations $\psi_1 = 0, \psi_2 = 0, \psi_3 = 0$ determine $L^+ \bigcup L^-$.

The methods of integrals searching of the system (2.1) represent the classical problem of analytical mechanics. From traditional methods it is possible to mark the full integral method [35,36], from modern – method L-A-pare and method of orbits of a co-adjoint representation of group [37,38,39].

2.3 Decomposition of the Bellman-Lyapunov function in Taylor series in a neighbourhood of the origin of coordinates

From consideration of differential continuations structure of the Hamilton-Jacoby equation (1.5)

$$D_{x^i}(\varphi) = \varphi_{x^i} + V_{ij}\varphi_{V_j} = 0 \qquad (2.6.a)$$

$$D_{x^k} \circ D_{x^i}(\varphi) = \varphi_{x^i x^k} + V_{ijk}\varphi_{V_j} + V_{ij}\varphi_{V_j x^k} + V_{kl}\varphi_{V_l x^i} + \\ + V_{kl}V_{ij}\varphi_{V_j V_l} = 0 \qquad (2.6.b)$$

$$D_{x^s} \circ D_{x^k} \circ D_{x^i}(\varphi) = D_{x^s}(\varphi_{x^i x^k}) + V_{iksj}\varphi_{V_j} + V_{ikj}D_{x^s}(\varphi_{V_j}) + V_{isj}\varphi_{V_j x^k} + \\ + V_{ij}D_{x^s}(\varphi_{V_j x^k}) + V_{ksl}\varphi_{V_l x^i} + V_{kl}D_{x^s}(\varphi_{V_l x^i}) + V_{ksl}V_{ij}\varphi_{V_j V_l} + V_{kl}V_{isj}\varphi_{V_j V_l} + \\ + V_{kl}V_{ij}D_{x^s}(\varphi_{V_j V_l}) = 0 \quad \text{etc.} \qquad (2.6.c)$$

where $D_{x^i} = \frac{\partial}{\partial x^i} + V_{ij}\frac{\partial}{\partial V_j}$, we can see that in the case of existing of the Bellman-Lyapunov function the coefficients of it decomposition in Taylor series in the origin of coordinates are determined by the system (2.6) uniquely. For each subsystem (2.6), defining a p-jet, $j_0^p(V)$, the number of equations coincides with the number of unknown variables (derivatives φ_{x^i}, φ_{V_j} by condition are equal 0). Flexons V_{ij} are calculated from the quadratic Riccati system (2.6.b) and condition of positive determinancy of the matrix (V_{ij}) , derivatives of the function V, beginning with third order, are defined from the system of the linear inhomogeneous equations. If the function V is analitical it can be represented by Taylor decomposition

$$V = \sum_{i^1,\ldots,i^n=0}^{\infty} \frac{1}{i^1!\ldots i^n!} V_{(i^1,\ldots,i^n)}(0)(x^1)^{i^1}\ldots(x^n)^{i^n} \qquad (2.7)$$

where

$$V_{(i^1,\ldots,i^n)}(0) = \left.\frac{\partial^{(i^1+\ldots+i^n)}V}{\partial(x^1)^{i^1}\ldots\partial(x^n)^{i^n}}\right|_{x=0}$$

In spite of the fact that the jet $j_0^\infty(V)$ is given by the system (2.6) in implicit kind, the solution of any finite subsystem (2.6) by use of computer algebra does not represent difficulties (there is an effective algorithm of solving of such system).

Example 2 (Duffing equation, rigid spring).

$$\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = -x^1 - 2(x^1)^3 + u \end{cases} \qquad J = \int_0^\infty ((x^1)^2 + (x^2)^2 + u^2)dt$$

$u_{opt} = -\frac{1}{2}V_2$ – optimal control on L^+ ;

$\varphi = (x^1)^2 + (x^2)^2 - \frac{1}{4}(V_2)^2 - (2(x^1)^3 + x^1)V_2 + x^2V_1 = 0$ – Hamilton-Jacoby equation.

Let's write initial defining equations to calculate derivatives of the Bellman-Lyapunov function in the origin of coordinates up to 4 order (indexes at φ mean the full derivatives in respect to appropriate variables):

$$\begin{cases} \varphi_{11}|_0 = 2 - \frac{1}{2}(V_{12})^2 - 2V_{12} = 0 \\ \varphi_{12}|_0 = -\frac{1}{2}V_{22}V_{12} - V_{22} + V_{11} = 0 \\ \varphi_{22}|_0 = 2 - \frac{1}{2}(V_{22})^2 + 2V_{12} = 0 \end{cases}$$

Positive definite solution of the given system is the matrix with elements:

$$V_{11} = 2\sqrt{4\sqrt{2} - 2},\ V_{12} = 2\sqrt{2} - 2,\ V_{22} = 2\sqrt{2\sqrt{2} - 1}.$$

$$\begin{cases} \varphi_{111}|_0 = \frac{3}{2}(2 - 2\sqrt{2})V_{112} - 3V_{112} = 0 \\ \varphi_{112}|_0 = -2\sqrt{2}V_{122} - \sqrt{2\sqrt{2} - 1}V_{112} + V_{111} = 0 \\ \varphi_{122}|_0 = -\sqrt{2\sqrt{2} - 1}V_{122} - \sqrt{2}V_{222} = 0 \\ \varphi_{222}|_0 = -3\sqrt{2\sqrt{2} - 1}V_{222} + 3V_{122} = 0 \end{cases}$$

This linear system has a unique trivial solution:

$$V_{111} = V_{112} = V_{122} = V_{222} = 0.$$

$$\begin{cases} \varphi_{1111}|_0 = -4\sqrt{2}V_{1112} + (96 - 96\sqrt{2}) = 0 \\ \varphi_{1112}|_0 = V_{1111} - 3\sqrt{2}V_{1122} + (36\sqrt{2} - 48) - 24\sqrt{2\sqrt{2} - 1} = 0 \\ \varphi_{1122}|_0 = -2\sqrt{2}V_{1222} - 2\sqrt{2\sqrt{2} - 1}V_{1122} + 24\sqrt{2} - 48 = 0 \\ \varphi_{1222}|_0 = 3V_{1122} - 3\sqrt{2\sqrt{2} - 1}V_{1222} - \sqrt{2}V_{2222} = 0 \\ \varphi_{2222}|_0 = -4\sqrt{2}V_{1112} - 48(2\sqrt{2} - 2) = 0 \end{cases}$$

Solution of the last system is:

$$V_{1111} = \frac{1}{573}((636 - 573\sqrt{2})\sqrt{2\sqrt{2} - 1} + 1312 - 44\sqrt{2}),$$

$$V_{1112} = 12\sqrt{2} - 24,$$

$$V_{1122} = \frac{1}{191}(137 - 3\sqrt{2}),$$

$$V_{1222} = \frac{1}{191}((-30\sqrt{2} - 9)\sqrt{2\sqrt{2} - 1} + 20 + 3\sqrt{2}),$$

$$V_{2222} = \frac{1}{191}(30\sqrt{2} + 9).$$

We obtain an initial segment of Taylor series of the Bellman-Lyapunov function:

$$V = \frac{1}{2}V_{11}(x^1)^2 + V_{12}x^1x^2 + \frac{1}{2}V_{22}(x^2)^2 + \frac{1}{24}V_{1111}(x^1)^4 + \frac{1}{6}V_{1112}(x^1)^3x^2 +$$

$$+\frac{1}{4}V_{1122}(x^1)^2(x^2)^2 + \frac{1}{6}V_{1222}x^1(x^2)^3 + \frac{1}{24}V_{2222}(x^2)^4 + \dots$$

2.4 Evolutionary equation

There is a possibility to deformate smoothly any smooth cut

$$S_0 : R^n \to T^*R^n \bigcap \varphi^{-1}(0) : (x^i) \mapsto (x^i, S_0^j(x^k))$$

with positive main minors of the Jacoby matrix $(S^j_{0x^k})$, passing through, $0 \in T^*R^n$, of stratification $T^*R^n \bigcap \varphi^{-1}(0) \to R^n$ on a semi-infinite segment $[0, \infty]$, remaining in the class of cuts of an indicated stratification, in holonomic (representing in this case Lag) along an approaching evolutionary vector field

$$\frac{\partial}{\partial \tau} V_i = \Phi_i(x^j, V_{lx^k}, V_{mx^s x^t})$$

$$.. : \; V_i(x,\tau)\Big|_{\tau=0} = S_0^i(x) \tag{2.8}$$

where

$$\sum_{i=0}^{n} \Phi_i \frac{\partial}{\partial V_i} \text{ – projection on tangent stratification}$$

$$T(T^* R^n \bigcap \varphi^{-1}(0)) \text{ of evolutionary vector field}$$

$$Q = \sum_{k=1}^{n} \sum_{i<j} V_{lx^k}(V_{ix^j} - V_{jx^i})(V_{ix^j x^k} - V_{jx^i x^k}) \frac{\partial}{\partial V_l}$$

appropriate to minus - gradient, $(-grad\pi)$ of function

$$\pi = \frac{1}{2} \sum_{i<j} (V_{ix^j} - V_{jx^i})^2, \; that \; is$$

$$\sum_{i=1}^{n} \Phi_i \frac{\partial}{\partial V_i} = \sum_{i,j=1}^{n} \beta_{ij} \varphi_{V_j} \frac{\partial}{\partial V_i}, \; \beta_{ij} = -\beta_{ji} \in C^\infty(T^* R^n),$$

β_{ij} – any solution of the inequality (scalar product is more than zero)

$$\sum_{i,j,k=1}^{n} \sum_{s<t} \beta_{ij} \varphi_{V_j} V_{ix^k}(V_{sx^t} - V_{tx^s})(V_{sx^t x^k} - V_{tx^s x^k}) > 0, \; if \; (x^i, V_j) \neq 0.$$

Lagrange manifold of a nondegenerate Bellman-Lyapunov function is limiting solution at $\tau \to \infty$ equations (2.8). The more skillful methods of deformation of algebraic solutions of the differential equations in holonomic can be found in fundamental work[40].

3 Geometrical methods

In the given section the greater attention is given to invariant description of the potential function and its Lagrange manifold.

3.1 Class of system with invariant foliation of Bellman-Lyapunov function

System (1.1), (1.2) has invariant foliation of the Bellman-Lyapunov function, if the optimal stream in the state space keeps invariant surfaces of a level of the Bellman-Lyapunov function. The class of such systems is not empty and has good algebraic description. For systems of this class the following sentence takes place [41].

Subalgebra generated by the Bellman-Lyapunov function in algebra of smooth functions on R^n is differential concerning an infinitesimal operator defining optimal stream.

Let's designate:
$\varphi(x^i, V_j)$ – Hamiltonian,
$(-\omega(x^i, V_j))$ – Lagrangian ($\varphi \equiv \varphi_{V_i} V_i + \omega$),
$Id\varphi$ – Hamilton vector field,
$L^{(k)}_{Id\varphi}$ – k-th Lie derivative along $Id\varphi$ ($k = 0, 1, 2, \ldots$),
$i_{Id\varphi}$ – substitution operation of Hamilton field into form.
Sequences of the 2-forms

$$d\omega \wedge dV,\ d(L_{Id\varphi}\omega) \wedge dV, \ldots, d(L^{(k)}_{Id\varphi}\omega) \wedge dV, \ldots$$

and 1-forms

$$i_{Id\varphi}(d\omega \wedge dV),\ i_{Id\varphi}(d(L_{Id\varphi})\ wedgedV), \ldots, i_{Id\varphi}(d(L^{(k)}_{Id\varphi}\omega) \wedge dV), \ldots$$

are contained in a differential ideal of the Lagrangian manifold of the Bellman-Lyapunov function.

Sequences of the 2-forms and 1-forms from the previous sentence completely characterize a pair of Lagrange manifolds $L^{+} \bigcup L^{-}$ in a phase space, just, separatrix of points that are stable concerning the origin of coordinates of the Hamilton vector field $Id\varphi$ and separatrix of points that are unstable concerning the origin of coordinates.

Stated facts can be used for exact calculation of an optimal feedback of the optimal stabilization problem for the defined above class of systems.

Example 3.

$$\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = u \end{cases} \qquad J = \int\limits_0^\infty ((x^1)^2 + u^2)dt$$

$min_u\{x^2V_1 + uV_2 + (x^1)^2 + u^2\} = 0$ – Bellman equation.
$u_{opt} = -\frac{1}{2}V_2$
$\varphi = x^2V_1 - \frac{1}{4}(V_2)^2 + (x^1)^2$ – Hamiltonian of system.
$\omega = \varphi - \varphi_{V_i}V_i = (x^1)^2 + \frac{1}{4}(V_2)^2$ – minus Lagrangian.
$Id\varphi = \varphi_{V_i}\frac{\partial}{\partial x^i} - \varphi_{x^j}\frac{\partial}{\partial V_j} = x^2\frac{\partial}{\partial x^1} - \frac{1}{2}V_2\frac{\partial}{\partial x^2} - 2x^1\frac{\partial}{\partial V_1} - V_1\frac{\partial}{\partial V_2}$ – Hamilton vector field.
$\omega^{(1)} = L_{Id\varphi}\omega = 2x^1x^2 - \frac{1}{2}V_1V_2,\ \omega^{(2)} = L^{(2)}_{Id\varphi}\omega = 2(x^2)^2 + \frac{1}{2}(V_1)^2$ – first and second Lie derivatives from ω along $Id\varphi$.

Let's assume $\omega^{(1)} = A\omega,\ \omega(2) = B\omega,\ \varphi = 0,\ A, B\ -\ const.$
Solving the received system of the square equations concerning V_1, V_2, we shall obtain

$$\begin{cases} V_1 = \pm 2\sqrt{2}x^1 + 2x^2 \\ V_2 = 2x^1 \pm 2\sqrt{2}x^2 \end{cases}$$

Sign "+" corresponds to the Bellman-Lyapunov function .
We find $u_{opt} = -x^1 - \sqrt{2}x^2$.

Example 4 [14, page 47].

$$\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = sinx^1 + x^3 \\ \dot{x}^3 = u \end{cases} \qquad J = \int\limits_0^\infty ((x^1+x^2+x^3)^2 + (x^2+sinx^1+x^3+u)^2)dt$$

$$min_u\{x^2V_1 + (sinx^1 + x^3)V_2 + uV_3 + (x^1 + x^2 + x^3)^2 +$$

$$(x^2 + sinx^1 + x^3 + u)^2\} = 0 \text{ – the Bellman equation.}$$

$$u_{opt} = -\frac{1}{2}V_3 - x^2 - sinx^1 - x^3$$

$$\varphi = x^2V_1 + (sinx^1 + x^3)V_2 - (x^2 + sinx^1 + x^3)V_3 - \frac{1}{4}V_3^2 +$$

$$+(x^1 + x^2 + x^3)^2 \text{ – Hamiltonian.}$$

$$\omega = (x^1 + x^2 + x^3)^2 + \frac{1}{4}V_3^2 \text{ – Lagrangian.}$$

$$Id\varphi = x^2\frac{\partial}{\partial x^1} + (sinx^1 + x^3)\frac{\partial}{\partial x^2} - (x^2 + sinx^1 + x^3 + \frac{1}{2}V_3)\frac{\partial}{\partial x^3} - (cosx^1V_2 -$$

$$-cosx^1V_3 + 2x^1 + 2x^2 + 2x^3)\frac{\partial}{\partial V_1} - (V_1 - V_3 + 2x^1 + 2x^2 + 2x^3)\frac{\partial}{\partial V_2} - (V_2 - V_3 +$$

$$+2x^1 + 2x^2 + 2x^3)\frac{\partial}{\partial V_3} \text{ – Hamilton vector field.}$$

Let us assume, $KV = \omega$, K – $const$. If V – polinomial then V has a kind $V = C(x^1 + x^2 + x^3)^2$, C – $const$. Let's substitute V_1, V_2, V_3 in the equation $\varphi = 0$ and in result we shall obtain $C = \pm 1$ ("+" corresponds to the Bellman function).

It is possible to check up that functions $V_1 = V_2 = V_3 = \pm 2(x^1 + x^2 + x^3)$ satisfy to the equations $\omega^{(1)} = A\omega$, $\omega^{(2)} = B\omega$, A, B – $const$, where

$$\omega^{(1)} = L_{Id\varphi} = \frac{1}{2}(V_3)^2 - \frac{1}{2}V_2V_3 - 2V_3(x^1 + x^2 + x^3),$$

$$\omega^{(2)} = L^{(2)}_{Id\varphi}\omega = \frac{3}{2}(V_3)^2 + \frac{1}{2}V_1V_3 - 3V_3(x^1 + x^2 + x^3) - \frac{3}{2}V_2V_3 + \frac{1}{2}(V_2)^2 +$$

$$+3V_2(x^1 + x^2 + x^3) + 4(x^1 + x^2 + x^3)^2$$

– first and second Lie derivatives ω along $Id\varphi$.

We find $u_{opt} = -x^1 - 2x^2 - 2x^3 - sinx^1$.

3.2 Systems with Hamiltonian admitting group of symmetries along a level surface of the decision of the Hamilton-Jacoby equation

There are joint differential invariants of Hamiltonian φ and function $V(x)$, being a solution of the equation $\varphi = 0$. Let's calculate differential invariants of the first order by Laptev G.F. method [42,43].

On space of coframe stratification $H^{\infty}(R^n)$ over R^n there is a sequence of the 1-forms invariant concerning smooth automorphisms $H^{\infty}(R^n) \to H^{\infty}(R^n)$:

$$\omega^i = x^i_j dx^j,$$

$$\omega^i_j = d\tilde{x}^k_j x^i_k - x^i_{jk}\omega^k,$$

$$\omega^i_{jk} = dx^i_{jk} - x^i_{jl}\omega^l_k - x^i_{lk}\omega^l_j + x^l_{jk}\omega^i_l + x^m_{jk}x^i_{ml}\omega^l - x^i_{jkl}\omega^l, \text{etc.} \quad (3.1)$$

where

$$\tilde{x}^i_j x^j_k = \delta^i_k.$$

Let's make decomposition dV *and* $d\varphi$ along the invariant forms (3.1):

$$dV = V_i dx^i = V_i \tilde{x}^i_j \omega^j; \; I_j \equiv V_i \tilde{x}^i_j;$$

$$dV_l = dI_j x^j_l + I_j dx^j_l = dI_j x^j_l - I_j x^k_l \omega^j_k - I_j x^t_l x^j_{tk}\omega^k;$$

$$d\varphi = (\varphi_{x^i}\tilde{x}^i_k - \varphi_{V_l} I_j x^t_l x^j_{tk})\omega^k + \varphi_{V_l} x^j_l dI_j - \varphi_{V_l} I_j x^k_l \omega^j_k;$$

$$J^k_j \equiv \varphi_{V_l} I_j x^k_l; \; J^j \equiv \varphi_{V_l} x^j_l; \; J_k \equiv \varphi_{x^i}\tilde{x}^i_k - \varphi_{V_l} I_j x^t_l x^j_{tk};$$

I_j; J^k_j; J^j; J_k – differential invariants of the first order with additional foliation parameters. These invariants are dependent among themselves and therefore at their evaluation in initial algebra $C^{\infty}(T^*R^n)$ there will be nontrivial base invariants. Actually, there is one independent nontrivial joint differential invariant of functions φ and V, obtained by convolution J^k_j on the upper and lower indexes with regard of values I_j .

$$J \equiv J^j_j = \varphi_{V_l} I_j x^j_l = \varphi_{V_l} V_l$$

If the Hamiltonian φ admits group of symmetries with orbits on R^n $V = const$, then $J = \varphi_{V_l} V_l = F(V)$ – smooth function from V.

From here immediately follows, that the class of systems with invariant foliation of the Bellman-Lyapunov function is contained in a class of systems with Hamiltonian admitting group of symmetries along a level surface of a solution of the Hamilton-Jacoby equation.

4 Differential invariants of a quadratic Bellman-Lyapunov function

Let's calculate differential invariants of smooth function relatively to the standard operation of full linear group $GL(n, R)$ on R^n

$$GL(n, R) \times R^n \to R^n : (A, x) \mapsto Ax.$$

Let's raise function $V(x)$ with the help of canonical envelopment $x = Ax_0$, $x_0 = const \neq 0$ on group space $GL(n, R)$ and decompose it differential, and also all coeffuicients, obtained at each stage of expansion, according to the left-invariant forms of group $GL(n, R)$ $\omega = A^{-1}dA$, or $\omega^i_j = \tilde{a}^i_k da^k_j$, where $\tilde{a}^i_k a^k_j = \delta^i_j$.

$$dV = V_i dx^i = V_i x^j_0 a^i_k \omega^k_j; \quad J^j_k \equiv V_i x^j_0 a^i_k;$$

$$dJ^j_k = (V_{il} a^l_t a^i_k x^s_0 x^j_0 + V_i a^i_t x^j_0 \delta^s_k)\omega^t_s; \quad J^{sj}_{tk} \equiv V_{il} a^l_t a^i_k x^s_0 x^j_0;$$

$$dJ^{sj}_{tk} = (V_{il\alpha} a^\alpha_\sigma a^l_t a^i_k x^\beta_0 x^s_0 x^j_0 + V_{il} a^l_\sigma a^i_k x^s_0 x^j_0 \delta^\beta_t + V_{il} a^l_t a^i_\sigma x^s_0 x^j_0 \delta^\beta_k)\omega^\sigma_\beta;$$

$$J^{\beta sj}_{\sigma tk} \equiv V_{il\alpha} a^\alpha_\sigma a^l_t a^i_k x^\beta_0 x^s_0 x^j_0; \ etc.$$

J^j_k, J^{sj}_{tk}, $J^{\beta sj}_{\sigma tk}$ etc. – differential invariants of first, second, third and etc. orders of function $V(x)$ continued on group space. The indicated magnitudes are dependent among themselves, that is there are independent nontrivial base invariants of function $V(x)$, obtained with the help of convolution on appropriate to the upper and lower indexes of found invariants.

$I_1 \equiv J^j_j = V_i x^i$, $I_2 \equiv J^{sj}_{sj} = V_{il} x^l x^i$, $I_3 \equiv J^{\beta sj}_{\beta sj} = V_{il\alpha} x^\alpha x^l x^i, \ldots$ – base invariants of first, second, third and etc. orders of smooth function $V(x)$, among which there is only one (differentially) independent invariant I_1 .

For homogeneous square function V $I_1 = V_i x^i = 2V$ (known the Euler relation). Unfortunately, I_j, $j = 1, 2, \ldots$ will not form the full system of invariants, nevertheless they can be useful. By share solution of the Hamilton-Jacoby equation (1.5) linearly – square systems and equations $2V = V_i x^i$ is the pair square positive definite and negative definite functions. From the structure of the relation $2V = V_i x^i$ in common case it is possible to make a conclusion about presence of the smooth in a neighbourhood of zero vector field $\xi^i \frac{\partial}{\partial x^i}$, similar to the field of radial stretch $x^i \frac{\partial}{\partial x^i}$, enveloping the vector field on R^1 $2V \frac{\partial}{\partial V}$ with the help of potential function $V = V(x)$.

5 Bellman-Lyapunov function as equidistant function of Euclidean space

In canonical case the nondegenerate potential function has the kind $V(x) = (x^1)^2 + \ldots + (x^n)^2$.

Therefore for the standard Euclidean metric $\rho^2 = g_{ij} dx^i \otimes dx^j$, where $g_{ij} = \delta_{ij}$, and appropriate, $(\nabla(\rho^2) = 0)$ connectedness $\nabla : \bigwedge^1_{C^\infty(R^n)} \to \bigwedge^1_{C^\infty(R^n)} \otimes \bigwedge^1_{C^\infty(R^n)} : dx^i \mapsto dx^j \otimes \tilde{\omega}^i_j$,
where $\tilde{\omega}^i_j = 0$, $\bigwedge^1_{C^\infty(R^n)}$ – module of 1-forms on R^n,

$$\nabla(dV) = 2\rho^2 \tag{3.2}$$

Function V satisfying (3.2) is called equidistant[44].

The equation (3.2) has an invariant character (function V by internally is joined to Riemannian space with the metric ρ^2). By virtue of said, nondegenerate potential function $V(x)$ is uniquely determined as equidistant function of some plane Riemannian space.

From the equation (3.2) follows

$$dV_j = -\tilde{\omega}^i_j V_i + 2g_{ji} dx^i \tag{3.3}$$

where

$$\tilde{\omega}^i_j = -\Gamma^i_{jk} dx^k, \ \Gamma^i_{jk} = \frac{1}{2} g^{it}(g_{tjx^k} + g_{tkx^j} - g_{kjx^t}), \ \Gamma^i_{jk} = \Gamma^i_{kj},$$

$$d\tilde{\omega}^i_k + \tilde{\omega}^j_k \wedge \tilde{\omega}^i_j = 0 \ .$$

Let's substitute (3.3) in the differential of the equation (2.4.a) taken in regard (2.4.b); we shall obtain

$$(f^i_{x^k} + \Gamma^i_{jk} f^j) V_i + 2 f^j g_{jk} + \omega_{x^k} = 0 \tag{3.4}$$

Let us assume, that the matrix $f^i_{x^k}$ is not degenerate (in some neighbourhood of the origin of coordinates $R^n \times R^m$); the condition of nondegeneracy of a vector field is invariant concerning replacement of coordinates; therefore in the canonical system of coordinates $\{\tilde{x}^i\}$ (where $\tilde{\Gamma}^i_{jk} = 0$) $\tilde{f}^i_{\tilde{x}^k}$ – nonsingular matrix. Coefficients at V_i in the equation (3.4) represent the covariant derivative $f^i_{,k}$ of vector field $f^i \frac{\partial}{\partial x^i}$; in canonical coordinates the following equality takes place $\tilde{f}^i_{,k} = \tilde{f}^i_{\tilde{x}^k}$. Therefore matrix of coefficients $f^i_{,k} = f^i_{x^k} + \Gamma^i_{jk} f^j$ in the equation (3.4) nonsingular (under condition of nonsingularity $f^i_{x^k}$). Let $\overline{f}^k_l$ – inverse matrix for $f^i_{,k}$ ($f^i_{,k} \overline{f}^k_l = \delta^i_l$). Then

$$V_l = -2 g_{jk} f^j \overline{f}^k_l - \omega_{x^k} \overline{f}^k_l \tag{3.4.a}$$

Let's substitute (3.4.a) in $u_{opt} = u_{opt}(V_i, x^j)$, we shall obtain $\tilde{u}_{opt} = \tilde{u}_{opt}(g_{ij}, \Gamma^i_{jk}, x^l)$ (it is supposed, that equations (2.4.b) are locally solvable concerning u in vicinity of origin of coordinates (matrix $(f^i_{u^k u^l} V_i + \omega_{u^k u^l})|_{0 \in T^* R^n \times R^m}$ is nondegenerate)).

Let's substitute in the differential of the equation (3.4) dV_i from (3.3) and obtained from (2.4.b) $du^j = -B^{jk}(f^i_{u^k x^l} V_i + \omega_{u^k x^l}) dx^l - B^{jk} f^i_{u^k} dV_i$, where B^{jk} – matrix is inverse for $B_{kr} \equiv f^i_{u^k u^r} V_i + \omega_{u^k u^r}$ ($B^{jk} B_{kr} = \delta^j_r$), we shall obtain the equations

$$[((f^i_{x^k x^s} + \Gamma^i_{jkx^s} f^j + \Gamma^i_{jk} f^j_{x^s})(-2 g_{j_1 k_1} f^{j_1} \overline{f}^{k_1}_i - \omega_{x^{k_1}} \overline{f}^{k_1}_i) + 2 f^j_{x^s} g_{jk} +$$

$$+2f^j g_{jkx^s} + \omega_{x^k x^s}) - B^{lk_1}((f^i_{x^k u^l} + \Gamma^i_{jk} f^j_{u^l})(-2g_{j_1 k_2} f^{j_1} \overline{f}^{k_2}_i - \omega_{x^{k_1}} \overline{f}^{k_2}_i) +$$
$$+2f^j_{u^l} g_{jk} + \omega_{x^k u^l}) \times (f^{i_1}_{u^{k_1} x^s}(-2g_{BC} f^B \overline{f}^C_{i_1} - \omega_{x^C} \overline{f}^C_{i_1}) + \omega_{u^{k_1} x^s} +$$
$$+f^{i_2}_{u^{k_1}}(\Gamma^A_{i_2 s}(-2g_{BC} f^B \overline{f}^C_A - \omega_{x^C} \overline{f}^C_A) + 2g_{i_2 s}) +$$
$$+(f^i_{x^k} + \Gamma^i_{jk} f^j)(\Gamma^A_{is}(-2g_{BC} f^B \overline{f}^C_A - \omega_{x^C} \overline{f}^C_A) +$$
$$+2g_{is})]\Big|_{u=\tilde{u}_{opt}(g_{ij}, \Gamma^k_{ls}, x^r)} = 0 \qquad (3.5)$$

The equations (3.5) together with following

$$-\Gamma^i_{ksx^t} + \Gamma^j_{kt}\Gamma^i_{js} + \Gamma^i_{ktx^s} - \Gamma^j_{ks}\Gamma^i_{jt} = 0 \quad \text{(zero curvature)} \qquad (3.6)$$

determine (ambiguously) plane Riemannian metric associated with the problem of the optimum stabilization. We can get rid of the equations (3.6) by taking $g_{ij} = \delta_{kl} h^k_{x^i} h^l_{x^j}$, where $h^k(x)$ – apropriate functions of a transformation of coordinates, but the equations (3.5) look too vast even at a possibility of their unique solution with additional defining conditions on the metric.

To synthesis of quasioptimal control can be useful the metric $g_{ij} = \frac{1}{2}V_{ij}$ (in an exactness appropriate for the linearly-quadratic problem).

to be continued ...

References

[1] Bellman R. Dynamic Programming, Moscow, "Foreign literature", 1960. Russian

[2] Boltyanskiy V.G. Mathematical methods of optimum control, Moscow, "Science", 1966. Russian

[3] Pontryagin N.R.,Boltyanskiy V.G.,Gamkrelidze R.V.,Mishchenko E.F. Mathematical theory of optimum processes, Moscow, Phismatgis, 1961. Russian

[4] Letov A.M. Mathematical theory of control procedures , Moscow, "Science", 1981. Russian

[5] Kalman R., Falb P., Arbib M. Sketches under the mathematical theory of systems, Moscow, "World ", 1971. Russian

[6] Merriem K. Theory of optimization and calculation of feedback control systems, Moscow, "World", 1967. Russian

[7] Kuncevich V.M., Lychak M.M. Synthesis of systems of automatic control with the help of of Lyapunov functions, Moscow, "Science", 1977. Russian

[8] Chaki F. Modern theory of control, Moscow, "World", 1975. Russian

[9] Fleming W., Rishel R. Optimum control of determined and stochastic systems, Moscow, "World", 1978. Russian

[10] Yang L.Ch. Lectures on variation calculus and theory of optimum control, Moscow, "World", 1974. Russian

[11] Li E.B., Markus L. Fundamentals of the theory of optimum control, Moscow, "Science", 1972. Russian

[12] Krasovskiy A.A. Systems of automatic control of flight and their analytical designing, Moscow, "Science", 1973. Russian

[13] Krasovskiy N.N. Theory of movemet control, Moscow, "Science", 1968. Russian

[14] Kolesnikov A.A. Analytical designing of optimum nonlinear systems, Taganrog, 1984. Russian

[15] Antomonov Yu.G. Synthesis of optimum systems, Kiev, "Naukova dumka", 1972. Russian

[16] Geldner K., Kubik S. Nonlinear control systems, Moscow, "World", 1987. Russian

[17] Lobry K. Dinamical polysystems and control theory in "Mathematical methods in the theory of systems", Moscow, "World", 1979. Russian

[18] Brokket R.U. Lie algebras and Lie groups in the theory of control, in "Mathematical methods in the theory of systems", Moscow, "World", 1979. Russian

[19] Belyaev E.I. Numerically-analytical method of synthesis of the optimum and suboptimal control algorithms by nonlinear objects, thesis, Gorky, 1986. Russian

[20] Sage A.P., Wyat Ch.S. Optimum Control of systems, Moscow, "Radio and communication", 1982. Russian

[21] Ray W. Methods of control by technological processes, Moscow, "World", 1983. Russian

[22] Kondrat'ev G.V. Means of exterior forms in research of nonlinear systems, in "Modern problems of automatic control: Procedings of VII All-Union conferences - seminars of school of the young scientist and experts", Minsk, 1987. Russian

[23] Kondrat'ev G.V. Theorem of Rashevsky-Chjou as a criterion of weak controllability of finite-dimensional systems linear on control, in "Mathematical modelling in the problems of mechanics and control".

[24] Kondrat'ev G.V. Sufficient condition of controllability of nonlinear systems, in "Control Systems, transformation and mapping of an information", Ryazan, 1984. Russian

[25] Kondrat'ev G.V. Controllability of nonlinear systems, in "Control by multiply connected systems: Proceedings of 5-th All-Union Conferences", Tbilisi, 1984. Russian

[26] Kondrat'ev G.V. Construction of areas of an accessibility of finite-dimensional dynamic systems, in "Optimum Control in mechani-

cal systems: Proceedings of All-Union conferences", Kazan, 1985. Russian

[27] Kondrat'ev G.V., Misevich P.V. Construction of informative area for nonlinear systems with the help of Foker-Plank-Kolmogorov equation, in "Modern problems of automatic control: Proceedings of X All-Union conferences - seminars of entists and experts", Omsk, 1989. Russian

[28] Grasse K.A. Structure of the boundary of the attainable set in sertain nonlinear systems, "Math. Syst. Theory", 18, No. 1, 1985. Russian

[29] Gorokhovik S.Ya. Sufficient conditions of local controllability of nonlinear control systems, in "Control by multiply connected systems: Proceedings of V All-Union Conferences", Tbilisi, 1984. Russian

[30] Levitt M.Yu. About some problems of local and global controllability of n onlinear systems, in "Control by multiply connected systems: thesises of the reports of V All-Union Conferences", Tbilisi, 1984. Rusdsian

[31] Furi M., Nistri P., Pera M.P., Zezza P.L. Topological methods for the global controllability of nonlinear systems, J. Optim. Theory and Appl.", 45, No. 2, 1985.

[32] Arnold V.I. Mathematical methods of classical mechanics, Moscow, "Science", 1974. Russian

[33] Dubrovin V.A., Novikov S.P., Fomenko A.T. Modern geometry, Moscow, "Science", 1986. Russian

[34] Rashevsky P.K. Geometric theory of the equations with partial derivatives, Moscow, Phismath, 1947. Russian

[35] Gantmakher F.R. Lectures on analytical mechanics, Moscow, "Science", 1966. Russian

[36] Dobronravov V.V. Fundamentals of analytical mechanics, Moscow, "Higher School", 1976. Russian

[37] Arnold V.I., Kozlov V.V., Neishtatd AI. Mathematical aspects of classical and celestial mechanics, Totals of science and engineering, Dynamic systems, v. 3, 1985. Russian

[38] Perelomov A.M. Integrable systems of classical mechanics and Lie algebras, Moscow, "Science", 1990. Russian

[39] Kirillov A.A. Elements of the theory of representations, Moscow, "Science", 1978. Russian

[40] Gromov M. Differential relations with partial derivatives, Moscow, "World", 1990. Russian

[41] Kondrat'ev G.V. Class of systems with invarient foliation of the Bellman- Lyapunov function, in "Systems of data proseccing and control", N. Novgorod, 1995. Russian

[42] Laptev G.F. Basic infinitesimal structures of higher orders on a smooth manifold, Proceedingss of geometric seminar, v. 1, 1966. Russian

[43] Evtushik L.E., Lumiste Yu.G., Ostianu N.M., Shirokov A.P. Differential-geometric structures on manifolds, Totals of science and engineering, Problems of geometry, volume 9, 1979. Russian

[44] Sinyukov N.S. Geodesic mappings of Riemannian spaces, Moscow, "Science", 1979. Russian

G.Kondrat'ev, A.Balabanov
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G.Kondrat'ev
State Technical University of
Nizhniy Novgorod (NSU),
Computing mashinery Departament,
str. Minin, 24,
603000, N.Novgorod, Russia,
phone: (8312) 36-82-28; Fax 360-569
e–mail: *nstu@nnpi.nnov.su*

A.Balabanov,
Technical University of Moldova,
Automatica Departament,
Stefan cel Mare 168,
2012, Kishinev, Moldova,
phone: (373-2) 497694, 497014;
fax: (373-2) 497014, 497022
e–mail: *besliu@cc.acad.md* or *ast@utm.usm.md*