Computer Science Journal of Moldova, vol.5, no.3(15), 1997

Two problems concerning dynamic flows in node and arc capacitated networks

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Abstract

This paper presents two problems concerning dynamic flow in node and arc capacitated networks: (1) maximal dynamic flow; (2) feasible dynamic flows.

1 Maximal dynamic flows in arc capacitated networks

Let $G = (N, A)$ be a connected digraph with $N = \{s, \ldots, x, \ldots, s'\}$, $|N| = n$ the node set and $A = \{(x, y)/x, y \in N\}, |A| = m \leq n(n-1)$ the arc set. Let $\mathbf{N}$ be the natural number set and $P\{0, 1, \ldots, p\}$ the set of periods. Let us state the time function $h : A \to \mathbf{N}$ and the capacity function $c : A \times P \to \mathbf{N}$, where $h(x, y)$, represents the arc transit time and $c(x, y; t)$ the arc capacity at time t for $(x, y) \in A, t \in P$. Let us designate by s the sourse and by s' the sink of the network $G = (N, A, h, c)$.

The maximal dynamic flows problem for p time periods may be formulated as follows. Let us determine the function $f : A \times P \to \mathbf{N}$, which should satisfy the following relations:

$$\sum_{t=0}^{p}(\sum_{y} f(x, y; t) - \sum_{y} f(x, y; t')) = v(p),\ x = s, \tag{1a}$$

$$\sum_{y} f(x, y; t) - \sum_{y} f(x, y; t') = 0,\ x \neq s, s';\ t, t' \in P, \tag{1b}$$

$$\sum_{t=0}^{p}(\sum_{y} f(x,y;t) - \sum_{y} f(x,y;t')) = -v(p),\ x = s', \tag{1c}$$

$$0 \le f(x,y;t) \le c(x,y;t),\ (x,y) \in A, t \in P, \tag{2}$$

$$\max v(p), \tag{3}$$

where $t' = t - h(y,x)$.

Ford and Fulkerson [5] have shown that a dynamic flow for p time periods in the static network $G = (N, A, h, c)$ can be represented as a static flow in the dynamic netwoek $G(p) = (N(p), A(p))$, where

$$N(p) = \{x(t)/x \in N, t \in P\}$$
$$A(p) = \{(x(t), y(t')/(x,y) \in A; t, t' \in P, t' = t + h(x,y)\}$$
$$c(x(t), y(t')) = c(x,y;t),\ (x,y) \in A;\ t, t' \in P,\ t' = t + h(x,y).$$

In references [2], [3] it is shown that a dynamic flow for p time periods in the static network $G = (N, A, h, c)$ is equivalent with a dynamic flow for the same time periods in the static network $G'(p) = (N'(p), A'(p), h', c')$ and can be represented as a static flow in the dynamic network $G^*(p) = (N^*(p), A^*(p), h^*, c^*)$.

The networks $G'(p)$ and $G^*(p)$ may be constructed as follows. Let $d(s,x)$ be the length of the shortest route from the sorce s to the node x and $d(x, s'$ the length of the shortest route from the node x to the sink s', with respect to $h(x,y)$. Computing $d(s,x),\ d(x,s')$ for all $x \in N$ is performed by means of the usual shortest path algorithms and is not discussed in this paper. Let us consider:

$$P(x) = \{t/t \in P,\ d(s,x) \le t \le p - d(x,s')\},\ x \in N,$$

$$P(x,y) = \{t/t \in P,\ d(s,x) \le t \le p - (h(x,y) + d(x,s'))\},\ (x,y) \in A.$$

The network $G'(p) = (N'(p), A'(p), h', c')$ may be constructed as follows:

$$N'(p) = \{x/x \in N,\ P(x) \ne \emptyset\},$$

$$A'(p) = \{(x,y)/(x,y) \in A,\ P(x,y) \neq \emptyset\},$$

and h', c' are the restrictions to $A'(p)$.

The network $G^*(p) = (N^*(p), A^*(p), h^*, c^*)$ is constructed from the network $G'(p)$ in the following manner:

$$N^*(p) = \{x(t)/x \in N'(p), t \in P(x)\}$$

$$A^*(p) = \{(x(t), y(t')/(x,y) \in A'(p); t \in P(x,y), t' = t + h(x,y)\}$$

$$c^*(x(t), y(t')) = c(x,y;t),\ (x,y) \in A'(p);\ t \in P(x,y),\ t' = t + h(x,y).$$

The network $G'(p)$ is, in general, a partial subnetwork of G and the network $G^*(p)$ is allways a partial subnetwork of $G(p)$.

A dynamic flow problem is said to be stationary if the network parameters such as capacities, arc traversal times, and so on, are constant over time ($h : A \to \mathbf{N},\ c : A \to \mathbf{N}$, and so on).

In the references [3], [5] are presented algorithms for maximum value dynamic flow problem. In the stationary case it does not require the construction of time-expanded network $G(p)$ or $G^*(p)$ for solving this problem for any p. Show that maximum value dynamic flow in the stationary case can be generated from a maximum value and minimum time static flow $F'(p)$ in network $G'(p)$. The maximum dynamic flow algorithm dor the stationary case is as follows.

Stage 1 *Perform the maximum value and minimum time static flow F' in static network $G'(p)$.*

Stage 2 *Decompose f' into a set of fllow paths $w'_1, \ldots, w'_k$ from s to s' respectively carrying $g'_1, \ldots, g'_k$ flow from s to s'.*

Stage 3 *For $i = 1, \ldots, k$ send g'_i flow units from s to s' along flow path w'_i starting out from s at time periods 0 and repeat it after each time period as long as there is enough time left in the horizon for the flow along the path to arrive at the sink.*

The flow cunstructed above is referred to as a maximal temporally repeated flow.

2 Maximal dynamil flows in node and arc capacitated networks

In Section 1, we studied the maximal dynamic flow problem with capacity function $c : A \times P \to \mathbf{N}$. In the present section we extended capacity function to $k : N \times P \to \mathbf{N}$. Our objective is to find a maximal dynamic flow for p time periods from the source s to the sink s' in G subject to both arc and node capacities. The problem may be formulated as follows. Let us determine the function $f : A \times P \to \mathbf{N}$, with should satisfy the (1), (2), (3) and

$$\sum_y f(x, y; t) \le k(x; t),\ x \ne s',\ t \in P \qquad (2b)$$

$$\sum_y f(y, s'; t) \le k(s'; t),\ t \in P \qquad (2c)$$

This problem can be converted to the arc capacity case by the following procedure. Let $\widetilde{G} = (\widetilde{N}, \widetilde{A}, \widetilde{h}, \widetilde{c})$ be the network derived from $G = (N, A, h, c, k)$ in accordance with the following rules. To each node $x \in N$ there correspond two nodes $x_1, x_2 \in \widetilde{N}$ such that if $(x, y) \in A$, then $(x_2, y_1) \in \widetilde{A}$. In addition, for each $x \in N$, there is an $(x_1, x_2) \in \widetilde{A}$. The arc time $\widetilde{h}$ and the arc capacity $\widetilde{c}$ defined on $\widetilde{A}$ is given by

$$\widetilde{h}\ (x_2, y_1) = h(x, y),\ \widetilde{c}\ (x_2, y_1; t) = c(x, y; t),\ (x, y) \in A, t \in P,$$

$$\widetilde{h}\ (x_1, x_2) = 0,\ \widetilde{c}\ (x_1, x_2; t) = k(x; t),\ x \in N,\ t \in P.$$

With this representation, it is clear that for any dynamic flow f for p time periods from s to s' in G that does not exceed the node capacities, there corresponds an equivalent dynamic flow $\widetilde{f}$ for p time periods from s_1 to s'_2 in $\widetilde{G}$ by taking

$$\widetilde{f}\ (x_2, y_1; t) = f(x, y; t),\ (x, y) \in A,\ t \in P,$$

$$\widetilde{f}\ (x_1, x_2; t) = \sum_y f(x, y; t),\ x \ne s',\ t \in P,$$

$$\widetilde{f}\,({s_1}', {s_2}'; t) = \sum_y f(y, s'; t),\ t \in P,$$

and vice versa.

It is clear that a dynamic flow for p time periods in the static network $G = (N, A, h, c, k)$ is equivalent with a dynamic flow for the same time periods in the static network $G'(p) = (N'(p), A'(p), h', c', k')$ and can be represented as a static flow in the dynamic network $G^*(p) = (N^*(p), A^*(p), h^8, c^*, k^*)$. The networks $G'(p) = (N'(p), A'(p), h', c', k')$ and $G^*(p) = (N^*(p), A^*(p), h^*, c^*, k^*)$ may be constructed as $G'(p) = (N'(p), A'(p), h', c')$ and $G'(p) = (N'(p), A'(p), h', c')$ from Section 1 and in addition k' is the restriction to $A'(p)$, $k^*(x(t)) = k(x; t)$, $x \in N'(p)$, $t \in P(x)$.

In the present section we consider that node and arc capacities remains unchanged through time, i.e., the problem is stationary

$$c(x, y; t) = c(x, y),\ (x, y) \in A,\ t \in P, \tag{4a}$$

$$k(x; t) = k(x),\ (x) \in N,\ t \in P, \tag{4b}$$

The arcs in A and the nodes in N are arranged in some order, and $h = (h(x, y))$, $c = (c(x, y))$, $f = (f(x, y))$, $k = (k(x))$ denote the time, arc cpacity, flow and node capacity vectors in which these quantities are ordered in the same order as arcs in A and N.

A maximum dynamic flow for stationary case can be found with the algorithm presented in Section 1 with the remark that in Stage 1 perform the maximum value and minimum time static flow $\widetilde{f}'$ in static network $\widetilde{G}'(p)$ is performed.

3 Feasible dynamic flows in node and arc capacitated networks

In Scetion 2, we studied the maximum dynamic flow problem in a node and arc capacitated networks. In the present section, we extended the feasibility theorems to node and arc capacitated networks.

Associate with s, nonnegative real number $q(t)$ called the supply of source s at time t and with s', nonnegative real number $q'(t)$ called the demand of sink s' at time t, $t \in P$.

The objective is to determine the existence of a flow in $G = (N, A, h, c, k)$ so that the demands at the sink can be fulfilled from the supplies at the source s satisfying the constraints

$$\sum_y f(s, y; t) - \sum_y f(y, s; t') \leq q(t),\ t \in P, \tag{5a}$$

$$\sum_y f(x, y; t) - \sum_y f(y, x; t) = 0,\ x \neq s, s',\ t \in P, \tag{5b}$$

$$\sum_y f(y, s'; t) - \sum_y f(s', y; t) \leq q'(t),\ t \in P, \tag{5c}$$

$$0 \leq f(x, y; t) \leq k(x; t),\ (x, y) \in A,\ t \in P, \tag{6a}$$

$$\sum_y f(x, y; t) \leq k(x; t),\ x \neq s',\ t \in P, \tag{6b}$$

$$\sum_y f(x, y; t) \leq k(x; t),\ t \in P, \tag{6b}$$

where $t = t - h(y, x)$.

If such solution exists, we say that the condtraints (5), (6) are feasible. Otherwise, they are infeasible.

If $X^*(p)$ is a subset of $N^*(p)$ and write $\overline{X}^* = N^*(p) - X^*(p)$, then a path from a node in $X^*(p)$ to a node in $\overline{X}^*(p)$ is called a path from $X^*(p)$ to $\overline{X}^*(p)$. A generalized static disconnecting set from $X^*(p)$ to $\overline{X}^*(p)$ in dynamic network $G^*(p) = (N^*(p), A^*(p), c^*, k^*)$ is a collection od nodes and arcs of $G^*(p)$ such that every path from $X^*(p)$ to $\overline{X}^*(p)$ has at least a node or an arc belonging to the collection. Denote by $D^*(X^*(p), \overline{X}^*(p))$ a generalized static disconnecting set from $X^*(p)$ to $\overline{X}^*(p)$ in $G^*(p) = (N^*(p), A^*(p), c^*, k^*)$. Then the capacity $c^*(D^*(X^*(p), \overline{X}^*(p)))$ is defined to be the sum of the capacities of its constituient elements:

$$c^*(D^*(X^*(p),\overline{X}^*(p)))$$
$$= \sum(c^*(x(t),y(t')) : (x(t),y(t')) \in D^*(X^*(p),\overline{X}^*(p)) + \qquad (7)$$
$$+ \sum(k^*(x(t)) : x(t)) \in D^*(X^*(p),\overline{X}^*(p))$$

Let us consider:

$$S^*(p) = \{s(t),\ t \in P(s)\},\ \ \overline{S}^*(p) = \{s'(t),\ t \in P(s')\},$$
$$T^*(p) = \{t|s(t) \in S^*(p) \cap \overline{X}^*(p)\},\ \ \overline{T}^*(p) = \{t|s' \in \overline{S}^*(p) \cap \overline{X}^*(p)\}$$

.

The following result can be found in [1].

Theorem 1 *The constraints (5), (6) are feasible if and only if*

$$\sum_{\overline{T}^*(p)} q'(t) - \sum_{T^*(p)} q(t) \leq \min_i c^*(D_i^*(X^*(p),\overline{X}^*(p))) \qquad (8)$$

holds gor every subset $X^*(p) \subset N^*(p)$, where $D_i^*(X^*(p),\overline{X}^*(p))$ is the i-th generalized static disconnecting set from $X^*(p)$ to $\overline{X}^*(p)$ in $G^*(p) = (N^*(p), A^*(p), C^*, k^*)$.

Let the network $G'(p) = (N'(p), A'(p), c', k')$ and $Q(x) \subset P(x)$, $\overline{Q}(x) = P(x) - Q(x),\ x \in N'(p)$. We define $X'(p) = \{x(t)|x \in N'(p),\ t \in Q(x)\}$, $\overline{X}'(p) = \{x(t)|x \in N'(p),\ t \in \overline{Q}(x)\}$ and denote by $D'(X'(p),\overline{X}'(p))$ a generalized dynamic disconnecting set from $X'(p)$ to $\overline{X}'(p)$ in $G'(p) = (N'(p), A'(p), c', k')$.

We have the following theorem

Theorem 2 *The constrains (5), (6) are feasible if and only if*

$$\sum_{\overline{T}'(p)} q'(t) - \sum_{T'(p)} q(t) \leq \min_i c'(D_i'(X'(p),\overline{X}'(p))) \qquad (9)$$

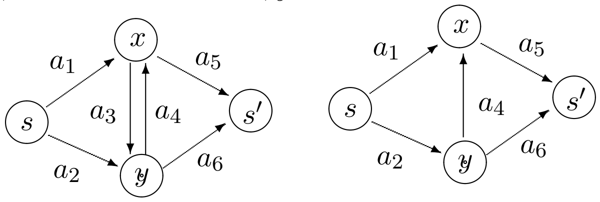
holds for every subset $Q(x) \subset P(x),\ x \in N'(p)$ where $D_i'(X'(p),\overline{X}'(p))$ is the i-th generalized dynamic disconnecting set from $X'(p)$ to $\overline{X}'(p)$ in $G'(p) = (N'(p), A'(p), c', k')$.

Proof. The result folows directly from Theorem 1 and the fact that a dynamic flow for p time periods in the static network $G'(p) = (N'(p), A'(p), c', k')$ can be represented as a static flow in the dynamic network $G^*(p) = (N^*(p), A^*(p), c^*, k^*)$.

We illustrate the above results by the following example.

4 Example

Consider the network $G = (N, A, h, c, k)$ of Fig. 1a) where node s is the source, s' is the sink and x, y are the intermediate nodes.



a) b)

Fig.1

We have $A = (a_1, a_2, a_3, a_4, a_5, a_6)$, $h = (3, 1, 1, 1, 1, 3)$, $c = (3, 2, 1, 2, 2, 3)$, $N = (s, x, y, s')$, $k = (5, 3, 1, 4)$. For $p = 5$ the network $G'(5) = (N'(5), A'(5), h', c', k')$ is shown in fig.1b) with $N'(5) = N$, $A'(5) = (a_1, a_2, a_3, a_4, a_5, a_6)$, $h' = (3, 1, 1, 1, 3)$, $c' = (3, 2, 2, 2, 3)$, $k' = (5, 3, 1, 4)$.

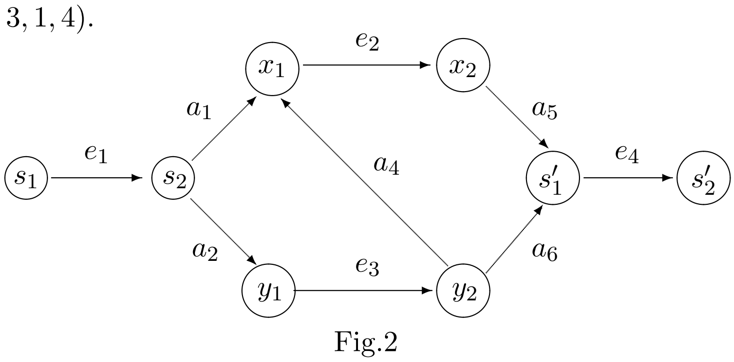


Fig.2

The network $G'(5)$ can be converted into an equivalent arc capacity network

$\widetilde{G}'(5) = (\widetilde{N}'(5), \widetilde{A}'(5), \widetilde{h}', \widetilde{c}')$ with $\widetilde{N}'(5) = (s_1, s_2, x_1, x_2, y_1, y_2, s'_1, s'_2)$, $\widetilde{A}'(5) = (e_1, a_1, a_2, e_2, e_3, a_4, a_5, a_6, e_4)$, $\widetilde{h}' = (0, 3, 1, 0, 0, 1, 1, 3, 0)$, $\widetilde{c}' = (5, 3, 2, 3, 1, 2, 2, 3, 4)$ and is shown in Fig.2.

The network $G^*(5) = (N^*(5), A^*(5), c^*, k^*)$ is shown in Fig.3 with

$$\begin{aligned} A^*(5) &= (a_1(0), a_1(1), a_2(0), a_2(1), a_2(2), a_4(1), a_4(2), \\ &\quad a_4(3), a_5(2), a_5(3), a_5(4), a_6(1), a_6(2)), \\ c^* &= (3, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3), \\ N^*(5) &= (s(0), s(1), s(2), x(2), x(3), x(4), \\ &\quad y(1), y(2), y(3), s'(3), s'(4), s'(5)), \\ k^* &= (5, 5, 5, 3, 3, 3, 1, 1, 1, 4, 4, 4). \end{aligned}$$

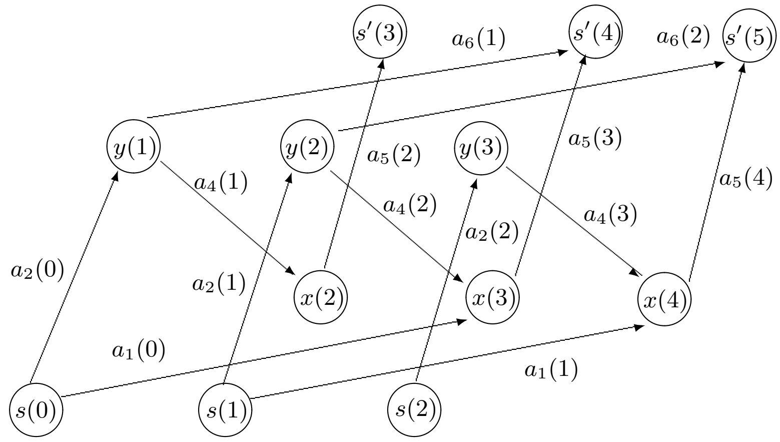


Fig.3

The temporally repeated flow in network $G^*(5)$ generated from the maximum value and minimum time static flow $\widetilde{f} = (3, 2, 1, 2, 1, 0, 2, 1, 3)$ from $\widetilde{G}'(5)$ which is equivalent with the maximum value and minimum time static flow $F' = (2, 1, 0, 2, 1)$ from $G'(5)$ is $f^* = (2, 2, 1, 1, 0, 0, 0, 0, 0, 2, 2, 1, 1)$. It is interesting to observe that without nodal capacity constraints, the maximum static flow $\widetilde{f}^* = (2, 2, 2, 2, 0, 0, 0, 0, 0, 2, 2, 2, 2)$ has the value $\overline{v}^* = 8$, whereas with nodal capacity the maximum static flow value is reduced to $v^* = 6$.

The suppllies and demands are given by $q(0) = 3$, $q(1) = 3$, $q(2) = 1$, $q'(3) = 1$, $q'(4) = 2$, $q'(5) = 3$. In this case the problem is feasible with flow $f^* = (2,2,1,1,0,1,0,0,1,2,2,0,1)$. If $q'(3) = 2$ then the problem is infeasible. To verify that condition (8) is indeed violated for some $X^*(5) \subset N^*(5)$, let $X^*(5) = \{s(0), s(1), s(2), x(3), x(4), y(1), y(2)\}$ and

$D_1^*(X^*(5), \overline{X}^*(5)) = \{(x(3), s'(4)), (x(4), s'(5), (y(1), y(2))\}$.

We have $T^*(5) = \emptyset$, $\overline{T}^*(5) = \{3,4,5\}$ and $q'(3) + q'(4) + q'(5) = 2{+}2{+}3 = 7 > c^*(D_1^*(X^*(5), \overline{X}^*(5))) = c^*(x(3), s'(4)) + c^*(x(4), s'(5)) + k^*(y(1)) + k^*(y(2)) = 2 + 2 + 1 + 1 = 6$. For $Q(s) = \{0,1,2\}$, $Q(x) = \{3,4\}$, $Q(y) = \{1,2\}$, $Q(s') = \emptyset$ we obtain the same generalized disconnecting set $D'_1(X'(5), \overline{X}'(5)) = \{(x(3), s'(40),\ (x(4), s'(5)), (y(1), y(2))\}$ and the condition (9) is indeed violated for some $Q(x) \subset P(x)$, $x \in \mathbf{N}'(5)$.

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Received October 15, 1997