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dedicated to the 55th anniversary of the foundation of Vladimir Andrunachievici Institute of Mathematics and Computer Science

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Preface

Vladimir Andrunachievici Institute of Mathematics and Computer Science (IMCS), in this spring celebrates its 55th anniversary from the foundation. With all certainty we can say that IMCS is a centre of excellence for advance research in the domains of pure mathematics, the applied one and computer science.

Founded by the illustrious mathematician Vladimir Andrunachievici, it is an organization with a valuable human potential and with remarkable results in all the mentioned domains. Currently the famous scientific schools in the theory of algebraic rings (founded by acad. V. Andrunachievici), in the qualitative theory of differential equations (founded by acad. C. Sibirschi), in the theory of quasigroups (founded by Professor V. Belousov), in functional analysis (founded by cor.m. I Gorhberg) and in mathematical logic (founded by Dr. A. Kuznetov) continue their successful work and development. More than 30 doctors in habilitations and 400 PhDs were trained in the Institute.

During these 55 years over 3500 scientific papers and about 155 monographs have been published. The Institute publishes the journal "Buletinul Academiei de Stiinte a Republicii Moldova. Matematica" (from 1989 to this day 90 volumes were out), also the journal "Computer Science Journal of Moldova" (from 1993 to this day 80 volumes were out). Other journal in collaboration with colleagues from Poland, "Quasigroups and related systems", has published 27 volumes.

In the last 25 years the Institute took part in executing more than 60 projects with partners from advanced research centres in different countries of Europe, Asia, America.

The **Fifth Conference of the Mathematical Society of Moldova** is the international conference dedicated to this anniversary and is a continuation of the tradition of reviewing new achievements in mathematics, informatics and information technology and information exchange with colleagues from major research centers across borders.

The organizers of the IMCS-55 edition are: Mathematical Society of the Republic of Moldova, Vladimir Andrunachievici Institute of Mathematics and Computer Science in cooperation with Tiraspol State University and Information Society Development Institute.

During the IMCS-55, the solutions for the most current problems that appear in various fields of mathematics and computer science as well as the socio-economic aspect of modern society are also addressed. The conference considers discussions between the research groups of universities, research institutes and the commercial IT field. In this regard, it is planned to establish scientific priorities for applying the solutions found by researchers to specific problems in society, national economy and abroad as well.

The volume includes 77 papers, covering topics in major areas of mathematics and informatics. It is divided in 3 chapters: Pure Mathematics, Applied Mathematics and Computer Science.

We express our sincere thanks to all people who contributed to the success of this event, especially from the Scientific Committee, who spent time carefully reviewing all the proposals submitted to IMCS-55 to insure a qualitative improvement of the papers, and from the Organizing Committee which ensured its high level.

Mitrofan Choban
Inga Titchiev

Section 1

Pure Mathematics

A computer-friendly alternative to Morse-Novikov theory

Dan Burghelea

Abstract

Classical Morse-Novikov theory relates the "dynamics elements" of a flow provided by a "Morse-Smale vector field" X equipped with a Lyapunov closed differential one form (=action) on a closed manifold M to the algebraic topology of the underlying manifold M.

The Alternative to Morse Novikov (AMN) theory I propose extends considerably the class of spaces and flows the theory applies, providing the same relations. The new theory is based on different intermediate concepts which happens to be "computer-friendly" as opposed to the ones in classical theory which might not be so.

There are substantial applications in topology, geometry, as well as in Data Analysis.

In this talk I will review the classical Morse-Novikov theory and introduce the audience to the Alternative to Morse-Novikov theory. If the time permits a few comments on possible applications will be inserted.

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Algebraic Methods for Bloch-Iserles Hamiltonian Systems

Vasile Brînzănescu

Abstract

We shall prezent the study of complete integrability of the Bloch-Iserles Hamiltonian systems by using algebraic methods to describe the invariant tori.

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Gauss-Bonnet-Chern formulas on singular manifolds

Sergiu Moroianu

Abstract

We give formulas for the Euler characteristic in terms of curvature invariants on Riemannian manifolds with three types of singularities: conical metrics, incomplete edge metrics, and complete fibered boundary metrics. The boundary contribution in each case involves a new polynomial in the curvature tensor, called the odd Pfaffian. As consequences, we obtain obstructions for the existence of certain flat Riemannian cobordisms. This work is joint with Daniel Cibotaru (Fortaleza).

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Non-discrete topologizations of some countable groups

Vladimir Arnautov, Galina Ermakova

Abstract

If G is a countable group such that for any finite subset $M \subseteq G$ there exists an infinite set $S \subseteq G$ such that $m \cdot g = g \cdot m$ for any $m \in M$ and any $g \in S$, then the group G admits a non-discrete Hausdorff group topology.

Keywords: countable group, group topology, Hausdorff topologies, lattice of group topologies, coatom in a lattice.

1 Introduction

The article is devoted to the study of the possibility of specifying non-discrete group topologies on countable groups.

The research on this issue was initiated in [1], in which there was indicated a criterion for existence of non-discrete Hausdorff group topologies on a countable group. Later in [2] it was proved that any infinite commutative group admits non-discrete Hausdorff group topology.

Finally, an example of a countable group which does not admit non-discrete Hausdorff group topologies is given in [3].

It is known that for any group the set of all group topologies is a complete lattice in which the discrete topology is the largest element and the anti-discrete topology is the smallest element.

Properties of the lattice of all topologies of a countable group which admits a non-discrete Hausdorff group topology are studied in the article [4].

2 Main Results

Theorem. If for any finite subset of elements of a countable group G there is an infinite set of elements in the group G, each of which commutes with any element of S, then the group G admits a non-discrete Hausdorff group topology.

Corollary 1. If the center Z of a countable group G is infinite, then on the group G there exists a non-discrete Hausdorff group topology.

From ([5], Theorem 3.1.) and Theorem 5 it follows

Corollary 2. If G is a countable group such that for any finite subset $M \subseteq G$ there exists an infinite set $S \subseteq G$ such that $m \cdot g = g \cdot m$ for any $m \in M$ and any $g \in S$ for any $m \in M$ and any $g \in S$, then the following statements are true:

1. The group G admits a continuum of non-discrete Hausdorff group topologies, in each of which the topological group has a countable filter of neighborhoods of the unit and such that any two of which are comparable with each other;

2. The group admits a continuum of non-discrete Hausdorff group topologies, in each of which the topological group has a countable filter of neighborhoods of the unit and such that it is a discrete topology for any two of these topologies;

3. The lattice of all group topologies on a group G contains 2 to the power of continuum of coatoms.

We construct an example of a countable group in which the center is a finite set, but for any finite subset, the group has an infinite subset of elements, each of which commutes with any element of the set.

example. If G is a finite, simple, non-commutative group (as the group G one can take, for example, a group of invertible matrices of order 2×2 over the two-element field), then its center is a one-element set $\{e\}$. Consider a group $\tilde{G}$ that is a direct sum of a countable number of groups, each of which is isomorphic to the group G.

It is easy to verify that the center of the group $\tilde{G}$ is a singleton set $\{\tilde{e}\}$. Since for any element $\tilde{g}$ the set $\{i | pr_i(\tilde{g}) \neq e\}$ is a finite set, then for any finite subset $M \subseteq \tilde{G}$ there is an infinite subset of elements, each of which commutes with any element from the set M.

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On Einstein six-dimensional planar Hermitian submanifolds of Cayley algebra

Galina Banaru

Abstract

A criterion in terms of second fundamental form for six-dimensional planar Hermitian submanifolds of Cayley algebra to be Einstein manifolds is established.

Keywords: almost Hermitian manifold, Kähler manifold, Einstein manifold, six-dimensional Hermitian submanifold of Cayley algebra, planar submanifold.

1 Introduction

The existence of Brown-Cray 3-vector cross products in Cayley algebra [7], [8] gives a lot of important and substantive examples of almost Hermitian (AH-) manifolds. As it is well known, a 3-vector cross product in Cayley algebra induces a 1-vector cross product, or what is the same in this case, an almost Hermitian structure on its six-dimensional oriented submanifold (see, for example, [2], [9], [11]). Such almost Hermitian structures (in particular, Kähler, nearly Kähler, special Hermitian, Hermitian etc) were studied by such remarkable geometers as Alfred Gray, Vadim Feodorovich Kirichenko, Kouei Sekigawa and Luc Vranchen. As the most important results in this domain, we mark out a complete classification of nearly Kähler [10], Kähler [11] and locally symmetric Hermitian structures [12] on six-dimensional submanifolds of the octave algebra obtained by Vadim Feodorovich Kirichenko. In the present paper, we consider six-dimensional Hermitian planar submanifolds of Cayley algebra, i.e. six-dimensional planar submanifolds with integrable almost Hermitian structure. We shall study the case when such submanifolds of the octave algebra are Einstein.

2 Preliminaries

An almost Hermitian manifold is a $2n$-dimensional manifold M^{2n} equipped with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX,\ JY \rangle = \langle X,\ Y \rangle\,, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of vector fields on M^{2n} [13]. We state that all considered manifolds, tensor fields and similar objects are assumed to be C^{∞}-smooth.

The specification of an almost Hermitian structure on an arbitrary manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [13]. Its elements are the frames adapted to the structure (A-frames). These frames look as follows:

$$(p,\ \varepsilon_1,\ \ldots, \varepsilon_n,\ \varepsilon_{\hat{1}},\ \ldots,\ \varepsilon_{\hat{n}}\,),$$

where ε_a are the eigenvectors corresponding to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponding to the eigenvalue $-i$. Here the index a ranges from 1 to n, $\hat{a} = a + n$.

The fundamental form of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY \rangle\,, \quad X, Y \in \aleph(M^{2n}).$$

By direct computation, it is easy to obtain that the matrices of the operator of the almost complex structure, of the Riemannian metric and of the fundamental form written in an A-frame look as follows, respectively:

$$\left(J_j^k\right) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right); \quad (g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right);$$

$$(F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right), \tag{1}$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$.

An almost Hermitian manifold is called Hermitian, if its almost complex structure is integrable. The following identity characterizes the Hermitian structure [13]:

$$\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = 0,$$

where$X, Y, Z \in \aleph(M^{2n})$. The first group of the Cartan structural equations of a Hermitian manifold written in an A-frame looks as follows [2], [13]:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b,$$
$$d\omega_a = -\omega^b_\alpha \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b,$$

where $\left\{B^{ab}{}_c\right\}$ and $\{B_{ab}{}^c\}$ are components of the Kirichenko tensors of M^{2n} [1]; $a, b, c = 1, 2, \ldots, n$; $\{\omega^k\}$ are the components of the displacement forms and $\{\omega^k_j\}$ are the components of the Levi-Civita connection forms.

3 The main results

At first, let us use the Cartan structural equations of a Hermitian structure on a six-dimensional Hermitian submanifold of Cayley algebra $M^6 \subset \mathbf{O}$ [3]:

$$d\omega^a = \omega^a_b \wedge \omega^b + \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hc}\omega^c \wedge \omega_b;$$

$$d\omega_a = -\omega^b_a \wedge \omega_b + \frac{1}{\sqrt{2}}\varepsilon_{abh}D^{hc}\omega_c \wedge \omega^b; \tag{2}$$

$$d\omega^a_b = \omega^a_c \wedge \omega^c_b - \left(\frac{1}{2}\delta^{ah}_{bg}D_{hd}D^{gc} + \sum_\varphi T^\varphi_{\hat{a}\hat{c}}T^\varphi_{bd}\right)\omega_c \wedge \omega^d.$$

Here $\varphi = 7, 8$; $a, b, c, d, g, h = 1, 2, 3$; $\hat{a} = a + 3$; $k, j = 1, 2, 3, 4, 5, 6$; $\varepsilon_{abc} = \varepsilon_{abc}^{123}$, $\varepsilon^{abc} = \varepsilon_{123}^{abc}$ are the components of third-order Kronecker tensors; $\delta_{bg}^{ah} = \delta_b^a \delta_g^h - \delta_g^a \delta_b^h$; $D^{hc} = D_{\hat{h}\hat{c}}$; $D_{cj} = \mp T_{cj}^8 + i T_{cj}^7$, $D_{\hat{c}j} = \mp T_{\hat{c}j}^8 - i T_{\hat{c}j}^7$, and $\left\{ T_{kj}^{\varphi} \right\}$ are the components of the second fundamental form of the immersion of the submanifold M^6 into the octave algebra.

A Hermitian submanifold $M^6 \subset \mathbf{O}$ is planar if and only if the following conditions are fulfilled [4], [5]:

$$T_{ab}^8 = \mu T_{ab}^7; \quad T_{\hat{a}\hat{b}}^8 = \bar{\mu} T_{\hat{a}\hat{b}}^7; \quad \mu \in C; \quad \mu \ - const. \tag{3}$$

Really, if a six-dimensional submanifold of Cayley algebra M^6 is planar (i.e. M^6 is a submanifold of a hyperplane in $\mathbf{O}$), then the components T_{kj}^8 and T_{kj}^7 are linearly dependent at an arbitrary point of M^6. The converse chain of arguments is evident. Remark that we do not consider the case even one of the components T_{kj}^{φ} vanishes. Apropos, six-dimensional Kähler submanifolds of the octave algebra are planar. In this case $\mu = i$ [3]. As examples of non-Kähler six-dimensional planar Hermitian submanifolds of Cayley algebra we can consider introduced by Vadim Feodorovich Kirichenko locally symmetric Hermitian submanifolds $M^6 \subset \mathbf{O}$ [6], [12].

As it is known, the manifold is called Einstein if its Ricci tensor satisfies the condition:

$$ric = \varepsilon g, \quad \varepsilon - const.$$

Taking into account (1), we conclude that a Hermitian submanifold $M^6 \subset \mathbf{O}$ is an Einstein manifold if and only if

$$ric_{\hat{a}b} = \varepsilon\ \delta_b^a\ ,\ ric_{a\hat{b}} = \varepsilon\ \delta_a^b.$$

Using (2) we obtain

$$-\sum_{\psi} T_{\hat{a}\hat{c}}^{\varphi} T_{cb}^{\varphi} = \varepsilon\ \delta_b^a; \quad -\sum_{\psi} T_{ac}^{\varphi} T_{\hat{c}\hat{b}}^{\varphi} = \varepsilon\ \delta_a^b. \tag{4}$$

Finally, we use the formulae (3) that characterize namely six-dimensional planar submanifolds of Cayley algebra. We get

$$-\left(1+|\mu|^2\right)\ T^7_{\hat{a}\hat{c}}\,T^7_{cb}=\varepsilon\ \delta^a_b;\quad -\left(1+|\mu|^2\right)\ T^7_{ac}\,T^7_{\hat{c}\hat{b}}=\varepsilon\ \delta^b_a. \qquad (5)$$

Now, we can state our first result.

Theorem 1. *A six-dimensional planar Hermitian submanifold of the octave algebra is an Einstein manifold if and only if its second fundamental form satisfies the conditions (5).*

As we already have noted, the condition $\mu = i$ is a criterion for a six-dimensional planar Hermitian submanifold of Cayley algebra to be Kähler. Substituting this condition in (5), we obtain our second result.

Theorem 2. *A six-dimensional Kähler submanifold of the octave algebra is an Einstein manifold if and only if the following equalities are fulfilled:*

$$-2T^7_{\hat{a}\hat{c}}\,T^7_{cb}\ =\ \varepsilon\ \delta^a_b;\quad -2T^7_{ac}\,T^7_{\hat{c}\hat{b}}\ =\ \varepsilon\ \delta^b_a.$$

At the end, we observe that the constant ε is negative both for Kähler and for non-Kähler six-dimensional planar Hermitian submanifolds of the octave algebra.

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On some almost contact metric hypersurfaces of Kählerian manifolds

Mihail Banaru

Abstract

It is proved that a minimal hypersurfaces of a Kählerian manifold, equipped with an almost contact metric Kirichenko–Uskorev structure, is totally umbilical if and only if it is totally geodesic.

Keywords: Kählerian manifold, almost contact metric structure, Kirichenko–Uskorev structure, minimal hypersurface, second fundamental form.

1 Introduction

It is known that the almost contact metric structure is one of the most important differential-geometrical structures on manifolds. As the most meaningful examples of almost contact metric structures we can consider the cosymplectic structure, the nearly cosymplectic structure and the Kenmotsu structure. These structures and their numerous generalizations are profoundly studied from the point of view of differential geometry as well as from the point of view of modern theoretical physics.

In [5] Vadim F. Kirichenko and Ivan V. Uskorev have introduced a new class of almost contact metric structure. Namely, they have defined the almost contact metric structure with a close contact form as the structures of cosymplectic type.

It is evident that a trivial example of Kirichenko–Uskorev (KU-) structure is the cosymplectic structure. As the non-trivial examples of the almost contact metric KU-structure we can consider the nearly cosymplectic and Kenmotsu structures.

In this paper we consider almost contact metric minimal hypersurfaces in Kählerian manifolds. The main result is the following:

Theorem 1. *A minimal hypersurfaces of a Kählerian manifold, equipped with an almost contact metric Kirichenko–Uskorev structure, is totally umbilical if and only if it is totally geodesic.*

2 Preliminaries

Let N be an odd-dimensional smooth oriented manifold, η be a differential 1-form called a contact form, ξ be a vector field called a characteristic vector, Φ be an endomorphism of the module $\aleph(N)$ of smooth vector fields on N called a structure endomorphism. In this case the triple $\{\Phi,\, \xi,\, \eta\}$ is called an almost contact structure on the manifold N if the following conditions are fulfilled:

$$\eta(\xi) = 1,\; \Phi(\xi) = 0,\; \eta \circ \Phi = 0,\; \Phi^2 = -id + \xi \otimes \eta.$$

If in addition there is a Riemannian metric $\langle \cdot, \cdot \rangle$ on the manifold N such that

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta\,(X)\,\eta\,(Y)\,,\; X, Y \in \aleph(N),$$

then the tensor system $\{\Phi,\, \xi,\, \eta,\, \langle \cdot, \cdot \rangle\}$ is called an almost contact metric structure on this manifold [4].

We remind also that an almost Hermitian manifold is an even-dimensional manifold M^{2n}, equipped with a Riemannian metric $g = \langle \cdot,\, \cdot \rangle$ and an almost complex structure J. These objects must satisfy the following condition:

$$\langle JX,\; JY \rangle = \langle X,\; Y \rangle\,, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on the manifold M^{2n} [4]. The fundamental form F of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY \rangle\,, \quad X, Y \in \aleph(M^{2n}).$$

An almost Hermitian structure is Kählerian if $\nabla F = 0$, where ∇ is the Riemannian connection of the metric $g = \langle\cdot,\ \cdot\rangle$ [4].

In some papers (see for example, [1],[2] and [7]) the following Cartan structural equations of an almost contact metric structure on a hypersurface of a Kählerian manifold of dimension at least six were considered:

$$\begin{gathered} d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + i\sigma^\alpha_\beta \omega^\beta \wedge \omega + i\sigma^{\alpha\beta}\omega_\beta \wedge \omega; \\ d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta - i\sigma^\beta_\alpha \omega_\beta \wedge \omega - i\sigma_{\alpha\beta}\omega^\beta \wedge \omega; \\ d\omega = -i\sigma^\alpha_\beta\, \omega^\beta \wedge \omega_\alpha + i\sigma_{n\beta}\omega \wedge \omega^\beta - i\sigma^\beta_n \omega \wedge \omega_\beta. \end{gathered} \tag{1}$$

Here $\{\omega^\alpha\}$, $\{\omega_\alpha\}$ are the components of the displacement forms ($\omega^n = \omega$), and $\{\omega^\alpha_\beta\}$ are the components of the Riemannian connection forms; σ is the second fundamental form of the immersion of the hypersurface N^{2n-1} into a Kählerian manifold M^{2n}, $n \geq 3$. Note that $\omega_\alpha = \omega^{\hat{\alpha}}$; $\alpha,\ \beta = 1, \ldots n-1$; $a,\ b = 1, \ldots n$; $\hat{a} = a + n$.

We note that all considered manifolds, tensor fields etc are assumed to be of the class C^∞ .

3 Proof of the theorem

Vadim F. Kirichenko and Ivan V. Uskorev have proved [5] that the condition

$$d\omega = 0$$

is a criterion for an arbitrary almost contact metric structure to be of cosymplectic type. Taking in account the Cartan structural equations (1), we conclude that the fulfillment of the following equalities

$$1)\ \sigma^\alpha_\beta = 0;\ \ 2)\ \sigma^\beta_n = 0;\ \ 3)\ \sigma_{n\beta} = 0$$

is also a criterion for an arbitrary almost contact metric structure on a hypersurface of a Kählerian manifold M^{2n}, $n \geq 3$, to be Kirichenko–Uskorev. Evidently, only the components $\sigma_{\alpha\beta}$, $\sigma_{\hat{\alpha}\hat{\beta}}$ and σ_{nn}, can not vanish. That is why the matrix of the second fundamental form of the immersion of the Kirichenko–Uskorev hypersurface N^{2n-1} into a Kählerian manifold M^{2n}, $n \geq 3$, looks as follows:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} \sigma_{\alpha\beta} & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & 0 \\ \hline 0\dots0 & \sigma_{nn} & 0\dots0 \\ \hline 0 & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & \sigma_{\hat{\alpha}\hat{\beta}} \end{array}\right), \ p,s = 1,...,2n-1.$$

Now let N^{2n-1} be a minimal hypersurface of a Kählerian manifold M^{2n}, $n \geq 3$. Then in accordance with the definition [6]

$$g^{ps}\sigma_{ps} = 0.$$

Taking into account that the matrix of the contravariant metric tensor is as follows [3]:

$$(g^{ps}) = \left(\begin{array}{c|c|c} 0 & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & I_{n-1} \\ \hline 0\dots0 & 1 & 0\dots0 \\ \hline I_{n-1} & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & 0 \end{array}\right), \quad p,s = 1,...,2n-1,$$

we obtain for a hypersurface with Kirichenko–Uskorev almost contact metric structure of a Kählerian manifold M^{2n}, $n \geq 3$:

$$g^{ps}\sigma_{ps} = g^{\alpha\beta}\sigma_{\alpha\beta} + g^{\hat{\alpha}\hat{\beta}}\sigma_{\hat{\alpha}\hat{\beta}} + g^{\hat{\alpha}\beta}\sigma_{\hat{\alpha}\beta} + g^{\alpha\hat{\beta}}\sigma_{\alpha\hat{\beta}} + g^{nn}\sigma_{nn} =$$

$$= g^{\hat{\alpha}\beta}\sigma_{\hat{\alpha}\beta} + g^{\alpha\hat{\beta}}\sigma_{\alpha\hat{\beta}} + g^{nn}\sigma_{nn} = \sigma_{nn}.$$

We get

$$g^{ps}\sigma_{ps} = 0 \Leftrightarrow \sigma_{nn} = 0 \ \Leftrightarrow \sigma(\xi,\xi) = 0.$$

So, we can represent more exact matrix of the second fundamental form of the immersion of the minimal KU-hypersurface N^{2n-1} into a Kählerian manifold M^{2n}, $n \geq 3$:

$$(\sigma_{ps}) = \left(\begin{array}{cc|c|cc} \multicolumn{2}{c|}{\multirow{2}{*}{\Large 0}} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\sigma_{\alpha\hat{\beta}}$}} \\ & & 0 & & \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\sigma_{\hat{\alpha}\beta}$}} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{\Large 0}} \\ & & 0 & & \end{array}\right). \qquad (2)$$

At the end of this section, let a hypersurface with Kirichenko–Uskorev almost contact metric structure N^{2n-1} be a totally umbilical submanifold of a Kählerian manifold M^{2n}, $n \geq 3$. Then $\sigma_{ps} = \lambda g_{ps}$, $\lambda - const.$ Using (2), we obtain $\lambda = 0$, consequently the matrix (σ_{ps}) vanish. That is why in this case the hypersurface N^{2n-1} is totally geodesic, Q.E.D.

4 Conclusion

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Configurations of invariant straight lines of the type $(2, 2, 1, 1)$ for a family of cubic systems

Cristina Bujac, Dana Schlomiuk, Nicolae Vulpe

Abstract

We construct all possible configurations of invariant straight lines for the class $\mathbf{CSL}^{2r2c\infty}_{(2,2,1,1)}$. We prove that there are exactly 14 distinct such configurations and present corresponding examples for the realization of each one of the detected configurations.

Keywords: cubic system, affine transformation, invariant straight line, infinite and finite singularities, multiplicity of an invariant line and singularity, configuration of invariant straight lines.

1 Introduction

Consider real polynomial differential cubic systems

$$\frac{dx}{\partial t} = P(x,y), \quad \frac{dy}{\partial t} = Q(x,y), \tag{1}$$

where $P, Q \in R[x,y]$, i.e. are polynomials in x, y with real coefficients and $\max(\deg P, \deg Q) = 3$.

In this work we consider a particular case of systems which possess invariant straight lines. A line $f(x,y) = ux + vy + w = 0$ over C is an invariant line if and only if there exists $K(x,y) \in \mathbb{C}[x,y]$ which satisfies the following identity in $\mathbb{C}[x,y]$: $uP(x,y) + vQ(x,y) = (ux + vy + w)K(x,y)$.

It is well known that a system (1) could possess at most 9 invariant straight lines, including the line at infinity. Such systems were investigated in [6], where the authors detected 23 configurations. Systems possessing one line less, i.e. 8 invariant straight lines, where the line at infinity is considered, have been made in [1], [2], [3], [4] and 51 distinct configurations have been detected. Here we continue this investigation for system in $\mathbf{CSL}_7$, i.e. the class of systems having 7 invariant straight lines, including the infinite one.

We say that a cubic system belonging to $\mathbf{CSL}_7$ possesses a *configuration of the type* $(2,2,1,1)$ if there exists two couples of parallel lines and two additional straight lines, every set with different slope (see [7] for the definition of *configuration of invariant straight lines* and *type of configuration*). Such family of systems we will denote by $\mathbf{CSL}^{2r2c\infty}_{(2,2,1,1)}$. We note that all configurations of the straight lines are presented on the Poincaré disc.

2 Main results

Theorem 1. *A cubic system in $CSL^{2r2c\infty}_{(2,2,1,1)}$ could possess only one of the following 14 configurations of invariant straight lines Config. 7.1d–Config. 7.14d given in* Figure 1. *Moreover applying the group of real affine transformations and a time rescaling any system in this class could be brought to one of the following 8 canonical systems which have the indicated configurations, correspondingly:*

$$\begin{cases} \dot{x} = (r-2)x^3 + 2sx^2y + rxy^2, \qquad r(r-1)(r-2)(r-3) \neq 0, \\ \dot{y} = -sx^3 + (r-3)x^2y + sxy^2 + (r-1)y^3 \end{cases}$$

$$\Downarrow$$

Config. 7.1d $\Leftrightarrow s \neq 0$; *Config. 7.2d* $\Leftrightarrow s = 0$.

$$\begin{cases} \dot{x} = \left(r^3 - 9r + 9\right)x + (r-3)sx^2 + (r-2)x^3 + rxy^2 + 2sx^2y \\ \qquad -(r-3)(2r-3)xy, \qquad (r-1)(r-2)(r-3) \neq 0, \\ \dot{y} = (r-3)(r-1)r(2r-3) - r(2r-3)sx + (r-3)sxy \\ \qquad -3(r-1)\left(r^2 - 3r + 3\right)y + (r-3)(2r-3)x^2 - sx^3 \\ \qquad +(r-3)x^2y + sxy^2 + (r-1)y^3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \textit{Config.7.3d} \Leftrightarrow s \neq 0,\ r < 1; \ \textit{Config.7.4d} \Leftrightarrow s \neq 0, 1 < r < 2; \\ \textit{Config.7.5d} \Leftrightarrow s \neq 0, r > 2; \ \textit{Config.7.6d} \Leftrightarrow s = 0, r < 1; \\ \textit{Config.7.7d} \Leftrightarrow s = 0, 1 < r < 2, \ \textit{Config.7.8d} \Leftrightarrow s = 0, r > 2. \end{cases}$$

$$\begin{cases} \dot{x} = (r-2)x^3 + rxy^2 + (r-2)x/(2r-3)^2 - xy, \\ \dot{y} = (r-3)x^2y + (r-1)y^3 + (r-2)y/(2r-3)^2 - y^2; \\ \qquad (r-2)(r-3)(2r-3) \neq 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \textit{Config.7.6d} \Leftrightarrow r < 1; \quad \textit{Config.7.7d} \Leftrightarrow 1 < r < 2; \\ \textit{Config.7.8d} \Leftrightarrow r > 2; \quad \textit{Config.7.11d} \Leftrightarrow r = 1. \end{cases}$$

Configurations of invariant straight lines of the type $(2,2,1,1)$

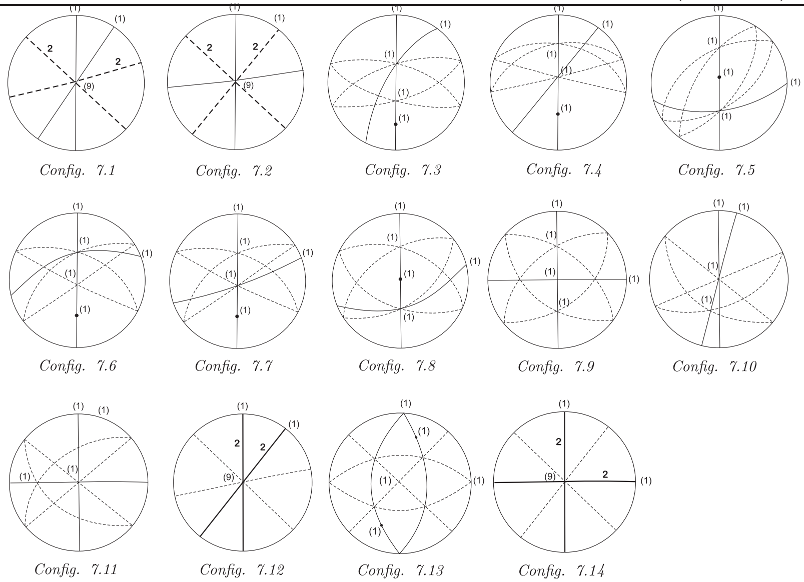

Figure 1. Configurations of invariant straight lines for the class $\mathbf{CSL}^{2r2c\infty}_{(2,2,1,1)}$

$$\begin{cases} \dot{x} = x\big[(r-2)(x^2-a)+ry^2\big], & r(r-1)(r-2)(r-3)(2r-3)\neq 0,\\ \dot{y} = y\big[(r-3)x^2+(r-1)(a+y^2)\big] \end{cases}$$

$$\Downarrow$$

Config. 7.9d $\Leftrightarrow a \neq 0$; *Config. 7.2d* $\Leftrightarrow a = 0$.

$$\{\ \dot{x} = (x^2-1)(x+vy), \quad \dot{y} = (y^2+1)(vx-y), \quad v\neq 0$$

$$\Downarrow$$

Config. 7.13d..

$$\begin{cases} \dot{x} = & x^2\left(2r-s^2-3\right)+(r-1)\left(s^2+1\right)x-\\ & -2(r-1)sxy+(r-2)x^3+rxy^2+2sx^2y,\\ \dot{y} = & (r-1)\left(s^2+1\right)y+xy\left(2r-s^2-3\right)-\\ & -2(r-1)sy^2+(r-3)x^2y+(r-1)y^3-\\ & -sx^3+sxy^2, \qquad r(r-1)\neq 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \textit{Config.7.5d} & \Leftrightarrow s\neq 0, r<1;\ \textit{Config.7.4d} \Leftrightarrow s\neq 0, 1<r<2;\\ \textit{Config.7.3d} & \Leftrightarrow s\neq 0, r>2;\ \ \textit{Config.7.10d} \Leftrightarrow s\neq 0, r=2;\\ \textit{Config.7.8d} & \Leftrightarrow s=0, r<1;\ \ \textit{Config.7.7d} \Leftrightarrow s=0, 1<r<2;\\ \textit{Config.7.6d} & \Leftrightarrow s=0; r>2,\ \ \textit{Config.7.11d} \Leftrightarrow s=0, r=2. \end{cases}$$

$$\begin{cases} \dot{x} = -x - sx^2 + xy + 2sx^2y - x^3 + xy^2, \\ \dot{y} = -y - sxy + y^2 - sx^3 + sxy^2 - 2x^2y \end{cases}$$

$$\Downarrow$$

Config. 7.10d $\Leftrightarrow s \neq 0$; *Config. 7.11d* $\Leftrightarrow s = 0$.

$$\begin{cases} \dot{x} = (1 + 2s^2 + sv)x^3 + (s + v)x^2y, \\ \dot{y} = -sx^3 + s(2s + v)x^2y + vxy^2 - y^3, \\ \qquad (s + v)(3s + v) \neq 0 \end{cases}$$

$$\Downarrow$$

Config. 7.12d $\Leftrightarrow s \neq 0$; *Config. 7.14d* $\Leftrightarrow s = 0$.

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Some classes of polynomial bidimensional cubic differential systems with the symmetry axis

Iurie Calin, Valeriu Baltag

Abstract

The autonomous bidimensional polynomial cubic systems of differential equations with pure imaginary eigenvalues of the Jacobian matrix at the singular point $(0,0)$ are considered in this paper. For these cubic systems the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions to having the symmetry axis, which passes through the origin of the coordinates of the phase plane, were obtained. Also, for the mentioned systems $GL(2,\mathbb{R})$-invariant sufficient center conditions for the origin of the coordinates of the phase plane were established.

Keywords: polynomial differential systems, invariant, comitant, transvectant, center conditions, symmetry axis.

1 Preliminaries

Let us consider the cubic system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= P_1(x,y) + P_2(x,y) + P_3(x,y) = P(x,y),\\ \frac{dy}{dt} &= Q_1(x,y) + Q_2(x,y) + Q_3(x,y) = Q(x,y),\end{aligned} \tag{1}$$

where $P_i(x,\ y)$, $Q_i(x,\ y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

Definition 1 [1]. *Let $\varphi(x,y)$ and $\psi(x,y)$ be homogeneous polynomials in x and y with real coefficients of the degrees $\rho_1 \in \mathbb{N}^*$ and*

$\rho_2 \in \mathbb{N}^*$, *respectively, and* $j \in \mathbb{N}^*$. *The polynomial*

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1-j)!(\rho_2-j)!}{\rho_1!\rho_2!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{\partial^j\varphi}{\partial x^{j-i}\partial y^i}\frac{\partial^j\psi}{\partial x^i\partial y^{j-i}} \qquad (2)$$

is called the transvectant of index j *of polynomials* φ *and* ψ.

Remark 1 [2]. *If polynomials* φ *and* ψ *are* $GL(2,\mathbb{R})$*-comitants* [3] *of the degrees* $\rho_1 \in \mathbb{N}^*$ *and* $\rho_2 \in \mathbb{N}^*$, *respectively, for the system* (1), *then the transvectant of index* $j \le \min\{\rho_1,\rho_2\}$ *is a* $GL(2,\mathbb{R})$*-comitant of the degree* $\rho_1+\rho_2-2j$ *for the system* (1). *If* $j > \min\{\rho_1,\rho_2\}$, *then* $(\varphi,\psi)^{(j)} = 0$.

The $GL(2,\mathbb{R})$-comitants of the first degree with respect to the coefficients of the system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ i = 1,2,3.$$

By using the comitants R_i and S_i ($i = 1,2,3$), and the transvectant (2) the following $GL(2,\mathbb{R})$-invariants [3] of the system (1) were constructed:

$$I_1 = S_1, \qquad I_2 = (R_1,R_1)^{(2)}, \qquad I_4 = (R_1,S_3)^{(2)},$$
$$I_{18} = ((R_3,R_1)^{(2)},S_3)^{(2)}, \qquad I_{20} = ((R_2,R_2)^{(2)},S_3)^{(2)},$$
$$I_{22} = ((S_3,S_2)^{(1)},S_2)^{(1)}, \qquad I_{38} = (((R_2,R_1)^{(2)},R_1)^{(1)},S_2)^{(1)},$$
$$I_{61} = (((R_3,S_3)^{(2)},R_1)^{(1)},S_3)^{(2)},$$
$$I_{111} = ((((R_3,R_1)^{(2)},R_1)^{(1)},S_2)^{(1)},S_2)^{(1)},$$
$$I_{112} = (((R_2,R_1)^{(2)},R_1)^{(1)},(R_2,S_3)^{(2)})^{(1)},$$
$$I_{125} = ((((R_2,R_1)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)},$$
$$I_{174} = (((R_2,S_3)^{(2)},S_3)^{(1)},(R_2,S_3)^{(2)})^{(1)},$$
$$I_{278} = (((((R_3,R_1)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)}.$$

For the cubic systems (1) with $I_1 = 0$, $I_2 > 0$, $2S_2 + 3S_3 \not\equiv 0$ the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions to having the symmetry axis, which passes through the origin of the coordinates of the phase plane of the system, were established in this paper. In addition, for the systems (1) with $I_1 = 0$, $I_2 > 0$ sufficient $GL(2, \mathbb{R})$-invariant center conditions for the origin of the coordinates of the phase plane of the system, were obtained.

2 $GL(2, \mathbb{R})$-invariant conditions for the systems (1) with $I_1 = 0$, $I_2 > 0$, $2S_2 + 3S_3 \not\equiv 0$ to having the symmetry axis

Theorem 1. *The system* (1) *with* $I_1 = 0$, $I_2 > 0$, $S_3 \equiv 0$ *and* $S_2 \not\equiv 0$ *has the symmetry axis, which passes through the origin of the coordinates of the phase plane of the system* (1), *if and only if the following conditions are fulfilled:* $I_{38} = I_{111} = I_{125} = I_{278} = 0.$

Theorem 2. *The system* (1) *with* $I_1 = 0$, $I_2 > 0$, $S_2 \equiv 0$ *and* $S_3 \not\equiv 0$ *has the symmetry axis, which passes through the origin of the coordinates of the phase plane of the system* (1), *if and only if the following conditions are fulfilled:* $I_4 = I_{18} = I_{20} = I_{61} = I_{112} = I_{174} = 0.$

Theorem 3. *The system* (1) *with* $I_1 = 0$, $I_2 > 0$, $S_2 \not\equiv 0$ *and* $S_3 \not\equiv 0$ *has the symmetry axis, which passes through the origin of the coordinates of the phase plane of the system* (1), *if and only if the following conditions are fulfilled:* $I_4 = I_{22} = I_{38} = I_{111} = I_{125} = I_{278} = 0.$

3 Sufficient $GL(2, \mathbb{R})$-invariant center conditions for some classes of systems (1) with $I_1 = 0$, $I_2 > 0$

Theorem 4. *If the system* (1) *with* $I_1 = 0$, $I_2 > 0$, $S_3 \equiv 0$ *fulfills the conditions*

$$I_{38} = I_{111} = I_{125} = I_{278} = 0,$$

then the origin of the coordinates of the phase plane of the system (1) *is a singular point of the center type.*

Theorem 5. *If the system* (1) *with* $I_1 = 0$, $I_2 > 0$, $S_2 \equiv 0$ *fulfills the conditions*

$$I_4 = I_{18} = I_{20} = I_{61} = I_{112} = I_{174} = 0,$$

then the origin of the coordinates of the phase plane of the system (1) *is a singular point of the center type.*

Theorem 6. *If the system* (1) *with* $I_1 = 0$, $I_2 > 0$ *and* $S_2 \not\equiv 0$ *fulfills the conditions*

$$I_4 = I_{22} = I_{38} = I_{111} = I_{125} = I_{278} = 0,$$

then the origin of the coordinates of the phase plane of the system (1) *is a singular point of the center type.*

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A class of bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with the symmetry axis

Iurie Calin, Stanislav Ciubotaru

Abstract

A class of the autonomous bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with pure imaginary eigenvalues of the Jacobian matrix at the singular point $(0,0)$ is considered in this paper. For this class of differential systems the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions to having the symmetry axis, which passes through the origin of the coordinates of the phase plane, were obtained. Also, for the mentioned systems $GL(2,\mathbb{R})$-invariant sufficient center conditions for the origin of the coordinates of the phase plane of the system, were established.

Keywords: polynomial differential systems, invariant, comitant, transvectant, center conditions, symmetry axis.

1 Preliminaries

Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\begin{aligned} \frac{dx}{dt} &= \mathbf{P}_1(x,y) + \mathbf{P}_4(x,y) = \mathbf{P}(x,y), \\ \frac{dy}{dt} &= \mathbf{Q}_1(x,y) + \mathbf{Q}_4(x,y) = \mathbf{Q}(x,y), \end{aligned} \tag{1}$$

where $\mathbf{P}_i(x,y)$, $\mathbf{Q}_i(x,y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

Definition 1 [1]. *Let $\varphi(x,y)$ and $\psi(x,y)$ be homogeneous polynomials in x and y with real coefficients of the degrees $\rho_1 \in \mathbb{N}^*$ and $\rho_2 \in \mathbb{N}^*$, respectively, and $j \in \mathbb{N}^*$. The polynomial*

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1-j)!(\rho_2-j)!}{\rho_1!\rho_2!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{\partial^j\varphi}{\partial x^{j-i}\partial y^i}\frac{\partial^j\psi}{\partial x^i\partial y^{j-i}} \qquad (2)$$

is called the transvectant of index j of polynomials φ and ψ.

Remark 1 [2]. *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants* [3,4] *of the degrees $\rho_1 \in \mathbb{N}^*$ and $\rho_2 \in \mathbb{N}^*$, respectively, for the system* (1)*, then the transvectant of index $j \le \min\{\rho_1,\rho_2\}$ is a $GL(2,\mathbb{R})$-comitant of the degree $\rho_1+\rho_2-2j$ for the system* (1)*. If $j > \min\{\rho_1,\rho_2\}$, then $(\varphi,\psi)^{(j)} = 0$.*

The $GL(2,\mathbb{R})$-comitants of the first degree with respect to the coefficients of the system (1) have the form

$$R_i = \mathbf{P}_i(x,y)y - \mathbf{Q}_i(x,y)x,\ \ S_i = \frac{1}{i}\left(\frac{\partial \mathbf{P}_i(x,y)}{\partial x} + \frac{\partial \mathbf{Q}_i(x,y)}{\partial y}\right),\ \ i = \overline{1,4}.$$

By using the comitants R_i and S_i $(i = \overline{1,4})$ and the transvectant (2), the following $GL(2,\mathbb{R})$-invariants [3,4] of the system (1) were constructed:

$$I_1 = S_1, \quad I_2 = (R_1,R_1)^{(2)}, \quad I_3 = (((S_4,R_1)^{(2)},R_1)^{(1)},(S_4,R_1)^{(2)})^{(1)},$$
$$I_4 = ((((R_4,R_1)^{(2)},R_1)^{(2)},R_1)^{(1)},((R_4,R_1)^{(2)},R_1)^{(2)})^{(1)},$$
$$J_1 = ((S_4,S_4)^{(2)},R_1)^{(2)}, \quad J_2 = (((R_4,S_4)^{(2)},R_1)^{(2)},R_1)^{(2)},$$
$$J_3 = ((((R_4,R_4)^{(2)},S_4)^{(3)},S_4)^{(2)},R_1)^{(2)},$$
$$J_4 = ((((R_4,R_4)^{(4)},R_4)^{(2)},S_4)^{(2)},R_1)^{(2)},$$
$$J_5 = ((((R_4,R_4)^{(4)},R_4)^{(1)},S_4)^{(3)},R_1)^{(2)},$$
$$J_6 = ((((((R_4,R_4)^{(2)},R_4)^{(1)},S_4)^{(3)},R_1)^{(2)},R_1)^{(2)},R_1)^{(2)},$$
$$J_7 = ((((R_4,R_4)^{(2)},S_4)^{(3)},R_1)^{(2)} \cdot ((R_4,S_4)^{(3)},S_4)^{(2)},R_1)^{(2)},$$
$$J_8 = (((R_4,S_4)^{(3)} \cdot (R_4,S_4)^{(3)},S_4)^{(3)},(((R_4,S_4)^{(3)},S_4)^{(2)},R_1)^{(1)})^{(1)},$$
$$J_9 = 198\,J_1^2J_4 + 45\,J_1^2J_5 + 900\,J_1J_7 - 200\,I_2J_8,$$
$$J_{10} = 1619\,I_2J_1^2J_4 + 1260\,I_2J_1^2J_5 + 20475\,J_1^2J_6 + 9450\,I_2J_1J_7 + 6300\,I_2^2J_8.$$

In the paper [5] the center-focus problem for the class of the autonomous bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with $I_3 = I_4 = 0$ was investigated. For this class of the systems the necessary and sufficient $GL(2,\mathbb{R})$-invariant center conditions for the origin of the coordinates of the phase plane of the system were established.

In this paper a class of the autonomous bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with $I_1 = 0$, $I_2 > 0$ and $I_3 = 0$ is considered. The conditions $I_1 = 0$, $I_2 > 0$ mean that the eigenvalues of the Jacobian matrix at the singular point $(0, 0)$ are pure imaginary, i.e., the system has the center or a weak focus at $(0, 0)$. For the system (1) with $I_1 = 0$, $I_2 > 0$, $I_3 = 0$ and $S_4 \not\equiv 0$ the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions to having the symmetry axis, which passes through the origin of the coordinates of the phase plane of the system (1), were obtained. Also, for the system (1) with $I_1 = 0$, $I_2 > 0$ and $I_3 = 0$ sufficient $GL(2,\mathbb{R})$-invariant center conditions for the origin of the coordinates of the phase plane of the system (1), were established.

2 $GL(2,\mathbb{R})$-invariant conditions for the system (1) with $I_1 = 0$, $I_2 > 0$, $I_3 = 0$ and $S_4 \not\equiv 0$ to having the symmetry axis

Theorem 1. *The system* (1) *with* $I_1 = 0$, $I_2 > 0$, $I_3 = 0$ *and* $S_4 \not\equiv 0$ *has the symmetry axis, which passes through the origin of the coordinates of the phase plane of the system, if and only if the following conditions are fulfilled:* $J_2 = J_3 = J_9 = J_{10} = 0.$

3 Sufficient $GL(2,\mathbb{R})$-invariant center conditions for system (1) with $I_1 = 0$, $I_2 > 0$ and $I_3 = 0$

Theorem 2. *If the system* (1) *with* $I_1 = 0$, $I_2 > 0$, $I_3 = 0$ *fulfills the* *conditions*

$$J_2 = J_3 = J_9 = J_{10} = 0,$$

then the origin of the coordinates of the phase plane of the system (1) *is a singular point of the center type.*

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Levitan Almost Periodic Solutions of Linear Partial Differential Equation

David Cheban

Abstract

The known Levitan's Theorem states that the finite-dimensional linear differential equation

$$x' = A(t)x + f(t) \tag{1}$$

with Bohr almost periodic coefficients $A(t)$ and $f(t)$ admits at least one Levitan almost periodic solution if it has a bounded solution. The main assumption in this theorem is the separation among bounded solutions of homogeneous equations

$$x' = A(t)x \ . \tag{2}$$

In this paper we prove that infinite-dimensional linear differential equation (1) with Levitan almost periodic coefficients has a Levitan almost periodic solution, if it has at least one relatively compact solution and the trivial solution of equation (2) is Lyapunov stable.

Keywords: almost periodic solution; abstract differential equation.

1 Introduction

Let (X, ρ) be a complete metric space. Denote by $C(\mathbb{R}, X)$ the space of all continuous functions $\varphi : \mathbb{R} \to X$ equipped with the distance

$$d(\varphi, \psi) := \sup_{L>0} \min\{\max_{|t|\leq L} \rho(\varphi(t), \psi(t)), L^{-1}\}.$$

Let $h \in \mathbb{R}$ and $\varphi \in C(\mathbb{R}, X)$. Denote by φ^h the h-translation of function φ, i.e., $\varphi^h(t) := \varphi(t+h)$ for any $t \in \mathbb{R}$ and by $\mathfrak{N}_\varphi := \{\{h_k\} : \varphi^{h_k} \to \varphi\}$.

Definition 1. Let $\varepsilon > 0$. A number $\tau \in \mathbb{R}$ is called *ε-almost period* of the function φ if

$$\rho(\varphi(t+\tau), \varphi(t)) < \varepsilon$$

for all $t \in \mathbb{R}$. Denote by $\mathcal{T}(\varphi, \varepsilon)$ the set of ε-almost periods of φ.

Definition 2. A function $\varphi \in C(\mathbb{R}, X)$ is said to be *Bohr almost periodic* if the set of ε-almost periods of φ is *relatively dense* for each $\varepsilon > 0$, i.e., for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that $\mathcal{T}(\varphi, \varepsilon) \cap [a, a+l] \neq \emptyset$ for all $a \in \mathbb{R}$.

Definition 3. Let $\varphi \in C(\mathbb{R}, X)$ and $\psi \in C(\mathbb{R}, Y)$. A function $\varphi \in C(\mathbb{R}, X)$ is called *Levitan almost periodic* if there exists a Bohr almost periodic function $\psi \in C(\mathbb{R}, Y)$ such that $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$, where Y is some metric space (generally speaking $Y \neq X$).

B. M. Levitan (1939) studied the problem of existence of Levitan almost periodic solutions of equation

$$x' = A(t)x + f(t) \quad (x \in \mathbb{R}^n) \tag{3}$$

with the matrix $A(t)$ and vector-function $f(t)$ Levitan almost periodic.

Along with equation (3), consider the homogeneous equation

$$x' = A(t)x \; . \tag{4}$$

Theorem 1. (Levitan's theorem [2]) Linear differential equation (3) with Bohr almost periodic coefficients admits at least one Levitan almost periodic solution, if it has a bounded solution and each bounded on $\mathbb{R}$ solution $\varphi(t)$ of equation (4) is separated from zero, i.e.

$$\inf_{t \in \mathbb{R}} |\varphi(t)| > 0.$$

Open problem (V. V. Zhikov [3]). Can we state that every equation (3) admits at least one Levitan almost periodic solution if (3) has a bounded on $\mathbb{R}$ solution?

From our result [1] it follows the positive answer to this question. In this paper we generalize this result for infinite-dimensional equations (3).

2 Levitan almost periodic solutions of linear partial differential equations

Let Λ be some complete metric space of linear closed operators acting into Banach space $\mathfrak{B}$ and $f \in C(\mathbb{R}, \mathfrak{B})$. Consider a linear non-homogeneous differential equation

$$x' = A(t)x + f(t), \tag{5}$$

where $A \in C(\mathbb{R}, \Lambda)$. Along this equation (5) consider the corresponding homogeneous equation

$$x' = A(t)x \tag{6}$$

and its H-class, i.e., the following family of equations

$$x' = B(t)x, \tag{7}$$

where $B \in H(A)$. We assume that the following conditions are fulfilled for equation (6) and its H-class (7):

a. for any $u \in \mathfrak{B}$ and $B \in H(A)$ equation (7) has exactly one mild solution $\varphi(t, u, B)$ and the condition $\varphi(0, u, B) = v$;

b. the mapping $\varphi : (t, u, B) \to \varphi(t, u, B)$ is continuous in the topology of $\mathbb{R}_+ \times \mathfrak{B} \times C(\mathbb{R}; \Lambda)$.

Denote by $(H(A), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A)$. Under the above assumptions equation (6) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (H(A), \mathbb{R}, \sigma) \rangle$ over dynamical system $(H(A), \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$. Denote by ψ the mapping from $\mathbb{R}_+ \times \mathfrak{B} \times H(A)$ into $\mathfrak{B}$ defined by equality

$$\psi(t, u, (B, g)) := U(t, B)u + \int_0^t U(t - \tau, B^\tau)g(\tau)d\tau \ .$$

Theorem 2. Assume that the following conditions are fulfilled:

1. $\mathfrak{B}$ is a uniformly convex Banach space;

2. $\mathfrak{B}^s_A := \{u \in \mathfrak{B} |\ \sup\limits_{t\geq 0} |\varphi(t,u,A)| < \infty\}$ is a subspace of the Banach space $\mathfrak{B}$, where φ is a linear cocycle generated by equation (6);

3. the function $f \in C(\mathbb{R},\mathfrak{B})$ and operator-function $A \in C(\mathbb{R},\Lambda)$ are jointly Poisson stable;

4. there exits a relatively on $\mathbb{R}_+$ solution $\psi(t,u_0,A,f)$ of equation (5).

Then there exists at least one Levitan almost periodic solution of equation (5).

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On Topological AG-groupoids and Homogeneous Isotopies

Liubomir Chiriac, Natalia Lupasco, Natalia Josu

Abstract

This paper studies some properties of (n, m)-homogeneous isotopies of topological paramedial groupoids and associative groupoids with (n, m)-identities.

Keywords: groupoids with (n, m)-identities, homogeneous isotopies.

1 Introduction

In this work we study the (n, m)-homogeneous isotopies of paramedial groupoids and associative commutative groupoids with multiple identities. The results established in this paper are related to the results of M. Choban and L. Kiriyak in [2] and to the research papers [3].

2 Basic Notions

A non-empty set G is said to be a groupoid relative to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

If the groupoid G is a topological space and the multiplication operation $(a, b) \to a \cdot b$ is continuous, then G is called a topological groupoid.

An element $e \in G$ is called an identity if $ex = xe = x$ for every $x \in X$.

A quasigroup with an identity is called a loop. A groupoid G is called medial if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

A groupoid G is called paramedial if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

A groupoid G is called bicommutative if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

If a groupoid G contains an element e such that $e \cdot x = x (x \cdot e = x)$ for all x in G, then e is called a left (right) identity element of G.

A groupoid $(G, \cdot)$ is called a *groupoid Abel-Grassmann* or *AG*-groupoid if it satisfies the law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$. We shall use the notations and terminology from [1-3].

3 Groupoids with multiple identities

Consider a groupoid $(G, +)$. For every two elements a, b from $(G, +)$ we denote:

$$1(a, b, +) = (a, b, +)1 = a + b, \text{ and } n(a, b, +) = a + (n-1)(a, b, +),$$
$$(a, b, +)n = (a, b, +)(n-1) + b$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a, b)$ and $(a, b)n$ instead of $n(a, b, +)$ and $(a, b, +)n$.

Definition 1. *Let $(G, +)$ be a groupoid and let $n, m \geq 1$. The element e of the groupoid $(G, +)$ is called:*

- an (n, m)-zero of G if $e + e = e$ and $n(e, x) = (x, e)m = x$ for every $x \in G$;

- an $(n, \propto)$-zero if $e + e = e$ and $n(e, x) = x$ for every $x \in G$;

- an $(\propto, m)$-zero if $e + e = e$ and $(x, e)m = x$ for every $x \in G$.

Clearly, if $e \in G$ is both an $(n, \propto)$-zero and an $(\propto, m)$-zero, then it is also an (n, m)-zero. If $(G, \cdot)$ is a multiplicative groupoid, then the element e is called an (n, m)-identity. The notion of (n, m)-identity was introduced in [2].

Example 1. *Let $(G, \cdot)$ be an AG-groupoid, $e \in G$ and $ex = x$ for every $x \in G$. Then $(G, \cdot)$ is AG-groupoid with $(1, 2)$-identity e in G. Really, if $x \in G$, then $xe \cdot e = e \cdot ex = e \cdot x = x$.*

Let $(G,+)$ be a topological groupoid. A groupoid $(G,\cdot)$ is called a homogeneous isotope of the topological groupoid $(G,+)$ if there exist two topological automorphisms $\varphi,\psi:(G,+)\to(G,+)$ such that $x\cdot y=\varphi(x)+\psi(y)$, for all $x,y\in G$.

For every mapping $f:X\to X$ we denote $f^1(x)=f(x)$ and $f^{n+1}(x)=f(f^n(x))$ for any $n\geq 1$.

Definition 2. *Let $n,m\leq\infty$. A groupoid $(G,\cdot)$ is called an (n,m)-homogeneous isotope of a topological groupoid $(G,+)$ if there exist two topological automorphisms $\varphi,\psi:(G,+)\to(G,+)$ such that:*

1. $x\cdot y=\varphi(x)+\psi(y)$ for all $x,y\in G$;

2. $\varphi\varphi=\psi\psi$;

3. If $n<\infty$, then $\varphi^n(x)=x$ for all $x\in G$;

4. If $m<\infty$, then $\psi^m(x)=x$ for all $x\in G$.

Definition 3. *A groupoid $(G,\cdot)$ is called an isotope of a topological groupoid $(G,+)$ if there exist two homeomorphisms $\varphi,\psi:(G,+)\to(G,+)$ such that $x\cdot y=\varphi(x)+\psi(y)$ for all $x,y\in G$.*

Under the conditions of Definition 3 we shall say that the isotope $(G,\cdot)$ is generated by the homeomorphisms φ,ψ of the topological groupoids $(G,+)$ and write $(G,\cdot)=g(G,+,\varphi,\psi)$.

Example 2. *Let $(R,+)$ be a topological Abelian group of real numbers.*

1. If $\varphi(x)=3x$, $\psi(x)=x$ and $x\cdot y=3x+y$, then $(R,\cdot)=g(R,+,\varphi,\psi)$ is a commutative locally compact medial quasigroup. By virtue of Theorem 7 from [2], there exists a left invariant Haar measure on $(R,\cdot)$.

2. If $\varphi(x)=7x$, $\psi(x)=9x$ and $x\cdot y=7x+9y$, then $(R,\cdot)=g(R,+,\varphi,\psi)$ is a commutative locally compact medial quasigroup and on $(R,\cdot)$ as above, by virtue of Theorem 7 from [2], does not exist any left or right invariant Haar measure.

Example 3. *Denote by $Z_p=Z/pZ=\{0,1,...,p-1\}$ the cyclic Abelian group of order p. Consider the commutative group $(G,+)=(Z_5,+)$, $\varphi(x)=2x$, $\psi(x)=3x$ and $x\cdot y=2x+3y$. Then*

$(G,\cdot) = g(G,+,\varphi,\psi)$ *is a medial, paramedial and bicommutative quasigroup with* $(4,4)$*-identity in* $(G,\cdot)$*, which coincides with the zero element in* $(G,+)$*.*

4 Main Results

The following theorems have been proved:

Theorem 1. *If* $(G,+)$ *is a paramedial groupoid and* e *is an* $(1,p)$*-zero, then every* $(n,1)$*-homogeneous isotope* $(G,\cdot)$ *of the topological groupoid* $(G,+)$ *is AG-groupoid with* $(1,np)$*-identity* e *in* $(G,\cdot)$*.*

Theorem 2. *If* $(G,+)$ *is an associative and commutative groupoid and* e *is an* $(1,p)$*-zero, then every* $(n,1)$*-homogeneous isotope* $(G,\cdot)$ *of the topological groupoid* $(G,+)$ *is AG-groupoid with* $(1,np)$*-identity* e *in* $(G,\cdot)$ *and* $a\cdot b + c = a + c\cdot b$*, for all* $a,b,c \in G$ *and* $n,p \in N$*.*

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Convex families of sets on topological spaces

Mitrofan M. Choban and Ekaterina P. Mihaylova

Abstract

In the present article the E. Michael's [10] concept of convexity is extended on arbitrary topological spaces. Some problems about selections of set-valued mappings and distinct applications are examined.

Keywords: convex families of sets, metrizable families of sets, selection, set-valued mappings.

1 Introduction

All spaces are assumed to be T_0-spaces. By definition, any paracompact space is a Hausdorff space.

The set-valued mapping $F : X \longrightarrow Y$ is called a lower (upper) semicontinuous mapping if the set $F^{-1}(H) = \{x \in X : F(x) \cap H \neq \emptyset\}$ is an open (a closed) subset of the space X for any open (closed) subset H of the space Y.

Let E be a space. The set $\Delta(E) = \{(x, x) : x \in E\}$ is the diagonal of E. If $\Delta(X) \subset U \subset E \times E$, then U is called an entourage of the diagonal and $U^{-1} = \{(x, y) : (y, x) \in U\}$. If U, V are entourages, then $U \circ V = \{(x, y) : (x, z) \in U, (z, y) \in V \ \textit{for some}\ z \in E\}$. The entourage U of the diagonal is an uniform entourage of the space E if there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open entourages such that $V_{n+1} \circ V_{n+1} \subset V_n = V_n^{-1} \subset U$ for each $n \in \mathbb{N}$. Let A be a subset of E and U be an entourage of the diagonal of E. Then we define $U(A) = \{y \in E : (x, y) \in U \ \textit{for some}\ x \in E\}$. In particular, $U(x) = U(\{x\})$ for any point $x \in E$. Denote by $\mathcal{U}^*(E)$ the family of all entourages

U of $\Delta(E)$ for which there exists an open uniform entourage V of the space E such that $\Delta(E) \subset V \subset U$. In this case $\mathcal{U}^*(E)$ is the maximal continuous uniformity (not obligatory separated) on the space E.

The subfamily $\mathcal{V} \subset \mathcal{U}^*(E)$ is complete if: if $U, V \in \mathcal{V}$, then $U^{-1}, U \cap V \in \mathcal{V}$; if $\Delta(E) \subset U \subset V$ and $U \in \mathcal{V}$, then $V \in \mathcal{V}$; for any $U \in \mathcal{V}$ there exists $V \in \mathcal{V}$ such that $V^{-1} = V \subset V \circ V \subset U$; for each $x \in E$ and each $U \in \mathcal{U}^*(E)$ there exists $V \in \mathcal{V}$ such that $V(x) \subset U(x)$. In this case $\mathcal{V}$ is a continuous uniformity on the space E and the uniformities $\mathcal{U}^*(E)$, $\mathcal{V}$ generate the same topologies on the set E.

A (uniform) sub-convex structure on a topological space E assigns a complete family $\mathcal{U}$ of entourages of the diagonal $\Delta(E)$ and to each positive integer $n \in \mathbb{N}$ a subset M_n of E^n and a function $k_n : M_n \times P_n \longrightarrow E$ such that: (a) If $x \in M_1$, then $k_1(x, 1) = x$; (b) If $i, n \in \mathbb{N}$, $n \geq 2$ and $i \leq n$,then $\partial_{(n,i)}(M_n) \subset M_{n-1}$; (c) If $i, n \in \mathbb{N}$, $n \geq 2$, $i \leq n+1$, $x \in M_{n+1}$ and $t \in P_n$, then $k_{n+1}(x, \varrho_{(n,i)}(t)) = k_n(\partial_{(n+1,i)}(x), t)$; (d) If $i, n \in \mathbb{N}$, $i \leq n$, $x = (x_1, x_2, ..., x_{n+1}) \in M_{n+1}$, $t = (t_1, t_2, ..., t_{n+1}) \in P_n$, $x_i = x_{i+1}$, $x' = (x_1, x_2, ..., x_i, x_{i+2}, ..., x_{n+1}) = \partial_{(n+1,i+1)}(x)$ and $t' = (t_1, t_2, ..., t_i + t_{i+1}, x_{i+2}, ..., x_{n+1})$, then $k_{n+1}(x, t) = k_n(x', t')$; (e) If $x \in M_n$, then the mapping $k_{(n,x)} : P_n \longrightarrow E$, where $k_{(n,x)}(t) = k_n(x, t)$ for each $t \in P_n$, is continuous; (fs) For any uniform entourage $V \in \mathcal{U}$ and any number $n \in \mathbb{N}$ there exists a (uniform) open entourage W ($W \in \mathcal{U}$) such that for all $x, y \in M_n$ and $t \in P_n$ we have $(k_n(x, t), k_n(y, t)) \in V$ provided $(x, y) \in W^n$.

A (uniform) convex structure on a topological space E assigns a complete family $\mathcal{U}$ of entourages of the diagonal $\Delta(E)$ and to each positive integer $n \in \mathbb{N}$ a subset M_n of E^n and a function $k_n : M_n \times P_n \longrightarrow E$ such that $\{M_n, k_n, \mathcal{U} : n \in \mathbb{N}\}$ is a (uniform) sub-convex structure with the property: (f) For any uniform entourage $V \in \mathcal{U}$ there exists a (uniform) entourage W ($W \in \mathcal{U}$) such that $(k_n(x, t), k_n(y, t)) \in V$ for all $n \in \mathbb{N}$, $t \in P_n$ and $(x, y) \in W^n$ with $x, y \in M_n$.

A subset A of a space E with a sub-convex structure is admissible if $A^n \subset M_n$ for each $n \in \mathbb{N}$. Let A be an admissible set of a space E with sub-convex structure. Then 1-$conv(A) = \cup\{k_n(A^n \times P_n) : n \in \mathbb{N}\}$ is the 1-convex hull of A, 1-$conv(n$-1)-$conv(A))$ is the n-convex hull of

A and $conv(A) = \cup\{n\text{-}conv(A) : n \in \mathbb{N}\}$ is the convex hull of A. The set A is convex if $A = 1\text{-}conv(A)$.

2 Convex structures on spaces with a fixed pseudo-metric

A family $\mathcal{A}$ of subsets of a space E is called metrizable by the pseudo-metric ρ on E if ρ is a continuous pseudo-metric on the space E and for any $L \in \mathcal{A}$, any point $x \in L$ and any open subset U of E with $x \in U$ there exist an open subset V of E and a number $\epsilon > 0$ such that $x \in V$ and $V(x, \rho, \epsilon) \cap M \subseteq U$ provided $M \in \mathcal{A}$ and $M \cap V \neq \emptyset$. We observe that any set $L \in \mathcal{A}$ is metrizable by the metric ρ on L.

A family $\mathcal{A}$ of subsets of a space E is called a convex (sub-convex) metrizable family by the pseudo-metric ρ on E if ρ is a continuous pseudo-metric on the space E, a family $\mathcal{A}$ is metrizable by the pseudo-metric ρ, there exists a convex (sub-convex) structure $\{M_n, k_n, \mathcal{U} : n \in \mathbb{N}\}$ on E for which $conv(L) \in \mathcal{A}$ for any $L \in \mathcal{A}$ and ρ is a uniform pseudo-metric.

By virtue of reduction results from [2, 3], the following two theorems are true.

Theorem 1. *Let $\mathcal{A}$ be a convex metrizable family of subsets by the pseudo-metric ρ on a space E relatively to a convex structure $\{M_n, k_n, \mathcal{U} : n \in \mathbb{N}\}$ on the space E. Suppose that $F : X \longrightarrow E$ is a lower semicontinuous set-valued mapping of a paracompact space X into the space E and $cl_E(1\text{-}conv(F(x)))$ is complete metrizable by ρ for each $x \in X$. Then there exists a continuous single-valued mapping $f : X \longrightarrow E$ such that $f(x) \in cl_E(1\text{-}conv(F(x)))$ for each point $x \in X$.*

Theorem 2. *Let $\mathcal{A}$ be a sub-convex metrizable family of subsets by the pseudo-metric ρ on a space E relatively to a sub-convex structure $\{M_n, k_n, \mathcal{U} : n \in \mathbb{N}\}$ on the space E. Suppose that $F : X \longrightarrow E$ is a lower semicontinuous set-valued mapping of a paracompact space X with finite dimension $dim X < +\infty$ into the space E and $cl_E(1\text{-}conv(F(x)))$ is complete metrizable by ρ for each $x \in X$. Then there* *exists a continuous single-valued mapping* $f : X \longrightarrow E$ *such that* $f(x)) \in cl_E(1\text{-}conv(F(x)))$ *for each point* $x \in X$.

3 Constructions related to convexity

A term "convexity" is very large and it is used in various branches of mathematics (see [6, 7, 5, 9, 10]). Our interest is focused on such types of convexity that generate convexities like E. Michael's.

Let d be a pseudo-metric on the space E. If $a, b \in E$, then $[a, b]_d = \{x \in E : d(a, b) = d(a, x) + d(x, b)\}$ is the d-interval with the endpoints a, b. A subspace H of L is d-convex in sense of K. Menger (see [9]) if d is a metric on H and for any two distinct points $a, b \in H$ there exists a point $c \in H \cap [a, b]_d$ such that $c \notin \{a, b\}$. If the subspace (H, d) is d-convex and a complete metric space, then for any two distinct points $a, b \in H$ there exists a homeomorphic embedded $\psi : \mathbb{I} \longrightarrow H \cap [a, b]_d$ such that $\psi(0) = a$ and $\psi(1) = b$. This fact permits to introduce the following notion.

An M-convex structure on a topological space E assigns a non-empty subset M of E^2 and a mapping $\mu : \mathbb{M} \times \mathbb{I} \longrightarrow E$ with the properties: a) $\mu(x, x, t) = x$ for all $t \in \mathbb{I}$ and $(x, x) \in M$; b) $\mu(0, x, y)$ $= x$ and $\mu(1, x, y) = y$ for any $(x, y) \in M$; c) if $(x, y) \in M$, $t \in \mathbb{I}$ and $(\mu(x, y, t), x) \in M$, then $\mu(\mu(x, y, t), y, s) = \mu((x, y, t + s(1-t))$ for all $s \in \mathbb{I}$; d) for any $(x, y) \in M$ the mapping $\mu_{(x,y)} : \mathbb{I} \longrightarrow E$, where $\mu_{(x,y)}(t) = \mu(x, y, t)$, is continuous. The notion of continuous M-convex structure is due to I.N. Herstain and S. Milnor [7, 6]. In [1] R. Cauty has proved that each compact with continuous M-convex structure (with conditions (a) and (b)) has the point fix property. Distinct applications were indicated in [4].

An M-convex structure (M, μ) on a topological space E is called a geodesic structure (E. Michael [10]) if there is given a complete family $\mathcal{U}$ of entourages of the diagonal $\Delta(E)$ such that for any uniform entourage $V \in \mathcal{U}$ there exists a uniform entourage $W \in \mathcal{U}$ such that $(\mu(x, y, t), \mu(x', y', t)) \in V$ for all $t \in P_n$ provided and $(x, x') \in V$, $(y, y') \in W$ and $(x, y), (x', y') \in M$. Some convex structures on topological groups are studied too. In particular, we proved

Theorem 3. *Let H be a convex complete metrizable subgroup of a topological group E and $\psi : E \longrightarrow E/H$ be the quotient mapping of E onto quotient space E/H. Then for any continuous single-valued mapping $f : X \longrightarrow E/H$ of a paracompact space X into E/H there exists a continuous single-valued mapping $g : X \longrightarrow E$ such that $f = \psi \circ g$. In particular, if Y is a paracompact subspace of the space E/H, then there exists a subspace Z of E such that $\psi(Z) = Y$ and $\psi|Z : Z \longrightarrow Y$ is a homeomorphism.*

Corollary 1. *Let H be a convex complete metrizable by a left-invariant metric subgroup of a topological group E and $\psi : E \longrightarrow E/H$ be the quotient mapping of E onto quotient space E/H. Then for any continuous single-valued mapping $f : X \longrightarrow E/H$ of a paracompact space X into E/H there exists a continuous single-valued mapping $g : X \longrightarrow E$ such that $f = \psi \circ g$. In particular, if E is a k-space and E/H is a paracompact space or H is a normal subgroup of E, then there exists a subspace Z of E such that $\psi(Z) = E/H$, $\psi|Z : Z \longrightarrow Y$ is a homeomorphism and $g : Z \times H \longrightarrow E$, where $g(z, x) = z \cdot x$, is a homeomorphism of $Z \times H$ onto E.*

For H a linear closed subspace of a Banach space E Corollary 1 was proved by R. Bartle and L. M. Graves, E. Michael and P. Kenderov (see [10, 8]). Applying results of A. Nijenhuis [11], E. Michail ([10], Proposition 6.2) proved the existence of geodesic structures on every compact Riemannian manifold.

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Integrating factors of a cubic system with two parallel invariant lines and one invariant cubic

Dumitru Cozma, Angela Matei

Abstract

We determine the center conditions for a cubic differential system with two parallel invariant straight lines and one invariant cubic by constructing Darboux integrating factors.

Keywords: cubic differential system, invariant algebraic curves, Darboux integrability, the problem of the center.

1 Introduction

We consider the cubic differential system

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -x - gx^2 - dxy - by^2 - sx^3 - qx^2y - nxy^2 - ly^3 \equiv Q(x,y), \end{aligned} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point of a center or a focus type for (1). In this paper we obtain the center conditions for cubic system (1), with two parallel invariant straight lines and one irreducible invariant cubic, using the method of Darboux integrability.

The integrability conditions for cubic systems (1) with two invariant straight lines and one invariant conic were obtained in [1] and for some families of cubic systems with two invariant straight lines and one invariant cubic were determined in [2].

2 Algebraic solutions

We study the problem of the center for cubic system (1) with irreducible invariant algebraic curves, called *algebraic solutions* of (1).

Let the cubic system (1) have two parallel invariant straight lines $l_1 = 0$ and $l_2 = 0$ real or complex ($l_2 = \overline{l_1}$). Without loss of generality we can assume that the lines are parallel to the axis of ordinates (Oy)

$$l_{1,2} \equiv 1 + \frac{c \pm \sqrt{c^2 - 4m}}{2} x = 0. \tag{2}$$

In [1] it was proved the following Theorem.
Theorem 1. *The cubic system (1) has two parallel invariant straight lines of the form (2) if and only if the following conditions hold*

$$a = f = k = p = r = 0, \ m(c^2 - 4m) \neq 0. \tag{3}$$

According to [1] the cubic system (1) can have real invariant cubic curves of the form

$$\Phi \equiv a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^2 + y^2 = 0 \tag{4}$$

or

$$\begin{aligned} \Phi \equiv\ & a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \\ & + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0, \end{aligned} \tag{5}$$

where $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0, \ \ a_{ij} \in \mathbb{R}$.

3 Darboux integrating factors

It is known [1] that a singular point $O(0,0)$ is a center for system (1) if and only if the system has a holomorphic integrating factor of the form $\mu(x,y) = 1 + \sum_{k=1}^{\infty} \mu_k(x,y)$, where μ_k are homogeneous polynomials of degree k in some neighborhood of $O(0,0)$.

The function $\mu = \mu(x,y) \neq 0$ is an integrating factor of a system (1) if and only if

$$P(x,y) \cdot \frac{\partial \mu}{\partial x} + Q(x,y) \cdot \frac{\partial \mu}{\partial y} + \mu \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \equiv 0. \tag{6}$$

An integrating factor constructed from invariant algebraic curves $f_j(x,y)=0,\ j=\overline{1,q}$
$$\mu = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_q^{\alpha_q} \tag{7}$$
with $\alpha_j \in \mathbb{C}$ not all zero is called *a Darboux integrating factor.*

In this paper we give the conditions under which the cubic system (1) has Darboux integrating factors of the form (7) composed of two parallel invariant straight lines $l_1 = 0,\ l_2 = 0$ and one irreducible invariant cubic $\Phi = 0$, where $\alpha_j \in \mathbb{C}$.

Theorem 2. *The cubic system (1) has an integrating factor of the form (7), composed of two parallel invariant straight lines and one invariant cubic, if and only if one of the sets of conditions* (i)–(v) *is satisfied:*

(i) $a=f=k=p=r=0,\ l=[d(5b-c+4g)]/9,\ q=[d(c-2b-g)]/3,$ $m=2(c-2b-2g)(b+g),\ n=[(5b-c+4g)(2b-c+4g)]/3,$ $s=[(c-2b-4g)(2b-c+g)]/9,\ d^2=(4b+c+2g)(2b-c+4g);$

(ii) $a=f=k=l=p=r=0,\ g=b+c,\ n=b(2b+c),$ $m=-2n,\ s=-n,\ q=[16b^2(2b+c)]/(3d),\ 16b^2-3d^2=0;$

(iii) $q=s=0,\ m=3(c-3b-3g)(b+g),\ n=2(3b-c+3g)(b+g),$ $l=[(b+g)d]/3,\ 3(c-4g-3b)(3b+2g)+d^2=0;$

(iv) $l=-d(b+g)/6,\ q=[3d(b+g)]/2,\ s=[3(-3b^2-4bg-g^2)]/4,$ $3(3b+g)(c-g)+2d^2=0,\ (3(3b+g)^2+d^2)(b+g)-3(3b+g)n=0,$ $(27b^2+18bg+4d^2+3g^2)(b+g)+4m(3b+g)=0;$

(v) $l=[d(c-2g-2b)]/3,\ m=(2c-3g-3b)(3b+3g-c),\ s=-n,$ $n=(c-2g-3b)(c-3g-3b),\ q=d(3b-c+3g),\ d^2+3(3b+2g-c)(c-g)=0.$

Proof. First we consider the cubic curves of the form (4). Then identifying the coefficients of the monomials x^iy^j in (6), we obtain a system of fifteen equations for the unknowns a_{30}, a_{21}, a_{12}, a_{03}, α_1, α_2, α_3 and the coefficients of system (1). Solving the system we find the sets of conditions (i) and (ii) from Theorem 2.

Let system (1) have invariant cubics of the form (7). Then identifying the coefficients of the monomials x^iy^j in (6), we obtain a system of twenty equations for the unknowns a_{30}, a_{21}, a_{12}, a_{03}, a_{20}, a_{11}, a_{02},

a_{10}, a_{01}, α_1, α_2, α_3 and the coefficients of system (1). From this system we get the sets of conditions (iii), (iv) and (v). In each of these cases we have an integrating factor of the form (7):

In Case (i): $l_1 = 1 + 2(b+g)x$, $l_2 = 1 + (c - 2b - 2g)x$, $\Phi = 3(x^2+y^2) + (2b+4g-c)x^3 + 2dx^2y + (4b+c+2g)xy^2$ and $\alpha_1 = 0$, $\alpha_2 = (-3)/2$, $\alpha_3 = (-1)/2$.

In Case (ii): $l_1 = 1 - 2bx$, $l_2 = 1 + (2b+c)x$, $\Phi = 12(bx-1)(x^2 + y^2) - d(9x^2+y^2)y$ and $\alpha_1 = -1$, $\alpha_2 = 0$, $\alpha_3 = (-4)/3$.

In Case (iii): $l_1 = 1 + 3(b+g)x$, $l_2 = 1 + (c - 3b - 3g)x$, $\Phi_3 = (b+g)d^3y^3 + 3(3b+2g)(3(1+gx)^2 + 3d(1+gx)y + d^2y^2)$ and $\alpha_1 = (-1)/3, \alpha_2 = -1$, $\alpha_3 = -1$.

In Case (iv): $l_1 = 6(3b+g) - [3(3b+g)^2 + 4d^2]x$, $l_2 = 2 + 3(b+g)x$, $\Phi_3 = (b+g)(9bx+3gx-2dy)^3 - 6(3b+g)[3(3b+g)(3b+5g)x^2 - 12dgxy - 4d^2y^2 + 6(3b-g)x - 12dy - 12]$ and $\alpha_1 = -1$, $\alpha_2 = (-2)/3$, $\alpha_3 = -1$.

In Case (v): $l_1 = 3(c-g) + (d^2 + 3(c-g)^2)x$, $l_2 = 3(g-c) + (d^2 + 3g^2 - 3cg)x$, $\Phi_3 = (c-3b-2g)[3(3b-c+3g)(3b-c+2g)x^3 + 3d(c-3b-3g)x^2y + 3(c-g)(c-3b-3g)xy^2 + d(2b+2g-c)y^3] + 3(3b-c+2g)(3b-c+4g)x^2 - 3dgxy + 3(3b+2g-c)(c-g)y^2 + 3(3b+g-c)x - 3dy - 3$ and $\alpha_1 = (-1)/3$, $\alpha_2 = (-2)/3$, $\alpha_3 = -1$. □

We note that the center conditions (i) and (ii) where obtained in the paper [2].

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Construction of a hyperbolic 5-manifold with cusps

Florin Damian

Abstract

A method of construction of complete noncompact hyperbolic 5-manifolds with finite volume is given. Possibilities of metric reconstruction of such manifolds are discussed.

Keywords: hyperbolic 5-manifold, 24-cells, submanifold, isometry group, metric reconstruction, cusps.

In H^5 let $\mathcal{U}_{24}$ be a parabolic bundle of 24 hyperplanes which cut a regular 24-cells on the orthogonal to the bundle horosphere Σ^4. We note two significant features of this polyhedron: firstly it is self-dual, and secondly it is one of four-dimensional Euclidean parallelohedra. The second parabolic bundle is obtained as the image of the bundle $\mathcal{U}_{24}$ when reflected in the hyperplane orthogonal to the axis of symmetry of the bundle of hyperplanes and rotated to the dual position of initial 24-cells. Denote it by $\mathcal{U}'_{24}$. As a whole, each of these two bundles can be moved along their common symmetry line by sliding, bringing them close each to other or pushing them away. The presence of metric parameter makes it possible to obtain, for the dihedral angle between the hyperplanes of the first bundle and the second bundle, some values of the form $2\pi/k$.

As a result of the intersection of these two bundles, vertices of two types are formed. The combinatorial structure of a vertex of the first type is a regular pyramid over three-dimensional cube, where dihedral angles between hyperplanes at the apex of pyramid are equal to $2\pi/3$ and the angle ϕ at the base of the pyramid varies and depends on the position of the bundles $\mathcal{U}_{24}$ and $\mathcal{U}'_{24}$. The bundles can be moved until the one-dimensional edges of any bundle become parallel to their dual hyperplanes of the other bundle, and the new vertex becomes, therefore, improper, i.e. being on the absolute. In this case, the bundle of hyperplanes that defined a vertex and was elliptic becomes parabolic. The vertex figure will be a four-dimensional Euclidean regular pyramid over the cube K^3 with the value of dihedral angles at the apex being equal to $2\pi/3$, which implies the values of dihedral angles at the base of the pyramid being equal to $\pi/4$. Vertices of the second type also arise, but their structure is more difficult to be described, and we give only a final result: a four-dimensional simplex with dihedral angles of $2\pi/3$ at 2-faces incident to one edge and with angles of same value at 2-faces that have no common vertices with the specified edge, for the remained six 2-faces the angle is equal to ϕ. For $\phi = \pi/4$ such a simplex is Euclidean, and therefore the vertices of the second type are also on the absolute. The volume of such a noncompact five-dimensional polyhedron will be finite, since the formed vertices did not go beyond the absolute.

The identification of opposing facets in each of the bundles by the horospherical rotation (the reduction on Σ^4 is a parabolic translation) yields the construction of a complete noncompact hyperbolic 5-manifold with finite volume. The cycles of $(n-2)$-faces in the bundles $\mathcal{U}_{24}$ and $\mathcal{U}'_{24}$ are nonessential. It suffices to consider the reduction of this identification to Σ^4. A regular 24-cells is known as one of the parallelohedra in E^4, and the identification of its opposite faces by translations gives one of the richest Euclidean geometries on a four-dimensional torus. In the isometry group of this torus, the stabilizer of each point is isomorphic to the group F_4 which is the symmetry group of a regular 24-cells. Thus, the obtained 5-dimensional manifold has cusps over such a torus. In the study of complete noncompact hyperbolic manifolds with finite volume an important role is played by the geometry of its cusps, which can be in particular a Euclidean geometry on a torus, or a geometry on another Euclidean spatial form, different from the torus. We can check that the cycles of $(n-2)$-faces of the intersection of hyperfaces, belonging to two different bundles, are nonessential cycles of 3-faces (in each such cycle there are eight 3-faces).

Hyperplanes of reflections of a fundamental polyhedron which contain its symmetry axis, retain the above identification of facets. Reflections in these facets belong to the normalizer of the fundamental group (the so-called normalizing reflections). The hyperplanes themselves determine totally geodesic submanifolds on the manifold. The mentioned above four-dimensional submanifolds are geodesically embedded two-sided and have rather rich symmetry group. Cutting the 5-manifold along any such submanifold yields a connected manifold with two components of boundary. Different ways of gluing again boundaries give new manifolds, in particular non-orientable ones. The manifold admits cutting along a bundle of totally geodesic submanifolds, which allows us to build multisheeted coverings not only with cyclic slip group. The reconstruction of the latter makes it possible to build quite a lot of five-dimensional manifolds which do not cover the initial one. We only note the fact that synthetic methods in geometry (and combinatorial topology) remain rather effective in dimensions 4 and 5. Some examples of this approach in dimension 4 are proposed for Davis manifold. The described method can be also applied to the construction of noncompact hyperbolic 4- and 5-manifold with cusps (and with finite volume) over a Euclidean spatial form different from the torus.

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Units in quasigroups with non-classical Bol-Moufang type identities

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Abstract

We research the existence of units in quasigroups with some identities of non-classical Bol-Moufang type.

Keywords: quasigroup, left unit, right unit, Bol-Moufang type identity.
2000 Mathematics Subject Classification: 20N05

1 Introduction

Information about quasigroups can be found in [1, 6, 7]. An identity based on a single binary operation is of classical Bol-Moufang type "if both sides consist of the same three different letters taken in the same order but one of them occurs twice on each side"[3].

Here we will name Bol-Moufang type identities as "non-classical", if both sides of identity consist of the same three different letters taken in different order, but one of them occurs twice on each side. List of all non-classical type Bol-Moufang identities up to an equivalence is given in [2].

Notice, that both classical and non-classical Bol-Moufang type identities in [2] are named as Bol-Moufang identifies of generalized type.

Definition 1. *An element* $f \in Q$ *is a* left unit *of* $(Q, \cdot)$ *if and only if* $f \cdot x = x$ *for all* $x \in Q$*. An element* $e \in Q$ *is a* right unit *of* $(Q, \cdot)$ *if and only if* $x \cdot e = x$ *for all* $x \in Q$*. An element* $e \in Q$ *is a* (two–sided) unit *of* $(Q, \cdot)$ *if and only if it is both left and right unit. An element* $m \in Q$ *is a* middle unit *of* $(Q, \cdot)$ *if and only if* $x \cdot x = m$ *for all* $x \in Q$*.*

Definition 2. *A quasigroup is a left (right) loop if it has a left (right) unit. A quasigroup is a loop if it has both left and right units.*

We use Prover 9 [5] and Mace 4 [4].

2 Results

Commutative C-loops are defined with the help of identity

$$(y(xy))z = x(y(yz)). \tag{1}$$

The following quasigroup $(Q, *)$ (see Table 1) satisfies identity (1) and it has not left, right, and middle unit.

$*$	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

Table 1.

Commutative Alternative Loops are defined with the help of identity

$$((xx)y)z = z(x(yx)). \tag{2}$$

Quasigroup with identity (2) is a loop but it is not a group. Quasigroup with identity (2) does not have middle unit. See Table 2. Notice, the element 2 is left and right unit in Table 2.

Identity

$$((xx)y)z = (xx)(zy) \tag{3}$$

is called Commutative Nuclear Square identity.

Quasigroup with identity (3) is a commutative loop. This loop does not have middle unit. Quasigroup with Cayley Table 2 satisfies identity (3).

$*$	0	1	2
0	1	2	0
1	2	0	1
2	0	1	2

Table 2.

Identity
$$((yx)x)z = z(x(yx)) \tag{4}$$
is called Commutative Loop identity.

Quasigroup $(Q, *)$ with Cayley table (see Table 1) satisfies identity (4). Therefore quasigroup with this identity has no units.

Identity
$$x((xy)z) = (yx)(xz) \tag{5}$$
is called Cheban I identity (in memory of Andrei Martynovich Cheban).

The following quasigroup $(Q, \circ)$ (see Table 3) satisfies identity (5) and it has not left, right, and middle unit.

$\circ$	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

Table 3.

Identity
$$x((xy)z) = (y(zx))x \tag{6}$$
is called Cheban II identity.

Quasigroup $(Q, \circ)$ (see Table 3) satisfies identity (6) and it has not left, right, and middle unit.

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On the Coalitional Rationality and the Egalitarian Nonseparable Contribution

Irinel Dragan

Abstract

In earlier works, we introduced the Inverse Problem, relative to the Shapley Value, then relative to Semivalues. In the explicit representation of the Inverse Set, the solution set of the Inverse Problem, we built a family of games, called the almost null family, in which we determined more recently a game where the Shapley Value and the Egalitarian Allocation are coalitional rational. The Egalitarian Nonseparable Contribution is another value for cooperative transferable utilities games (TU games), showing how to allocate fairly the win of the grand coalition, in case that this has been formed. In the present paper, we solve the similar problem for this new value: given a nonnegative vector representing the Egalitarian Nonseparable Contribution of a TU game, find out a game in which the Egalitarian Nonseparable Contribution is kept the same, but it is coalitional rational. The new game will belong to the family of almost null games in the Inverse Set, relative to the Shapley Value, and it is proved that the threshold of coalitional rationality will be higher than the one for the Shapley Value. Some numerical examples are illustrating the procedure of finding the new game.

Keywords: Shapley Value, the Egalitarian Nonseparable Contribution, Inverse Set, Familly of almost null games, Coalitional Rationality.

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Riemann space in theory of the Navier-Stokes Equations

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Abstract

The Navier-Stokes system of incompressible fluids of equations is studied based on the 14-dimensional Riemann metrics. The connections between geometrical and analytical properties of the system are discussed.

Keywords: Riemann metrics, Navier-Stokes system, geodesics.

1 Introduction

A famous Navier-Stokes system of the equations

$$\vec{U}_t + (\vec{U} \cdot \vec{\nabla})\vec{U} - \mu \Delta \vec{U} + \vec{\nabla} P = 0, \quad (\vec{\nabla} \cdot \vec{U}) = 0, \tag{1}$$

which describes the flows of incompressible liquids and relates to various branches of modern mathematical physics, is a subject of intensive investigation. Numerous articles are devoted to studying of its solutions and properties. In this work we develop geometrical approach to construct new examples of solutions of given system and the relations with known results are considered.

2 14D-metrics and NS-equations

Let us introduce the 14-dim space with the metrics in the local coordinates $x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n$

$$^{14}ds^2 = 2\,dxdu + 2\,dydv + 2\,dzdw + 2\,dtdp + (-Uu - Vv - Ww)\,dt^2 +$$

$$+\Big(-u\,(U)^2 - uP - Up + u\mu\,U_x - vUV + v\mu\,U_y - wUW + w\mu\,U_z\Big)\,d\eta^2 +$$

$$d\eta d\xi + \Big(u\mu\,V_x + v\mu\,V_y - Vp - wVW + w\mu\,V_z - v\,(V)^2 - vP - uUV\Big)\,d\rho^2 +$$

$$+\Big(-wP - Wp + u\mu\,W_x + v\mu\,W_y + w\mu\,W_z - vVW - w\,(W)^2 - uUW\Big)\,dm^2 +$$

$$+2\,dmdn + 2\,d\rho d\chi, \tag{2}$$

where $U = U(x,y,z,t)$, $V = V(x,y,z,t)$, $W(x,y,z,t)$ and $P = P(x,y,z,t)$.

Theorema 1. *The Ricci-tensor of the metrics (2) has the components*

$$R_{44} = U_x + V_y + W_z,$$

$$R_{55} = 2\,UU_x - \mu\,U_{xx} + P_x + VU_y + UV_y - \mu\,U_{yy} + U_zW + UW_z - \mu\,U_{zz} + U_t$$

$$R_{66} = U_xV + UV_x - \mu\,V_{xx} + 2\,VV_y - \mu\,V_{yy} + P_y + V_zW + VW_z - \mu\,V_{zz} + V_t$$

$$R_{77} = U_xW + UW_x - \mu\,W_{xx} + V_yW + VW_y - \mu\,W_{yy} + 2\,WW_z - \mu\,W_{zz} + P_z + W_t$$

and it is equal to zero on solutions of the Navier-Stokes system of equations

The metrics D^{14} belongs to the class of spaces of Riemann extensions of affine connected spaces.

Namely, if we have the N-dim space in the coordinates $\vec{x}$ with the Affine Connection $\Pi^k_{ij}(\vec{x})$, then it is possible to construct the $2N$-dimensional space in the coordinates $\vec{x}, \vec{\Psi}_l$ equipped by the Riemann metrics of the form

$${}^{2n}ds^2 = -2\Pi^k_{ij}\Psi_k dx^i dx^j + 2dx^l d\Psi_l.$$

Wherein full system of geodesics consists of two parts

$$\frac{d^2x^k}{ds^2} + \Pi^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \quad and \quad \frac{\delta^2\Psi_k}{ds^2} + R^l_{kij}\frac{dx^i}{ds}\frac{dx^j}{ds}\Psi_l = 0,$$

where

$$\frac{\delta\Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Pi^l_{jk}\frac{dx^j}{ds}\Psi_l.$$

3 Geodesics

Part of geodesics of the metrics has the form

$$\frac{d^2}{ds^2}\xi(s) = 0, \quad \frac{d^2}{ds^2}\chi(s), \quad \frac{d^2}{ds^2}n(s) = 0,$$

$$\frac{d^2}{ds^2}\eta(s) = 0, \quad \frac{d^2}{ds^2}\rho(s), \quad \frac{d^2}{ds^2}m(s) = 0, \tag{3}$$

from where it follows

$$\xi(s) = a_1\, s,\ \chi(s) = a_2\, s,\ n(s) = a_3\, s,\ m(s) = a_4\, s,$$

$$\rho(s) = a_5\, s,\ \eta(s) = a_6\, s.$$

The remaining eight equations are of the form

$$\frac{d^2}{ds^2}x(s) - 1/2\,U\left(\frac{d}{ds}t(s)\right)^2 + 1/2\,{a_6}^2\mu\,\frac{\partial}{\partial x}U - 1/2\,{a_6}^2U^2 -$$

$$-1/2\,{a_6}^2P - 1/2\,{a_5}^2UV + 1/2\,{a_5}^2\mu\,V_x - 1/2\,{a_4}^2UW + 1/2\,{a_4}^2\mu\,W_x = 0,$$

$$\frac{d^2}{ds^2}y(s) - 1/2\,V\left(\frac{d}{ds}t(s)\right)^2 - 1/2\,{a_6}^2UV + 1/2\,{a_6}^2\mu\,U_y + 1/2\,{a_5}^2\mu\,\frac{\partial}{\partial y}V -$$

$$-1/2\,{a_5}^2V^2 - 1/2\,{a_5}^2P - 1/2\,{a_4}^2VW + 1/2\,{a_4}^2\mu\,W_y = 0,$$

$$\frac{d^2}{ds^2}z(s) - 1/2\,W\left(\frac{d}{ds}t(s)\right)^2 - 1/2\,{a_6}^2UW + 1/2\,{a_6}^2\mu\,U_z - 1/2\,{a_5}^2VW +$$

$$+1/2\,{a_5}^2\mu\,V_z + 1/2\,{a_4}^2\mu\,W_z - 1/2\,{a_4}^2W^2 - 1/2\,{a_4}^2P = 0,$$

$$\frac{d^2}{ds^2}t(s) - 1/2\,U{a_6}^2 - 1/2\,V{a_5}^2 - 1/2\,W{a_4}^2 = 0,$$

and the linear system of the second order ODE's to the coordinates $u(s),\ v(s),\ w(s),\ p(s)$.

Theorema 2. *Elimination of the variables u, v, w, p from the conditions to the non zero components of the Ricci-tensor*

$$R_{ij} - \lambda\, g_{ij} = 0,$$

lead to the condition on the

$$-\mu\, U_yUVP+\mu^2W_yU_xVW-\mu^2UVW_yU_z-P^2V^2-U^2P^2+\mu^2UWV_yU_z-$$

$$-\mu\, W_yPVW-\mu^2V_xW_yUW-\mu\, W_xPUW+\mu^2W_xV_yUW-\mu^2U_yW_xVW-$$

$$-P^2W^2+\mu\, PV_yW^2-\mu\, V_xUVP+\mu^2V_xU_yW^2+\mu^2W_xV^2U_z+$$

$$+\mu\, U_xPW^2-\mu^2U_xV_yW^2-\mu\, UWPU_z-\mu^2V_xVWU_z-\mu^2U_yUW_z-$$

$$-\mu^2UVW_xV_z-\mu\, VWPV_z+\mu^2W_yU^2V_z+\mu^2VWU_xV_z+\mu^2U^2V_y^2+$$

$$+\mu^2U_x^2V^2-\mu^2V_xUVU_x+V_y\mu^2U_xV^2-V_y\mu\, PV^2-U_x\mu\, U^2P+$$

$$+U_x\mu^2U^2V_y-\mu^2V_xUVV_y-\mu^2U_yUVU_x-\mu^2U_yUVV_y=0,$$

which can be used to the theory of the NS-equations.

4 Conclusion

As it was shown, the system of the Navier-Stokes equations, which describes a flow of incompressible liquids, admits geometric consideration on the basis of Riemann metrics of 14-dim space. In such a way, new possibilities for investigation of its properties arise.

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On extensions of d-lattice bimorphisms

Omer Gok

Abstract

In this note, we show that every d-lattice bimorphism on the cartesian product of a vector lattice with itself can be extended to a d-lattice bimorphism on the cartesian product of the Dedekind completion with itself.

Keywords: f-algebra, almost f-algebra, d-algebra, lattice ordered algebra, lattice bimorphism, Dedekind completion.

1 Introduction

Let E be an Archimedean vector lattice; let E^δ be its Dedekind completion; let F be a Dedekind complete vector lattice. In the paper, we prove that if $\psi : E \times E \to F$ is a d-lattice bimorphism, then we can extend ψ to the d-lattice bimorphism $\Psi : E^\delta \times E^\delta \to F$. In [11], the author proved that every orthosymmetric lattice bilinear map on the cartesian product of a vector lattice with itself can be extended to an orthosymmetric lattice bilinear map on the cartesian product of the Dedekind completion with itself. In [12], the author proved that every d-bimorphism on the cartesian product of a vector lattice with itself can be extended to a d-bimorphism on the cartesian product of the order continuous order bidual of vector lattice with itself. It was proved by using Arens multiplication [2] called triadjoint mapping. Extensions problem is given by Huijsmans in [9]. In [9], the author asked whether or not multiplication of d-algebra can be extended to its Dedekind completion. The same problem was asked for almost f-algebras and lattice ordered algebras in [9]. It is well-known that the multiplication in an Archimedean f-algebra A can be extended to multiplication to its Dedekind completion of A. It means that the Dedekind completion of A is an f-algebra with respect to this extended multiplication. A lot of mathematicians given in references were studied on these extensions problem [3], [4], [5], [6], [7], [8], [11], [12]. For notations, concepts and terminology that are not explained here we adhere to the standard books [1], [10], [13].

2 d-lattice bimorphisms and their extensions

A real vector lattice (Riesz space) A which is simultaneously an associative algebra with the property that $ab \in A^+$ for all $a, b \in A^+$, where A^+ is the positive cone of A, is said to be a lattice ordered algebra (or Riesz algebra). The lattice ordered algebra A is called an f-algebra whenever $x \wedge y = 0$ and $z \in A^+$ imply $zx \wedge y = xz \wedge y = 0$. The lattice ordered algebra A is called a d-algebra whenever $x \wedge y = 0$ and $z \in A^+$ imply $zx \wedge zy = xz \wedge yz = 0$. The lattice ordered algebra A is called an almost f-algebra whenever $x \wedge y = 0$ implies $xy = 0$. Any f-algebra is a d-algebra and almost f-algebra, but not conversely. Archimedean d-algebras have no positive squares and are not commutative. But, an Archimedean f-algebra is both commutative and associative and has positive squares. A vector lattice E is called Dedekind complete if every non-empty subset of E which is bounded from above has a supremum. A Dedekind complete vector lattice M is said to be a Dedekind completion of the vector lattice E whenever E is Riesz isomorphic to a majorizing order dense Riesz subspace of M. Every Archimedean vector lattice has a unique Dedekind completion. E^δ denotes the Dedekind completion of E. A vector lattice E is said to be universally complete if E is Dedekind complete and every pairwise disjoint positive vectors in E has a supremum in E. Every Archimedean vector lattice E have a universal completion denoted by E^u. This means that there exists a unique (up to a lattice isomorphism) universally complete vector lattice E^u such that E is Riesz isomorphic to an order dense Riesz subspace of E^u. E^δ is the order ideal generated by E in E^u. So, the Riesz subspace inclusions $E \subseteq E^{\delta} \subseteq E^{u}$ hold such that E is order dense in E^{u}. The vector lattice E^{u} is equal to $C^{\infty}(X)$ for some extremally disconnected compact Hausdorff topological space X. Extremally disconnected space X means that the closure of every open set of X is open.

Let E, F, G be vector lattices. A linear operator $T : E \to F$ is called a lattice homomorphism if $T(x \vee y) = Tx \vee Ty$ for every $x, y \in E$. A bilinear map $\psi : E \times F \to G$ is called positive if $| \psi(x, y) | \leq \psi(| x |, | y |)$ for all $x \in E$ and $y \in F$. The bilinear map $\psi : E \times F \to G$ is called lattice bimorphism or lattice bilinear map if whenever it is separately lattice homomorphisms for each variable or equivalently, $| \psi(x, y) | = \psi(| x |, | y |)$ for all $x \in E$ and $y \in F$. A bilinear map $\psi : E \times E \to F$ is called orthosymmetric if $x \wedge y = 0$ implies $\psi(x, y) = 0$ for all $x, y \in E$.

Let E, F be vector lattices and G be a Dedekind complete vector lattice. Grobler and Labuschagne in [8] proved that every lattice bimorphism $\phi : E \times F \to G$ can be extended to a lattice bimorphism $\varphi : E^{\delta} \times F^{\delta} \to G$. Our problem is that every d-lattice bimorphism $\psi : E \times E \to G$ can be extended to a d-lattice bimorphism $\Psi : E^{\delta} \times E^{\delta} \to G$.

Definition 1 [12]. *A lattice bilinear map $\psi : E \times E \to F$ is called d-lattice bimorphism if $x \wedge y = 0$ implies $\psi(x, z) \wedge (y, z) = 0$ for every $z \in E^{+}$.*

Theorem 1. *Let E be an Archimedean vector lattice, and let E^{d} be its Dedekind completion, and let F be a Dedekind complete vector lattice. If $\psi : E \times E \to F$ is a d-lattice bimorphism, then ψ can be extended to a d-lattice bimorphism $\Psi : E^{\delta} \times E^{\delta} \to F$.*

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On Generalized Brioschi's Formula and its Applications

Taras Goy

Abstract

In this paper, we consider determinants for some families of Hessenberg matrices having various translates of the Fibonacci numbers for the nonzero entries. These determinant formulas may also be rewritten as identities involving sums of products of Fibonacci numbers and multinomial coefficients.

Keywords: Fibonacci numbers, Lucas numbers, Hessenberg matrix, Brioschi's formula, multinomial coefficient.

1 Introduction

Formulas relating determinants to Fibonacci numbers have been an object of interest for a long time, especially from the viewpoint of applications. In some cases, this sequence arises as determinants for certain families of matrices having integer entries, while in other cases this sequence is the actual entries of the matrix whose determinant is being evaluated (see, e.g., [1–6, 8] for the complete bibliography).

Consider the $n \times n$ Hessenberg matrix having the form

$$H_n(a_1, a_2, \ldots, a_n) = \begin{bmatrix} k_1 a_1 & 1 & & & & \\ k_2 a_2 & a_1 & 1 & & 0 & \\ \vdots & \vdots & \ddots & \ddots & & \\ k_{n-1} a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \\ k_n a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix},$$

where $a_i \neq 0$ for at least one $i > 0$.

In [9], Zatorsky and Stefluk proved that

$$\det(H_n) = \sum_{\sigma_n = n} \frac{(-1)^{n-|s_n|}}{|s_n|} \left(\sum_{i=1}^{n} s_i k_i \right) m_n(s) a_1^{s_1} \cdots a_n^{s_n}, \tag{1}$$

where $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, $|s_n| = s_1 + \cdots + s_n$, $m_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ is the multinomial coefficient, and the summation is over all n-tuples $(s_1, \ldots, s_n)$ of integers $s_i \geq 0$ satisfying the Diophantine equation $\sigma_n = n$.

In the case $k_1 = \ldots = k_n = 1$ we have well-known *Brioschi's formula* [6, pp. 208–209].

Note that $s_1 + 2s_2 + \cdots + ns_n = n$ is partition of the positive integer n, where each positive integer i appears s_i times.

In the next section, we will investigate a particular case of determinants $\det(H_n)$, in which $k_i = i$. For the sake of brevity, we will use throughout the notation

$$\det(a_1, a_2, \ldots, a_n) = \det\big(H_n(a_1, a_2, \ldots, a_n)\big).$$

2 Fibonacci–Lucas multinomial identities

Let F_n denote the n-th Fibonacci number and L_n the n-th Lucas number, both satisfying the recurrence

$$w_n = w_{n-1} + w_{n-2},$$

but with the respective initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ (see [6] and the references given there).

Theorem 1. *For $n \geq 1$, the following formulas hold*:

$$\begin{aligned}
\det(F_0, F_1, \ldots, F_{n-1}) &= (-1)^{n-1}(L_n - 1), \\
\det(-F_0, -F_1, \ldots, -F_{n-1}) &= 2^n + (-1)^n - L_n, \\
\det(F_1, F_2, \ldots, F_n) &= (-1)^{n-1}\left(L_n - 1 - (-1)^n\right), \\
\det(F_2, F_3, \ldots, F_{n+1}) &= (-1)^{n-1} L_n, \\
\det(F_3, F_4, \ldots, F_{n+2}) &= (-1)^{n-1} L_n + 1.
\end{aligned}$$

Theorem 1 may be proved in the same way as Theorems 1 and 2 in [2].

Next, we focus on multinomial extensions of Theorem 1. Formula (1), coupled with Theorem 1 above, yields the following combinatorial identities expressing the Lucas numbers in terms of Fibonacci numbers.

Theorem 2. *For $n \geq 1$, the following formulas hold*:

$$L_n = 1 - n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_0^{s_1} F_1^{s_2} \cdots F_{n-1}^{s_n},$$

$$L_n = 2^n + (-1)^n - n \sum_{\sigma_n = n} \frac{1}{|s_n|} m_n(s) F_0^{s_1} F_1^{s_2} \cdots F_{n-1}^{s_n},$$

$$L_n = 1 + (-1)^n - n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_1^{s_1} F_2^{s_2} \cdots F_n^{s_n},$$

$$L_n = -n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_2^{s_1} F_3^{s_2} \cdots F_{n+1}^{s_n},$$

$$L_n = (-1)^n - n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_3^{s_1} F_4^{s_2} \cdots F_{n+2}^{s_n},$$

where $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, $|s_n| = s_1 + \cdots + s_n$, $m_n(s) = \frac{(s_1 + \cdots + s_n)!}{s_1! \cdots s_n!}$, and the summation is over nonnegative integers s_i satisfying equation $\sigma_n = n$.

3 Conclusion

In this paper, we evaluate several families of some Hessenberg matrices whose entries are Fibonacci numbers with sequential subscripts. In particular, we establish a connection between the Lucas and the Fibonacci sequences via Hessenberg determinants. Using generalized Brioschi's formula, we rewrite the obtained formulas as identities involving Lucas numbers, sums of products of Fibonacci numbers, and multinomial coefficients.

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On tiling H^3 by polyhedra with irrational relative to π dihedral angles

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Abstract

In the communication we construct a countable series of hyperbolic 3-manifolds whose fundamental polyhedron has all dihedral angles irrational relative to π.

Keywords: hyperbolic space, fundamental polyhedron, 3-manifold.

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In the communication we will construct a countable series of three-dimensional manifolds with finite volume and such that dihedral angles of fundamental polyhedra of these manifolds are irrational relative to π. On the hyperbolic plane ω consider a regular 2k-gon P with side h, where $ch(h) = 3$ and $k = 2, 3, \ldots$. It is not difficult to prove the existence of such a polygon. The measure of inner angle of the polygon P can be calculated by the formula:

$$sin(\alpha/2) = cos(\pi/(2k))/\sqrt{2}, \quad k = 2, 3, \ldots.$$

Using facts from number theory, one can show that for every $k = 3, 4, \cdots$, the angle α is irrational with respect to π (i.e. $\alpha \neq (m/n)\pi$, where m and n are integers); however the proof of this assertion is rather tedious. In the case $k = 2$ we have $\alpha = \pi/6$. At the vertices of the 2k-gon P draw straight lines $b_1, b_2, \ldots, b_{2k}$ orthogonal to the plane ω, numbering anticlockwise. At the center point of polygon P also draw the straight line b orthogonal to ω. Through pairs of neighbouring lines b_i and b_{i+1} draw planes β_i, $i = 1, 2, \ldots, 2k-1$, and the plane

β_{2k} goes through lines b_1 and b_{2k}. It is not difficult to see that there exist planes τ_1 and τ_2 such that they are situated in opposite directions from plane ω, are orthogonal to line b and parallel to each of lines b_i, $i = 1, 2, \ldots, 2k$. Let τ_1 lie below the plane ω and τ_2 lie above the plane ω. Both planes τ_1 and τ_2 intersect with planes β_i, $i = 1, 2, \ldots, 2k$. Denote lines of the intersection of plane τ_1 with planes β_i by a_i and lines of the intersection of plane τ_2 with planes β_i by c_i,

i.e. $a_i = \tau_1 \cap \beta_i$, $c_i = \tau_2 \cap \beta_i$, $i = 1, 2, \cdots, 2k$.

One can prove by means of hyperbolic geometry that dihedral angles at straight lines a_i and c_i $i = 1, 2, \ldots, 2k$, are equal to $\pi/2 - \alpha/2$. In the hyperbolic space H^3 consider the polyhedron R_k that appears in the intersection of planes τ_1, τ_2 and β_i, $i = 1, 2, \ldots, 2k$. The polyhedron R_k is a prism with all its vertices being on the absolute, i.e. it is unbounded but has a finite volume. Its faces τ_1 and τ_2 are regular polygons with all vertices being infinitely remote, τ_1 is the bottom base, τ_2 is the upper base of the prism. The lateral faces of prism R_k are squares with all vertices being infinitely remote. Any dihedral angle of the prism R_k is equal either to α (at vertical edges) or to $\pi/2 - \alpha/2$ (at horizontal edges). So, for $k > 2$ all the dihedral angles of the polyhedra R_k are irrational relative to π.

Now show that polyhedron R_k is a fundamental polyhedron for a group Γ_k without elements of finite order (i.e. torsion-free). In this case factorizing H_3 by the group Γ_k we obtain a hyperbolic manifold M_k which is non-compact but has a finite volume, i.e. $M_k = H^3/\Gamma_k$.

For the construction of group Γ_k we indicate the identification of faces of polyhedron R_k. We identify each lateral face of the polyhedron by a helical motion with opposite lateral face:

$$\beta_i \quad \varphi_i \quad \beta_{i+k}, \, i = 1, 2, \ldots, k,$$

where the translation distance is equal to the distance between two faces and rotation angle is equal to $\pi/2$. The bottom base τ_1 of the prism can be identified with the upper base τ_2 also by a helical motion ψ, where the translation distance is equal to the distance between the bases and rotation angle is equal to $(k-1) \times \pi/k$ (clockwise).

Write cycles of edges of prism induced by the above motions ψ and φ_i, $i = 1, 2, \ldots, k$, as follows: first indicate the initial edge, then in parentheses the faces that intersect along this edge, then motion which sends the edge into an equivalent edge, then in parentheses the faces that intersect along the edge, and so on until the cycle is closed. As a result we obtain the following cycles:

$$a_i(\beta_i \cap \tau_1) \quad \varphi_i \quad b_{i+k}(\beta_{i+k} \cap \beta_{i+k+1})$$

$$\varphi_{i+1}^{-1} \quad c_{i+1}(\beta_{i+1} \cap \tau_2) \quad \psi^{-1} \quad a_{i+k}(\beta_{i+k} \cap \tau_1)$$

$$\varphi_i^{-1} \quad b_i(\beta_i \cap \beta_{i+1}) \quad \varphi_{i+1} \quad c_{i+k+1}(\beta_{i+k+1} \cap \tau_2)$$

$$\psi^{-1} \quad a_i(\beta_i \cap \tau_1), \quad i = 1, 2, \ldots, k.$$

It is not difficult to see that the cycle of any edge is unessential, i.e. the sum of dihedral angles at edges of a cycle is equal to 2π. So the following theorem is true.

Theorem. *The motions ψ and φ_i, $i = 1, 2, \ldots, k$, generate in the hyperbolic space H^3 a discrete group Γ_k without elements of finite order, and the polyhedron R_k is a fundamental polyhedron for this group, i.e. $M_k = H^3/\Gamma_k$ is a three-dimensional manifold with finite volume. The manifolds M_k will have k cusps.*

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On Groupoids With Classical Bol-Moufang Type Identities

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Abstract

We count number of groupoids of order 2 with classical Bol-Moufang type identities which are listed in Fenvesh' article [8, 10] up to isomorphism and up to anti-isomorphism.

Keywords: groupoid, classical Bol-Moufang type identity, (12)-parastrophe, (12)-parastrophic identity.
2000 Mathematics Subject Classification: 20N05 20N02

1 Introduction

For groupoids the following natural problem is researched: how many groupoids with some identities of small order there exist? A list of numbers of semigroups of orders up to 8 is given in [15], of order 9 – in [7]; a list of numbers of quasigroups up to 11 is given in [13, 17].

An identity based on a single binary operation is of classical Bol-Moufang type if "both sides consist of the same three different letters taken in the same order but one of them occurs twice on each side"[8]. We use list of 60 Bol-Moufang type identities given in [10].

We continue the study of groupoids with Bol-Moufang type identities [12, 2, 16, 5, 3, 4, 9]. Some results presented in this paper are given in [3, 4, 9] by another name of identity. We use almost the same algorithm and program as in [3, 4, 9].

Various properties of Bol-Moufang type identities in quasigroups and loops are studied in [8, 14, 6, 1].

We recall, groupoid $(Q, *)$ is called a quasigroup, if the following conditions are true [2]: $(\forall u, v \in Q)(\exists!\, x, y \in Q)(u * x = v \,\&\, y * u = v)$.

Groupoid $(G, \cdot)$ is isomorphic to groupoid $(G, \circ)$ if there exists a permutation α of symmetric group S_G such that $x \circ y = \alpha^{-1}(\alpha x \cdot \alpha y)$ for all $x, y \in G$.

Groupoid $(G, \cdot)$ is anti-isomorphic to groupoid $(G, \circ)$ if there exists a permutation α of symmetric group S_G such that $x \circ y = \alpha^{-1}(\alpha y \cdot \alpha x)$ for all $x, y \in G$.

Below quadruple 22 12 means groupoid of order 2 with the following Cayley table:

$*$	1	2
1	2	2
2	1	2

and so on.

2 Results

2.1 (12)-parastrophes of identities

We recall, (12)-parastroph of groupoid $(G, \cdot)$ is a groupoid $(G, *)$, where operation "$*$ " is obtained by the following rule:

$$x * y = y \cdot x. \tag{1}$$

It is clear that for any groupoid $(G, \cdot)$ there exists its (12)-parastroph groupoid $(G, *)$.

Cayley table of groupoid $(G, *)$ is mirror image of the Cayley table of groupoid $(G, \cdot)$ relative to the main diagonal.

Suppose that an identity F is true in groupoid $(G, \cdot)$. Then we can obtain (12)-parastrophic identity (F^*) of the identity F replacing the operation "$\cdot$" on the operation "$*$" and changing the order of variables using rule (1).

It is clear that an identity F is true in groupoid $(G, \cdot)$ if and only if in groupoid $(Q, *)$ identity F^* is true.

Proposition 1. *The number of groupoids of a finite fixed order, in which the identity F is true, coincides with the number of groupoids in which the identity F^* is true.*

See [9] for more details.

Theorem 1. $(F_1)^* = F_3$, $(F_2)^* = F_4$, $(F_5)^* = F_{10}$, $(F_6)^* = F_6$, $(F_7)^* = F_8$, $(F_9)^* = F_9$, $(F_{11})^* = F_{24}$, $(F_{12})^* = F_{23}$, $(F_{13})^* = F_{22}$, $(F_{14})^* = F_{21}$, $(F_{15})^* = F_{30}$, $(F_{16})^* = F_{29}$, $(F_{17})^* = F_{27}$, $(F_{18})^* = F_{28}$, $(F_{19})^* = F_{26}$, $(F_{20})^* = F_{25}$, $(F_{31})^* = F_{34}$, $(F_{32})^* = F_{33}$, $(F_{35})^* = F_{40}$, $(F_{36})^* = F_{39}$, $(F_{37})^* = F_{37}$, $(F_{38})^* = F_{38}$, $(F_{41})^* = F_{53}$, $(F_{42})^* = F_{54}$, $(F_{43})^* = F_{51}$, $(F_{44})^* = F_{52}$, $(F_{45})^* = F_{60}$, $(F_{46})^* = F_{56}$, $(F_{47})^* = F_{58}$, $(F_{48})^* = F_{57}$, $(F_{49})^* = F_{59}$, $(F_{50})^* = F_{55}$.

Remark 1. *Identities F_6, F_9, F_{37}, F_{38} coincide with their* (12)*-parastrophe identities.*

Remark 2. *In quasigroup case analog of Theorem 1 is given in [11].*

Proposition 2. *Any from the following groupoids* 11 11, 22 22, 11 12, 12 22, 11 22, 12 12 *satisfies any from the identities F_1–F_{60}.*

Proof. It is possible to use direct calculations. □

2.2 Number of groupoids

We count number of groupoids of order two with classical Bol-Moufang type identities given in [10] including and number of non-isomorphic and number of non-isomorphic and non-anti-isomorphic groupoids of order 2. See Table 1. Notice, in some places Table 1 coincides with corresponding table from [9].

Table 1 is organised as follows: in the first column it is given name of identity in Fen'vesh list; in the second it is given abbreviation of this identity, if this identity has a name; in the third it is given identity; in the fourth column it is indicated the number of groupoids of order 2 with corresponding identity; in the fifth column – of non-isomorphic groupoids; and, in the sixth column – non-isomorphic and non-anti-isomorphic groupoids with corresponding identity.

Table 1: Number of groupoids of order 2 with classical Bol-Moufang identities

Na- me	Abb.	Ident.	2	n.- is.	n.- is., an.
F_1		$xy \cdot zx = (xy \cdot z)x$	10	6	5
F_2		$xy \cdot zx = (x \cdot yz)x$	9	6	5
F_3		$xy \cdot zx = x(y \cdot zx)$	10	6	5
F_4	middle Mouf.	$xy \cdot zx = x(yz \cdot x)$	9	6	5
F_5		$(xy \cdot z)x = (x \cdot yz)x$	11	7	6
F_6	extra ident.	$(xy \cdot z)x = x(y \cdot zx)$	10	7	5
F_7		$(xy \cdot z)x = x(yz \cdot x)$	9	6	5
F_8		$(x \cdot yz)x = x(y \cdot zx)$	9	6	5
F_9		$(x \cdot yz)x = x(yz \cdot x)$	10	6	5
F_{10}		$x(y \cdot zx) = x(yz \cdot x)$	11	7	6
F_{11}		$xy \cdot xz = (xy \cdot x)z$	8	5	4
F_{12}		$xy \cdot xz = (x \cdot yx)z$	9	7	6
F_{13}	extra ident.	$xy \cdot xz = x(yx \cdot z)$	9	6	5
F_{14}		$xy \cdot xz = x(y \cdot xz)$	10	6	5
F_{15}		$(xy \cdot x)z = (x \cdot yx)z$	11	7	6
F_{16}		$(xy \cdot x)z = x(yx \cdot z)$	11	7	6
F_{17}	left Mouf.	$(xy \cdot x)z = x(y \cdot xz)$	10	7	5

F_{18}		$(x \cdot yx)z = x(yx \cdot z)$	8	5	4
F_{19}	left Bol	$(x \cdot yx)z = x(y \cdot xz)$	9	6	5
F_{20}		$x(yx \cdot z) = x(y \cdot xz)$	9	6	5
F_{21}		$yx \cdot zx = (yx \cdot z)x$	10	6	5
F_{22}	extra ident.	$yx \cdot zx = (y \cdot xz)x$	9	6	5
F_{23}		$yx \cdot zx = y(xz \cdot x)$	9	6	5
F_{24}		$yx \cdot zx = y(x \cdot zx)$	8	5	4
F_{25}		$(yx \cdot z)x = (y \cdot xz)x$	9	6	5
F_{26}	right Bol	$(yx \cdot z)x = y(xz \cdot x)$	9	6	5
F_{27}	right Mouf.	$(yx \cdot z)x = y(x \cdot zx)$	10	7	5
F_{28}		$(y \cdot xz)x = y(xz \cdot x)$	8	5	4
F_{29}		$(y \cdot xz)x = y(x \cdot zx)$	11	7	6
F_{30}		$y(xz \cdot x) = y(x \cdot zx)$	11	7	6
F_{31}		$yx \cdot xz = (yx \cdot x)z$	8	5	4
F_{32}		$yx \cdot xz = (y \cdot xx)z$	9	6	5
F_{33}		$yx \cdot xz = y(xx \cdot z)$	9	6	5
F_{34}		$yx \cdot xz = y(x \cdot xz)$	8	5	4
F_{35}		$(yx \cdot x)z = (y \cdot xx)z$	9	6	5
F_{36}	RC ident.	$(yx \cdot x)z = y(xx \cdot z)$	9	6	5
F_{37}	C ident.	$(yx \cdot x)z = y(x \cdot xz)$	10	7	5
F_{38}		$(y \cdot xx)z = y(xx \cdot z)$	8	5	4

F_{39}	LC ident.	$(y \cdot xx)z = y(x \cdot xz)$	9	6	5
F_{40}		$y(xx \cdot z) = y(x \cdot xz)$	9	6	5
F_{41}	LC ident.	$xx \cdot yz = (x \cdot xy)z$	9	6	5
F_{42}		$xx \cdot yz = (xx \cdot y)$ z	12	7	5
F_{43}		$xx \cdot yz = x(x \cdot yz)$	8	5	4
F_{44}		$xx \cdot yz = x(xy \cdot z)$	9	6	5
F_{45}		$(x \cdot xy)z = (xx \cdot y)z$	9	6	5
F_{46}	LC ident.	$(x \cdot xy)z = x(x \cdot yz)$	11	7	6
F_{47}		$(x \cdot xy)z = x(xy \cdot z)$	8	5	4
F_{48}	LC ident.	$(xx \cdot y)z = x(x \cdot yz)$	10	7	5
F_{49}		$(xx \cdot y)z = x(xy \cdot z)$	9	6	5
F_{50}		$x(x \cdot yz) = x(xy \cdot z)$	11	7	6
F_{51}		$yz \cdot xx = (yz \cdot x)x$	8	5	4
F_{52}		$yz \cdot xx = (y \cdot zx)x$	9	6	5
F_{53}	RC ident.	$yz \cdot xx = y(zx \cdot x)$	9	6	5
F_{54}		$yz \cdot xx = y(z \cdot xx)$	12	7	6
F_{55}		$(yz \cdot x)x = (y \cdot zx)x$	11	7	6
F_{56}	RC ident.	$(yz \cdot x)x = y(zx \cdot x)$	11	7	6
F_{57}	RC ident.	$(yz \cdot x)x = y(z \cdot xx)$	10	7	6
F_{58}		$(y \cdot zx)x = y(zx \cdot x)$	8	5	4

F_{59}		$(y \cdot zx)x = y(z \cdot xx)$	9	6	5
F_{60}		$y(zx \cdot x) = y(z \cdot xx)$	9	6	5

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On functionally completeness of quasigroups

Izbash Olga, Izbash Vladimir

Abstract

A continuum of nonisomorphic functionally complete quasigroups is constructed.

Keywords: functionally complete algebra, functionally complete quasigroups , monoquasigroups.

By a functionally complete algebra A we mean a finite algebra with underlying set A and with basic operations $f_1, f_2, ..., f_k$ such that for every nonnegative integer n and for every function $f : A^n \to A$ there is a polynomial expression $p(x_1, x_2, ..., x_n)$ over A such that for every n-tuple $(a_1, a_2, ..., a_n)$ we have $p(a_1, a_2, ..., a_n) = f(a_1, a_2, ..., a_n)$, that is for each n all functions $f : A^n \to A$ are definable in terms of the operations $f_1, f_2, ..., f_k$. Every functionally complete algebra with a finite signature has no more than countable cardinality.

No semigroup (in particular, no group) can be functionally complete due to the fact that in every finite semigroup there is an idempotent. No loop, no ring can also be functionally complete. Functionally complete quasigroups not loops exist. Functionally complete quasigroups are of great importance in the context of cryptographic applications. One of the oldest algebraic questions, equally important in computer science, is to decide whether or not an equation or a system of equations has a solution over an algebra. For every finite functionally complete algebra this problem is in NP. In [1] it is proved that a finite algebra (A, ω) with a single operation is complete if and only if it has no proper subalgebras, automorphisms or congruences.

In [2] the following theorem is proved:

Theorem. *a) Any quasigroup* $(Q, \cdot)$ *for which* $2 \leq |Q| \leq \aleph_0$ *is isotopic to some quasigroup without subquasigroups and congruences. b) Any quasigroup* $(Q, \cdot)$ *for which* $5 \leq |Q| \leq \aleph_0$ *is isotopic to some quasigroup without subquasigroups, automorphisms and congruences.*

It is also shown that there is a continuum of nonisomorphic quasigroups without subquasigroups, without automorphisms and without congruences. All these quasigroups are functionally complete.

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Group-theoretic-free proof of Hall's theorem about finite sharply 2-transitive permutation group

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Abstract

In this article a scheme of group-theoretic-free proof of well-known Hall's theorem about description of finite sharply 2-transitive permutation group is given.

Keywords: sharply 2-transitive permutation group, loop transversals, normal subgroup.

1 Introduction

In the theory of finite multiply transitive permutation groups the M. Hall's theorem is well-known [4]. The proof of this theorem in his book is group-theoretic. The author of this article will show the another proof of this theorem without using the group theory. The scheme of the loop-theoretic proof of the Hall's theorem will be given below.

2 Definitions

Definition 2.1. *A set M of permutations on a set X is called **sharply 2-transitive**, if for any two pairs (a, b) and (c, d) of different elements from X there exists a unique permutation $\alpha \in M$ satisfying the following conditions*

$$\alpha(a) = c, \qquad \alpha(b) = d.$$

Definition 2.2. *[11] Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left cosets in G to H. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i, i.e. $t_i \in H_i$) is called a* ***left generalized transversal*** *in G to H.*

Definition 2.3. *[1] A left generalized transversal $T = \{t_i\}_{i\in E}$ in G to H which satisfies the following conditions: $t_{i_0} = e$ for some $i_0 \in E$ and $H = H_1$, is usually called a* ***left transversal*** *in G to H.*

Definition 2.4. *Let $T = \{t_i\}_{i\in E}$ be a left generalized transversal in G to H. Define the following operation on the set E:*

$$x \overset{(T)}{\cdot} y = z \quad \Leftrightarrow \quad t_x t_y = t_z h, \quad h \in H.$$

Theorem 1. *For an arbitrary left generalized transversal $T = \{t_i\}_{i\in E}$ in G to H the following statements are true:*

1. *There exists an element $a_0 \in E$ such that the system $\left(E, \overset{(T)}{\cdot}, a_0\right)$ is a left quasigroup with right unit a_0.*
2. *If $T = \{t_i\}_{i\in E}$ is a left transversal in G to H, then the system $\left(E, \overset{(T)}{\cdot}, 1\right)$ is a left loop with unit 1.*

3 Transversals with isotopic and isomorphic transversal operations

Theorem 2. *[11] For an arbitrary left generalized transversal $T = \{t_i\}_{i\in E}$ in G to H the following statements are true:*

1. *If $P = \{p_i\}_{i\in E}$ is a left generalized transversal in G to H such that for every $x \in E$:*

$$P = Th_0, \quad p_{x'} = t_x h_0,$$

where $h_0 \in H$ *is an arbitrary fixed element, then the transversal operation* $\left(E, \overset{(P)}{\cdot}\right)$ *is isotopic to the transversal operation* $\left(E, \overset{(T)}{\cdot}\right)$.

2. *If* $S = \{s_i\}_{i \in E}$ *is a left generalized transversal in* G *to* H *such that for every* $x \in E$*:*

$$S = \pi T, \quad s_{x'} = \pi t_x,$$

where $\pi \in G$ *is an arbitrary fixed element, then the transversal operation* $\left(E, \overset{(S)}{\cdot}\right)$ *is isotopic to the transversal operation* $\left(E, \overset{(T)}{\cdot}\right)$.

Lemma 3.1. *[8] Let* $T = \{t_x\}_{x \in E}$ *be a fixed loop transversal in* G *to* H *and* $h_0 \in N_{St_1(S_E)}(H)$*. Define the set of permutations:*

$$p_{x'} \overset{def}{=} h_0^{-1} t_x h_0 \quad \forall x \in E.$$

Then

1. $P = \{p_{x'}\}_{x' \in E}$ *is a loop transversal in* G *to* H*;*
2. *The transversal operations* $\left(E, \overset{(P)}{\cdot}, 1\right)$ *and* $\left(E, \overset{(T)}{\cdot}, 1\right)$ *are isomorphic, and the isomorphism is set up by the mapping* $\varphi(x) = h_0(x)$.

Lemma 3.2. *[10] Let* $T = \{t_x\}_{x \in E}$ *be a fixed loop transversal in* G *to* H*. Let* $h_0 \in N_{St_1(S_E)}(H)$ *be an element such that:*

$$t_{x'} \overset{def}{=} h_0^{-1} t_x h_0 \quad \forall x \in E.$$

Then $\varphi \equiv h_0 \in Aut\left(E, \overset{(T)}{\cdot}, 1\right)$.

4 A finite sharply 2-transitive group of permutations degree n

Theorem 3. *Let G be a sharply 2-transitive permutation group on a finite set of symbols E, i.e.*

1. *G is a 2-transitive permutation group on E;*
2. *only identity permutation id fixes two symbols from the set E.*

Then

1. *the identity permutation id with the set of all fixed-point-free permutations from the group G forms a transitive invariant subgroup A in the group G;*
2. *group G is isomorphic to the group of linear transformations*

$$G_K = \{\alpha \,|\, \alpha(x) = x \cdot a + b, \quad a, b \in E, \quad a \neq 0\}$$

of a some near-field $K = \langle E, +, \cdot, 0, 1 \rangle$.

5 A scheme of my proof of Hall's Theorem

Lemma 5.1. *[7] Let G be a sharply 2-transitive permutation group of degree n on a set E. Then the set T of all fixed-point-free permutations from G with the identity permutation id form a unique loop transversal in G to $H = St_0(G) = \{\alpha \in G | \alpha(0) = 0, 0 \in E\}$.*

Lemma 5.2. *Let T be the loop transversal in G to H from the previous Lemma. Then the following propositions are true:*

1. *$\forall a \in E$ the set $S_a = T \cdot t_a^{-1}$ (where $t_a \in T$) is a loop transversal in G to H.*
2. *$\forall a \in E$ the set $R_a = t_a \cdot T$ (where $t_a \in T$) is a loop transversal in G to H.*

Lemma 5.3. *The transversal T from the previous Lemmas is a group transversal in G to H.*

Lemma 5.4. *1. The group transversal operation $(T, \cdot)$ has a sharply transitive automorphism group.*

2. The group $(T, \cdot)$ is a primary cyclic group of order $n = p^m$, p is a prime number.

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On the theory of finite crystallographic groups of cyclical W_p-symmetry

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Abstract

In this paper there are studied the groups of W_p-symmetry transformations of the geometrical finite figures regularly weighted by "physical" scalar tasks. Some finite crystallographic groups of cyclical W_p-symmetry are analysed.

Keywords: groups, symmetry, quasi-homeomorphisms, wreath products.

1 Introduction

W_p-symmetry [1-5] is one of the main generalizations of classical symmetry. In this case the transformations of the qualities, attributed to the points, essentially depend on the choice of points. The set of transformations of W_p-symmetry of the given "indexed" geometrical figure $F^{(N)}$ forms a group $G^{(W_p)}$ with the operation $g_i^{(w_i)} \circ g_j^{(w_j)} = g_k^{(w_k)}$, where $g_k = g_i g_j$, $w_k = w_i^{g_j} w_j$ and $w_i^{g_j}(g_s) = w_i(g_j g_s)$. The group $G^{(W_p)}$ is called major, minor or V-middle if $w_0 < V = W' = W, w_0 = V < W' = W$ or $w_0 < V < W' = W$, respectively, where w_0 is the unit of group W, $W' = \{w | g^{(w)} \in G^{(W_p)}\} \subseteq W$ and $V = G^{(W_p)} \cap W = G^{(W_p)} \cap W'$ is the subgoup of W-identical transformations of the group $G^{(W_p)}$. If W' is a non-trivial subgroup of W, then the group $G^{(W_p)}$ is called W'-semi-major, W'-semi-minor or (W', V)-semi-middle according to the cases when $w_0 < V = W', w_0 = V < W'$ or $w_0 < V < W'$. If $W' \subset W$, but W' is not a group, the group $G^{(W_p)}$ is called W'-pseudo-minor or (W', V)-pseudo-middle when $w_0 = V \subset W'$ or $w_0 < V \subset W'$.

2 About general properties of the groups of W_p-symmetry

The cristallographic punctual groups $G^{(W_p)}$ of W_p-symmetry are subgroups of left direct wreath product of initial group P with the generating group G, accompanied with a fixed isomorphism $\varphi : G \to AutW$ (where W is the direct product of isomorphic copies of group P which are indexed by elements of group G). We will mention that the major W_p- symmetry group is constructed in the form of the left-direct wreath product of the initial definition group P with the generator group G, accompanied with a fixed isomorphism $\varphi : G \to AutW$.

Of all the specific properties of crystallographic punctual groups of W_p-symmetry, that relate to their overall structure, we shall mention only main ones.

1) The application μ of the group $G^{(Wp)}$ onto the set $W' = \{w|g^{(w)} \in G^{(W_p)}\} \subseteq W$ with the unit w_0 of the group W by the rule $\mu[g^{(w)}] = w$ is a left quasi-homomorphism with the kernel H, where H is the symmetry subgroup in $G^{(Wp)}$.

2) The application φ of the group $G^{(W_p)}$ onto the group G by the rule $\varphi[g^{(w)}] = g$ is homomorphic with the kernel V.

3) $V^g = wVw^{-1}$ for all components g and w of transformations $g^{(w)}$ of group $G^{(W_p)}$.

4) Any element of the class of residues Hg is combined in pairs with each element of residues class wV, but the elements of different classes Hg_i and Hg_j are combined with the elements of different classes w_iV and w_jV, respectively.

5) The group $G^{(W_p)}$ contains as its subgroup a group $G_1^{(W_1)}$ of P-symmetry (which is determined by initial group P of permutations, where $W_1 \leq DiagW \cong P$ and $W_1 \subset W'$) from the family with the generating group G_1 ($G_1 \leq G$), with the symmetry subgroup H (where $H \trianglelefteq G_1$ but $H \neq G$) and with the subgroup V_1 of W-identical transformations (where $V_1 = V \cap DiagW$).

3 On methods of deriving groups of W_p-symmetry of different types

Any W'-semi-major finite group of W_p-symmetry with initial group P and generating group G can be derived from G and P by the following steps: 1) we construct the direct product W of isomorphic copies of the group P which are indexed by elements of G; 2) we construct for each g of G the automorphism $\overleftarrow{g}$ of the group W, which acts on the elements w in W as g-moving to the left of the components; 3) we find in W such non trivial subgroups W' which verify the conditions $\overleftarrow{g}\,(W')W' = W'$; 4) we combine pairwise each g of group G with each w of subgroup W'; 5) we introduce into the set of all these pairs the operation $g_iw_i \circ g_jw_j = g_kw_k$, where $g_k = g_ig_j$, $w_k = w_i^{g_j}w_j$ and $w_i^{g_j}(g_s) = w_i(g_jg_s)$.

Any W'-semi-minor finite group of W_p-symmetry with initial group P, generating group G and symmetry subgroup H can be derived from G and P by the following steps: 1) we construct the left direct product W of isomorphic copies of the group P, indexed by elements of G; 2) we construct for each g of G the automorphism $\overleftarrow{g}$ of the group W, which acts on the elements w in W as g-moving to the left of the components; 3) we find in W such non trivial subgroups W' which verify the conditions $\overleftarrow{g}\,(W')W' = W'$; 4) we construct an exact natural left quasi-homomorphism μ with the kernel H of the group G onto the subgroup W' by the rule $\mu(Hg) = w$; 5) we combine pairwise each g of class Hg with $w = \mu(Hg)$; 6) we introduce into the set of all these pairs the operation $g_iw_i \circ g_jw_j = g_kw_k$, where $g_k = g_ig_j$, $w_k = w_i^{g_j}w_j$ and $w_i^{g_j}(g_s) = w_i(g_jg_s)$.

In the case of the pseudo-minor groups the method of deducing them from the given groups G and P is completely analogous to that for the semi-minor groups with the only difference: in this case W' is not a non-trivial subgroup but is a subset that contains the unit of the group W as an element.

4 Conclusion

From the non trivial crystallographic punctual groups G (cyclic and non cyclicals) with the order 2,3,4, 6 and the group P ($P \cong C_2$), we obtained: 1) 14 major groups, 2) 39 W'-semi-major groups; 3) 15 semi-minor groups, 4) 41 pseudo-minor groups of W_p-symmetry.

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Construction of linear binary codes using orthogonal systems of Latin squares

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Abstract

Using orthogonal systems of Latin squares we construct linear binary codes with the length of code words $(n^2+(r+2)n)$ which can correct up to $\frac{r+2}{2}$ errors, where $r \in \mathbb{N}$.

Keywords: quasigroup, Latin square, binary code, vector space, error, orthogonal binary groupoids.

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1 Introduction

Necessary information about linear codes, quasigroups and their orthogonality can be found in [2, 4, 5, 1, 6].

Let the incoming binary information be divided into blocks of constant length n^2 (binary vectors). In total there are 2^{n^2} possible different blocks. We can define on this set of blocks the operations $+$ (addition of vectors) and $\cdot$ (multiplication of vectors by a number from the field $GF(2)$). This set with these operations is transformed into linear vector space E^{n^2} of dimension n^2 over the field $GF(2)$.

We can associate with this vector space a linear binary code C with code words of length $n^2+(2+r)n$, i.e., we can define linear vector space $E^{n^2+(2+r)n}$ which contains 2^{n^2} code words.

2 Construction

The check matrix H of this code C [6, 1] we construct as follows.

Suppose that we have r mutually (in pairs) orthogonal Latin squares $L_1, L_2, \ldots, L_r$ and two selectors F and E defined on the set $\{1, 2, \ldots, n\}$.

$$F = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2 & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \ldots & n \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & n \end{pmatrix}$$

Notice, that evaluations of the number of mutually (in pairs) orthogonal Latin squares is given in [3, 5]. For example, if q is prime, then this number is equal to $(q-1)$.

We put in correspondence to every table from the set $K = \{F, E, L_1, L_2, \ldots, L_r\}$ a rectangular matrix D of size $n \times n^2$ as follows. Using table of size $n \times n$ we build n matrices $C_1, C_2, \ldots, C_n$ of size $n \times n$: matrix C_i consists of units at the locations of the element i in the matrix from the set K and zeros in other places, $i = 1, 2, \ldots, n$.

With each of the matrices D we associate row vectors v_i, $i = 1, 2, \ldots, n$ which consist of the rows of the matrix D, written one after another in order, starting from the first. The length of the vector v_i is n^2.

We associate each of the matrices $F, E, L_1, \ldots, L_r$ with matrices F', E', $L'_1, \ldots, L'_r$ in the way described above.

We form the check matrix H, writing the matrices F', E', $L'_1, \ldots, L'_r$ one under the other, and add the identity matrix to the resulting matrix on the right, i.e., we add a square matrix E_m ($m = (r+2)n$)) whose order is equal to the total number of rows in all matrices $F', E', L'_1, \ldots, L'_r$.

$$H = \begin{pmatrix} E' & \\ F' & \\ L_1' & \\ L_2' & E_m \\ \vdots & \\ L_r' & \end{pmatrix}.$$

The matrix H has dimension $((r+2)n) \times (n^2 + (r+2)n)$.

The coding is carried out according to the rule: each vector a of zeros and units of length n^2 is associated with a vector $b = aG$, where $GH^T = 0$, i.e., G is the generating code matrix.

Hamming distance of this code is $d = r + 3$. Hence the code is able to correct up to $\frac{r+2}{2}$ errors.

Due to the pairwise orthogonality of the squares $F, E, L_1, \ldots, L_r$ for each of the information symbols $\alpha_i, (i = 1, 2, \ldots, n^2)$ we obtain in accordance with the matrix H $r+2$ orthogonal checks, which makes it possible to fix $\frac{r+2}{2}$ errors.

Example 1. *We take matrices over the set $\overline{0, 2}$. It is easy to check that the cyclic group of order 3 $(Z_3, +)$, quasigroup $(Z_3, \circ)$ of the form $x \circ y = x + 2 \cdot y$, where $x, y \in R_3$, (here $(R_3, +, \cdot)$ means the ring residues modulo 3), selectors F and E are orthogonal in pairs. The matrix $D(Z_3, +)$ has the following form:*

$$D(Z_3, +) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

In this case $r = 2$, $m = (2+2)3 = 12$. The matrix H has dimension 12×21. The constructed code can correct two errors. The length of code words is equal to 21. The generating matrix G of this code has dimension 9×21. This code contains $2^9 = 512$ code words.

3 Concluson

Taking into consideration that concept of orthogonality admits different generalization in various directions we can say that the described construction can be generalised and improved.

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On Defining 4-Dimensional Finite Non-Commutative Associative Algebras Over $GF(2^s)$

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Abstract

Finite non-commutative associative algebras are attractive as carriers of the post-quantum public-key cryptoschemes. The 4-dimensional algebras defined over the fields $GF(2^s)$ represent practical interest to provide higher performance. Properties of several different algebras of such type are considered.

Keywords: non-commutative algebra, 4-dimensional associative algebra, post-quantum cryptoscheme.

1 Introduction

The m-dimensional finite non-commutative associative algebras (FNAAs; $m \geq 4$) represent practical interest as carriers of the hidden discrete logarithm problem (HDLP) that was proposed as the basic primitive of the public-key post-quantum cryptoschemes [1, 2]. The use of the 4-dimensional FNAA allows one to get higher performance of the cryptographic algorithms and protocols. To provide cheaper hardware implementation it is attractive to use the 4-dimensional FNAA set over the fields $GF(2^s)$. The finite algebra of quaternions represents the well known case of the 4-dimensional FNAA set over the fields $GF(p)$, where $p \geq 3$ is a prime, however there exists no quaternion algebra set over $GF(2^s)$. This is due to the fact that there exists only one square root from an arbitrary fixed element of the field $GF(2^s)$ for arbitrary fixed non-negative integer s.

In this paper we consider several different types of the 4-dimensional FNAA and some of their properties.

2 Preliminaries

The finite m-dimensional vector space with the additionally defined operation of multiplying arbitrary two vectors, which is distributive relatively the addition operation, represents the algebraic structure called the m-dimensional finite algebra. Suppose $\mathbf{e}_0$, $\mathbf{e}_1$, ... $\mathbf{e}_{m-1}$ are the basis vectors. The vector A of a vector space defined over the finite field $GF(2^s)$ can be denoted in the following two forms: $A = (a_0, a_1, \ldots, a_{m-1})$ and $A = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + \cdots + a_{m-1}\mathbf{e}_{m-1}$, where $a_0, a_1, \ldots, a_{m-1} \in GF(2^s)$ are called coordinates. The multiplication operation (denoted as $\circ$) of two vectors A and $B = \sum_{j=0}^{m-1} b_j\mathbf{e}_j$ is usually defined as follows

$$A \circ B = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_i b_j (\mathbf{e}_i \circ \mathbf{e}_j),$$

where each of the products $\mathbf{e}_i \circ \mathbf{e}_j$ is to be substituted by a single-component vector $\lambda\mathbf{e}_k$, where $\lambda \in GF(2^s)$ is called structural constant, indicated in the respective cell of so called basis vector multiplication table (BVMT). It is usually assumed that the intersection of the ith row and jth column defines the cell indicating the value of the product $\mathbf{e}_i \circ \mathbf{e}_j$. If the defined vector multiplication is non-commutative and associative, then we have FNAA. To define associative vector multiplication one should compose the BVMT that defines associative multiplication of all possible triples of the basis vectors $(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$:

$$(\mathbf{e}_i \circ \mathbf{e}_j) \circ \mathbf{e}_k = \mathbf{e}_i \circ (\mathbf{e}_j \circ \mathbf{e}_k)\,.$$

For arbitrary fixed natural number $s \geq 2$ a set of all binary polynomials of the degree $d < s$, in which the multiplication of two polynomials is defined modulo an irreducible binary polynomial ρ of the degree s, represents a field $GF(2^s)$. One can set different fields $GF(2^s)$ selecting different values ρ, all of such fields being isomorphic.

3 The FNAAs with global two-sided unit

The BVMT shown as Table 1 sets the non-commutative multiplication operation of the 4-dimensional vectors that are defined over an arbitrary fixed finite field, including the fields $GF(2^s)$. The respective 4-dimensional FNAA contains the global two-sided unit $E = (e_0, e_1, e_2, e_3)$ that is described as follows:

$$E = \left(\frac{\sigma}{\lambda\sigma - 1}, \frac{1}{1 - \lambda\sigma}, \frac{1}{1 - \lambda\sigma}, \frac{\lambda}{\lambda\sigma - 1} \right). \tag{1}$$

For some invertible vector $A = (a_0, a_1, a_2, a_3)$ the equation $X \circ A = E$, where $X = (x_0, x_1, x_2, x_3)$ is the unknown value, has solution $X = A^{-1}$ that can be obtained solving the following two independent systems of two linear equations with the unknowns (x_0, x_1) and (x_2, x_3) correspondingly:

$$\begin{cases} (\lambda a_0 + a_2)\, x_0 + (a_0 + \sigma a_2)\, x_1 = e_0; \\ (\lambda a_1 + a_3)\, x_0 + (a_1 + \sigma a_3)\, x_1 = e_1; \end{cases} \tag{2}$$

$$\begin{cases} (\lambda a_0 + a_2)\, x_2 + (a_0 + \sigma a_2)\, x_3 = e_2; \\ (\lambda a_1 + a_3)\, x_2 + (a_1 + \sigma a_3)\, x_3 = e_3. \end{cases} \tag{3}$$

Each of the systems (2) and (3) has the same main determinant Δ_A :

$$\begin{aligned} \Delta_A &= (\lambda a_0 + a_2)\,(a_1 + \sigma a_3) - (\lambda a_1 + a_3)\,(a_0 + \sigma a_2) = \\ &= (1 - \lambda\sigma)\,(a_1 a_2 - a_0 a_3)\,. \end{aligned} \tag{4}$$

From the condition $\Delta_A \neq 0$ we have the following invertibility condition of the elements of the considered 4-dimensional FNAA:

$$a_1 a_2 \neq a_0 a_3. \tag{5}$$

Table 1. The BVMT setting the 4-dimensional FNAA over $GF(2^s)$ ($\lambda\sigma \neq 1$).

$\circ$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_0$	$\lambda\mathbf{e}_0$	$\lambda\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{e}_1$
$\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\sigma\mathbf{e}_0$	$\sigma\mathbf{e}_1$
$\mathbf{e}_2$	$\lambda\mathbf{e}_2$	$\lambda\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\sigma\mathbf{e}_2$	$\sigma\mathbf{e}_3$

Another 4-dimensional FNAA can be set over the fields $GF(2^s)$ with the BVMT shown in Table 2 that is constructed using the unified method for defining the FNAAs of arbitrary even dimensions [3]. The FNAA defined by Table 2, where $\tau \neq 1$, contains the global bi-sided unit

$$E = \left(\frac{1}{1-\tau}, \frac{1}{1-\tau}, \frac{\tau}{\tau-1}, \frac{1}{\tau-1}\right).$$

Vectors $A = (a_0, a_1, a_2, a_3)$, coordinates of which satisfy the condition

$$a_0 a_1 \neq a_2 a_3, \tag{6}$$

are invertible.

From the invertibility conditions (5) and (6) one can derive the following formula for number Ω of the invertible elements in each of the described 4-dimensional FNAAs:

$$\Omega = 2^s\left(2^s-1\right)^2\left(2^s+1\right). \tag{7}$$

4 The FNAAs containing the set of 2^{2s} global right-sided units

Table 3 sets the 4-dimensional FNAA containing no global two-sided unit. This algebra is characterized in that it contains a large set of

Table 2. An alternative BVMT for setting the 4-dimensional FNAA over $GF(2^s)$, where $\tau \neq 1$.

$\circ$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_3$	$\mathbf{e}_0$	$\mathbf{e}_3$
$\mathbf{e}_1$	$\tau\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\tau\mathbf{e}_1$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1$
$\mathbf{e}_3$	$\tau\mathbf{e}_0$	$\mathbf{e}_3$	$\mathbf{e}_0$	$\tau\mathbf{e}_3$

global right-sided units which can be computed from the vector equation $A \circ X = A$. The last equation can be reduced to the following system of four linear equations with coordinates of the right operand x_0, x_1, x_2, x_3 as the unknown values:

$$\begin{cases} a_0x_0 + a_0x_1 + \lambda a_2x_2 + \lambda a_2x_3 = a_0; \\ a_1x_0 + a_1x_1 + \lambda a_3x_2 + \lambda a_3x_3 = a_1; \\ a_0x_2 + a_0x_3 + a_2x_0 + a_2x_1 = a_2; \\ a_1x_2 + a_1x_3 + a_3x_0 + a_3x_1 = a_3. \end{cases} \tag{8}$$

Performing the variable substitution $u_1 = x_0 + x_1$ and $u_2 = x_2 + x_3$, one can rewrite the system (8) in the following form

$$\begin{cases} a_0u_1 + \lambda a_2u_2 = a_0; \\ a_1u_1 + \lambda a_3u_2 = a_1; \\ a_0u_2 + a_2u_1 = a_2; \\ a_1u_2 + a_3u_1 = a_3. \end{cases} \tag{9}$$

It is easy to see that for arbitrary vector A the system (9) holds true for the values of the unknowns $u_1 = 1$ and $u_2 = 0$. Thus, every vector X, coordinates of which satisfy the conditions $x_0 + x_1 = u_1 = 1$, $x_2 + x_3 = u_2 = 0$, acts as a global right-sided unit. Evidently, the set of all 2^{2s} global right-sided units $R = (r_0, r_1, r_2, r_3)$ is described as follows:

$$R = (\alpha, 1 - \alpha, \beta, -\beta), \tag{10}$$

Table 3. The BVMT setting the 4-dimensional FNAA with 2^{2s} global right-sided units ($\lambda \in GF(2^s)$; $\lambda \neq 0$).

$\circ$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_2$	$\mathbf{e}_2$
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{e}_3$	$\mathbf{e}_3$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_2$	$\lambda\mathbf{e}_0$	$\lambda\mathbf{e}_0$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_3$	$\lambda\mathbf{e}_1$	$\lambda\mathbf{e}_1$

where $\alpha, \beta \in GF(2^s)$.

One can show that every vector $A = (a_0, a_1, a_2, a_3)$, coordinates of which satisfy the condition

$$(a_0 + a_1)^2 = \lambda (a_2 + a_3)^2 , \tag{11}$$

is a locally invertible element of the considered FNAA, i. e., a local two-sided unit E_A relates to such vector A. Besides, the vector A is a generator of finite cyclic group contained in the algebra. Evidently, the local unit E_A is the group unit of the mentioned cyclic group.

Local two-sided units are contained in the set (10) and every global right-sided unit is simultaneously a local two-sided unit relating to some invertible vector. All invertible vectors relating to some fixed local two-sided unit compose a finite multiplicative group. The considered FNAA is divided into 2^{2s} groups, since every global right-sided unit sets a group.

From the condition (11) one can find the number of invertible vectors Ω:

$$\Omega = 2^{3s} (2^s - 1) .$$

One can show every of the mentioned finite groups has the same order Ω'. Since every invertible vector is contained only in one of the groups, we have the following formula for the value Ω':

$$\Omega' = 2^s (2^s - 1) .$$

5 On some applications

The FNAAs described in Sections 2 can be used to implement post-quantum digital signature schemes proposed in [4] on the basis of the finite algebra of quaternions. The FNAAs described in Section 3 can be used as algebraic carrier of post-quantum public key-agreement scheme proposed in [5]. In both cases we will get higher performance of the cryptoschemes.

6 Conclusion

One can propose different types of the BVMT for setting 4-dimensional FNAAs over the finite fields $GF(2^s)$. Using such FNAAs as carries of the public-key post-quantum schemes one can get higher perfomance of the cryptoschemes.

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Finite Non-commutative Algebras as Carriers of Post-quantum Public-key Cryptoschemes

Alexandr Moldovyan and Nicolai Moldovyan

Abstract

Hidden logarithm problem defined in finite non-commutative associative algebras had been proposed as the base primitive of the post-quantum public-key cryptoschemes. Different types of the algebras are considered as the carriers of different post-quantum public-key cryptoschemes. Related research tasks are considered.

Keywords: finite non-commutative algebra, associative algebra, public-key cryptography, post-quantum cryptoscheme.

1 Introduction

Development of the post-quantum public-key cryptoschemes represents the modern challenge in the area of applied and theoretic cryptography [1, 2, 3]. Different forms of the hidden discrete logarithm problem (HDLP) defined over the finite non-commutative associative algebras (FNAAs) were proposed as the base primitive of the post-quantum public key-agreement [4, 5, 6] and digital signature schemes [7, 8, 9].

In this report we summarize the current results on application of the FNAAs as algebraic carrier of the post-quantum cryptoschemes and some research items connected with the development of the post-quantum cryptographic algorithm and protocols.

2 Preliminaries

For using as cryptographic primitive, the DLP is set in a finite cyclic group Γ of the order Ω containing a large prime divisor q as follows:

$$Y' = G^x,$$

where Y' is the public key; G is a generator of the group; and the non-negative integer $x < \Omega$ is the unknown (private key). Finding the value x, when the values G and Y' are known, is called DLP. The HDLP represents a universal cryptographic primitive on the basis of which different types of the public-key algorithms and protocols were developed, including various cryptographic standards. Unfortunately the DLP can be solved in polynomial time on the hypothetic quantum computer [10]. The last means the DLP-based cryptoschemes will be not secure in the comming age of quantum computations [11].

To provide security to quantum attacks the HDLP was proposed. The HDLP is defined so that one of the values G and Y' or both of them are masked (hidden). Thus, it is supposed the HDLP is set in some finite algebraic structure containing a large variety of finite cyclic groups. The FNAAs suite well as algebraic carriers of the HDLP. The main contribution to the security is introduced by the exponentiation operation G^x that is called the base operation. The operation used to hide the values G and Y' are called the masking operations. To provide possibility to design the HDLP-based public-key cryptoschemes one should use the masking operations that are mutually commutative with the base operation. Therefore, the automorphism-map operations and the homomorphism-map operations are attractive to be applied as masking operations. Using different types of the masking operations one can set different versions of the HDLP.

For the first time the HDLP was defined in the finite algebra of quaternions [4, 12] as follows:

$$Y = Q^w \circ G^x \circ Q^{-w} = \alpha\left(G^x\right), \tag{1}$$

where $Q \circ G \neq G \circ Q$; the variable V takes on all values in the quaternion algebra; $\alpha\left(V\right)$ is the automorphism-map operation. The form of HDLP descibed by the formula (1) was applied to design a public key-agreement scheme and commutative encryption algorithm [4, 12]. However, reducibility of the first form of the HDLP to the DLP in the finite field $GF(p^2)$ was shown in the paper [5].

Recently [7, 13] several new FNAAs and new versions of the HDLP were introduced and used to develop the post-quantum digital signature protocols. For example, in the digital signature scheme defined in the FNAA containing global two-sided unit the public key represents the triple of vectors (Y, Z, T) defined as follows [13]:

$$Y = Q \circ G^x \circ Q^{-1}, \quad Z = H \circ G \circ H^{-1}; \quad T = Q \circ E \circ H^{-1}, \qquad (2)$$

where $Q \circ G \neq G \circ Q$; $H \circ G \neq H \circ Q$; E is a randomly selected vector from the set of local units related to the non-invertible vector G. The HDLP consists in finding the value x in the case, when only the public key is known.

In the signature scheme defined in the FNAA containing a large set of global left-sided units the public key represents the pair of vectors (Y, Z) defined as follows [7]:

$$Y = H \circ G^x \circ D, \quad Z = J \circ G \circ T, \qquad (3)$$

where $D \circ G \neq G \circ D$; $D \circ H = L_1$; $D \circ J = L_2$; $T \circ J = L_3$; L_1, L_2, and L_3 are global left-sided units. The HDLP consists in finding the value x in the case, when only the values Y and Z are known.

3 Types of the FNAAs suitable for setting the HDLP

The FNAAs suitable for application as algebraic carriers of the cryptoschemes based on the HDLP can be classified by type of the masking operations that can be potentially used. The masking operations represent some homomorphism-map and automorphism-map operations and setting the operations of the last two kinds is connected with the types of the unit elements contained in a fixed FNAA, therefore the m-dimensional FNAAs ($m \geq 4$) can be classified by types of the units available in the frame of the fixed FNAA.

The FNAAs containing the global two-sided unit are attractive for using them as algebraic carriers of the digital signature and blind digital signature schemes. Setting of different types of HDLP is connected with using the automorphism-map operations.

The FNAAs containing the large set of the global single-sided (left-sided or right-sided) units can be applied as algebraic carriers of the digital signature and blind digital signature schemes, public key-agreement protocols, public-encryption and commutative-encryption algorithms. Setting different types of HDLP is connected with using the homomorphism-map operations of two different kinds.

4 Defining the FNAAs of different dimensions

One can define the m-dimensional FNAA for an arbitrary fixed value $m \geq 2$, however, the cases of even dimensions $m \geq 4$ represent the main interest for their application as algebraic carriers of the post-quantum cryptoschemes. The non-commutative multiplication operation in the FNAAs is defined with so called basis vector multiplication tables (BVMTs). For the case $m = 4$ different BVMTs can be selected using the exhaustive-search method that is not efficient for the cases $m \geq 6$.

In the last case it is interesting to use some unified methods for describing the BVMTs for different values m. The paper [14] considers a method for defining the FNAAs of arbitrary fixed dimension $m > 1$ and general description of the properties of the defined class of algebras.

The paper [7] had introduced a method for defining the FNAAs of arbitrary even dimension $m > 2$. Using that method one can define 4-dimensional FNAAs with global two-sided unit and 6-dimensional FNAAs containing sets of the global single-sided units.

The paper [15] had introduced a method for defining the FNAAs of arbitrary even dimension $m > 2$. Using that method one can define 4-dimensional FNAAs with global two-sided unit and 6-dimensional FNAAs containing sets of the global single-sided units.

Each of the unified methods is introduced as respective mathematical formula describing the class of the BVMTs defining the non-commutative associative multiplication operation. One of the merits of the unified methods for defining the FNAAs is the general proof of the associativity property for all given values m.

5 Conclusion

Currently the HDLP can be considered as universal post-quantum cryptographic primitive. Different FNAAs are proposed as algebraic carriers of different forms of the HDLP used to propose candidates for post-quantum signature schemes, public key-agreement proptocols, public-encryption and commutative encryption algorithms. The main research task connected with the post-quantum security estimation of the proposed candidates consists in the investigation of the reducibility of different forms of the HDLP to the DLP in a finite field.

Study of the structure of FNAAs and designing new forms of the HDLP are also important research tasks in the area of development of the post-quantum cryptoschemes based on computations in FNAAs.

Development of the HDLP-based cryptoschemes using the FNAAs defined over the finite field $GF(2^s)$ represents significant practical interest.

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Numerical approximation for a nonlocal Allen-Cahn equation supplied with non-homogeneous Neumann boundary conditions

Costică Moroşanu

Abstract

We propose a first-order implicit difference scheme to numerically solve the nonlocal Allen-Cahn equation subject to non-homogeneous Neumann boundary conditions.

Keywords: Nonlinear equations; Reaction-diffusion equations; Finite difference methods; thermodynamics; phase changes.

1 Introduction

Consider the following problem

$$\begin{cases} \alpha\xi \frac{\partial}{\partial t}\varphi(t,x) = \xi\Delta\varphi(t,x) + \frac{1}{2\xi}f(\varphi(t,x)) + \frac{1}{|\Omega|}\int\limits_{\Omega} f(\varphi(t,y))dy \\ \xi\frac{\partial}{\partial \mathbf{n}}\varphi(t,x) = w(t,x) \\ \varphi(0,x) = \varphi_0(x) \end{cases} \tag{1}$$

The nonlinear problem (1) (see [1]) was introduced to describe the motion of anti-phase boundaries in crystalline solids, and it has been widely applied to many complex moving interface problems, e.g., the *mixture of two non compressible fluids*, the *nucleation of solids*, the *vesicle membranes* (see for instance P. W. Bates, S. Brown and J. Han [2], J. Zhang and Q. Du [5] and the references therein).

2 Numerical method

We are concerned with the numerical approximation of the solution $\varphi(t,x)$ in (1), denoted by $V(t,x)$ in the sequel. We will work in one dimension and then $\Delta\varphi = \Delta V = V_{xx}$. Let $\Omega = [0,c] \subset I\!R_+$ and given a positive value T, we denote by V_j^i the approximate values in the point (t_i, x_j) of the unknown function $V(t,x)$ (see [4] for more details).

To approximate the partial derivative with respect to time, we employed a *first-order scheme*:

$$\frac{\partial}{\partial t} V(t_{i+1}, x_j) \approx \frac{V_j^{i+1} - V_j^i}{\varepsilon} \quad i = 1,2,\ldots,M-1, \quad j = 1,2,\ldots,N. \tag{2}$$

The Laplace operator (the *diffusion term*) will be approximated by a second order centered finite differences, namely:

$$V_{xx}(t_i, x_j) = \Delta_{dx} V_j^i \approx \frac{V_{j-1}^i - 2V_j^i + V_{j+1}^i}{dx^2} \; i = 1,2,...,M, \; j = 1,2,...,N, \tag{3}$$

(Δ_{dx} is the discrete Laplacian depending on the step-size dx).

Next, to approximate the nonlinear term (*reaction term*) $\frac{1}{2\xi} f(V(t,x)) = V(t,x) - V^3(t,x)$ in (1), we will involve an implicit formula, i.e.:

$$\left[V(t_i, x_j) - V^3(t_i, x_j)\right] \approx \left[V_j^i - (V_j^i)^3\right] \; i = 1,2,\ldots,M, \; j = 1,2,\ldots,N. \tag{4}$$

The *mass conserving term* will be approximated by

$$\frac{1}{|\Omega|} \int\limits_{\Omega} \left[V(t,y) - V^3(t,y)\right] dy \approx \frac{1}{|\Omega|} \sum_{j=1}^{N} \frac{V_j^i - (V_j^i)^3}{dx}. \tag{5}$$

First-order Implicit Backward Difference Formula (**1-IMBDF**). By replaying in (1) the approximations stated in (2)-(5); we get:

$$\begin{aligned} \frac{\alpha\xi}{\varepsilon} V_j^{i+1} + \xi \left[\frac{V_{j-1}^i - 2V_j^i + V_{j+1}^i}{dx^2}\right] + \frac{1}{2\xi}\left[\left(V_j^{i+1}\right)^3 - V_j^{i+1}\right] = \\ = \tfrac{\alpha\xi}{\varepsilon} V_j^i + \tfrac{1}{|\Omega|} \textstyle\sum_{j=1}^{N} \frac{V_j^i - (V_j^i)^3}{dx}, \end{aligned} \tag{6}$$

for $i = 1, 2, \ldots, M-1$ and $j = 1, 2, \ldots, N$.

In order to approximate the vector-solution V^i in (6) for time level i, $i = 2, 3, \ldots, M$, the usual iterative Newton method was involved (see [3] for details).

As a numerical experiment taking $T = 0.1$, $c = 2$, $M = 101$, $N = 201$, $\alpha = 1.0$, $\xi = .5$, Figure 1 shows the numerical results at $t = 0.002$, $t = 0.005$ and $t = 0.1$, corresponding to $w = 0$.

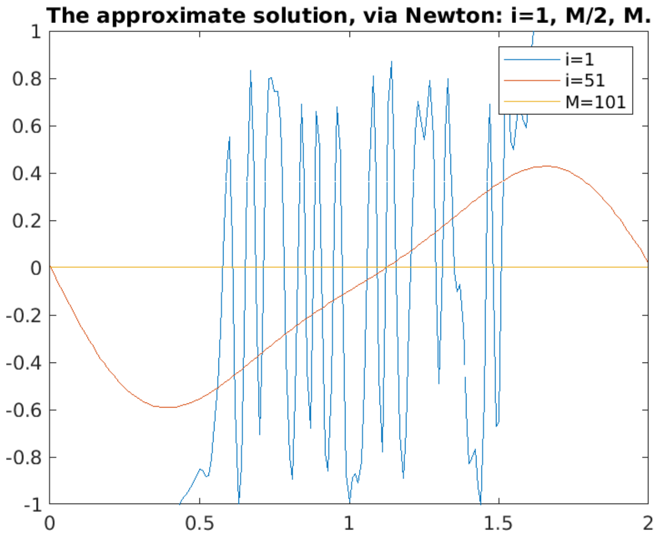

Figure 1. The approximate solutions V^i at different levels of time and $w = 0$.

3 Conclusions

As a novelty of this work we refer to the numerical scheme introduced by (6) in order to approximate the solution to the nonlocal reaction-diffusion problem (1) in presence of the cubic nonlinearity $f(\varphi(t,x)) = \varphi(t,x) - \varphi^3(t,x)$ and the *mass conserving term* $\frac{1}{|\Omega|} \int\limits_{\Omega} \left[\varphi(t,y) - \varphi^3(t,y)\right] dy$.

Not least, let's point out (as an open problem) that the approximate solutions obtained by implementing the numerical scheme (6) can be regarded as admissible for the corresponding boundary optimal control problem.

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Center-affine invariant stability conditions of unperturbed motion governed by critical cubic differential system

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Abstract

Center-affine invariant Lyapunov stability conditions of unperturbed motion governed by critical cubic differential system were obtained.

Keywords: differential system, comitant, transvectant, Lyapunov stability.

1 Introduction

The cubic differential system $s(1, 2, 3)$ is of particular interest not only in theoretical, but in a number of applied problems of differential equations. Therefore, the obtaining Lyapunov stability [1] invariant conditions for unperturbed motion governed by the differential system $s(1, 2, 3)$ is relevant. But, this problem turned out to be very difficult. We couldn't find the regularity in construction of Lyapunov series coefficients, which play an important role in determination of stability of unperturbed motion governed by differential systems $s(1, 2, 3)$ in critical case. Then we decided to examine the different particular cases of the differential system $s(1, 2, 3)$: when quadratic part or cubic part have the Darboux form [2]. This allowed us to understand how the problem of stability of unperturbed motion governed by the critical differential systems $s(1, 2, 3)$ can be solved in general terms.

2 Invariants and comitants of cubic differential system

Let us consider the system of differential equations

$$\frac{dx}{dt} = \sum_{i=1}^{3} P_i(x,y), \qquad \frac{dy}{dt} = \sum_{i=1}^{3} Q_i(x,y), \tag{1}$$

where $P_i(x,y)$, $Q_i(x,y)$ are homogeneous polynomials of degree $i = 1,2,3$ in the phase variables x and y. Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$.

Let f and φ be two center-affine comitants [3] of the system (1) of degree r and ρ respectively in the phase variables x and y. According to [4], [5] and [6] the polynomial

$$(f,\varphi)^{(k)} = \frac{(r-k)!(\rho-k)!}{r!\rho!} \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}} \tag{2}$$

also is a center-affine comitant of the system (1) and is called a transvectant of order k in polynomials f and φ.

The following $GL(2;\mathbb{R})$-comitants [3] have the first degree with respect to the coefficients of the system (1):

$$\begin{gathered} R_i = P_i(x,y)y - Q_i(x,y)x, \quad (i = 1,2,3), \\ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right), \quad (i = 1,2,3), \end{gathered} \tag{3}$$

Using the comitants (3) and the transvectant (2), in [7, 8] the following invariants and comitants of the system (1) were constructed:

$$\begin{gathered} K_1 = S_1,\ K_2 = R_1,\ K_3 = (R_1,R_1)^{(2)},\ K_4 = R_2,\ K_5 = S_2, \\ K_6 = (R_2,R_1)^{(1)},\ K_7 = (R_2,R_1)^{(2)},\ K_8 = R_3,\ K_9 = (R_3,R_1)^{(1)}, \\ K_{10} = (R_3,R_1)^{(2)},\ K_{11} = (K_{10},R_1)^{(1)},\ K_{12} = (K_{10},R_1)^{(2)}, \\ K_{13} = (K_7,R_1)^{(1)},\ K_{14} = (S_2,R_1)^{(1)},\ K_{15} = S_3, \\ K_{16} = (S_3,R_1)^{(1)},\ K_{17} = (S_3,R_1)^{(2)}. \end{gathered} \tag{4}$$

Later on, we will need the following comitants and invariants of system (1):

$$M_1 = 3K_1K_7 - 4K_1K_{14} + 2K_1^2K_5 - 6K_{13},$$

$$M_2 = -8K_1K_{11} + 6K_1K_2K_{17} + 4K_1^2K_{10} - 6K_1^2K_{16} + 3K_1^3K_{15} + 8K_2K_{12},$$

$$M_3 = 6K_1K_2K_7 + 4K_1K_2K_{14} - 2K_1^2K_2K_5 + 6K_1^2K_6 + 3K_1^3K_4 + 12K_2K_{13},$$

$$M_4 = -K_1K_5 + 2K_{14}, \quad M_5 = 3K_1K_5 + 6K_7 - 2K_{14},$$

$$M_6 = 4K_1K_2K_{11} - 4K_1^2K_2K_{10} + 2K_1^3K_9 - K_1^4K_8 - 2K_2^2K_{12},$$

$$M_7 = K_1K_{17} + 2K_{12}, \quad M_8 = -3K_1K_2K_7 + 2K_1^2K_6 - K_1^3K_4 + 2K_2K_{13},$$

$$M_9 = -3K_1K_7 + 4K_1K_{14},$$

$$M_{10} = 2K_1K_2K_{17} + 8K_1K_{11} + 4K_1^2K_{10} + 2K_1^2K_{16} + K_1^3K_{15} + 8K_2K_{12},$$

$$M_{11} = 2K_1K_2K_{17} + 8K_1K_{11} - 4K_1^2K_{10} - 2K_1^2K_{16} + K_1^3K_{15} - 8K_2K_{12},$$

$$M_{12} = 4K_1K_2K_{11} + 4K_1^2K_2K_{10} + 2K_1^3K_9 + K_1^4K_8 + 2K_2^2K_{12},$$

$$M_{13} = K_1K_{17} - 2K_{12},$$

$$M_{14} = -8K_1K_{11} + 6K_1K_2K_{17} - 4K_1^2K_{10} + 6K_1^2K_{16} + 3K_1^3K_{15} - 8K_2K_{12},$$

$$M_{15} = -8K_1K_{11} + 6K_1K_2K_{17} + 4K_1^2K_{10} + 3K_1^3K_{15} - 6K_1^2K_{16} + 8K_2K_{12}. \tag{5}$$

were K_i $(i = \overline{1,17})$ are from (4).

3 Invariant stability conditions of unperturbed motion

In [9, 10] the stability conditions of unperturbed motion for critical system (1) expressed through coefficients of this system were provided. Using these results, all cases of critical system (1) are separated by two sets of center-affine conditions:

$$\begin{aligned} &\text{a) } S_1 < 0,\ S_1^2 + 2K_3 = 0,\ M_8 \not\equiv 0; \\ &\text{b) } S_1 < 0,\ S_1^2 + 2K_3 = 0,\ M_8 \equiv 0. \end{aligned} \tag{6}$$

Taking into account the center-affine invariants and comitants (3)-(5), all invariant stability conditions of unperturbed motion for system (1) in the case a) from (6) are obtained. These conditions are large therefore not included here. We will bring here results obtained in case b) from (6).

Theorem 1. *If* $S_1 < 0$, $S_1^2 + 2K_3 = 0$ *and* $M_8 \equiv 0$, *then the stability of the unperturbed motion described by system* (1) *is characterized by one of the following 13 possible cases:*
I. $M_1 \not\equiv 0$, *then the unperturbed motion is unstable;*
II. $M_1 \equiv 0$, $M_{15} < 0$, *then the unperturbed motion is unstable;*
III. $M_1 \equiv 0$, $M_{15} > 0$, *then the unperturbed motion is stable;*
IV. $M_1 \equiv M_{15} \equiv 0$, $M_5 M_6 \not\equiv 0$, *then the unperturbed motion is unstable;*
V. $M_1 \equiv M_{15} \equiv M_5 \equiv 0$, $M_6 M_7 > 0$, *then the unperturbed motion is unstable;*
VI. $M_1 \equiv M_{15} \equiv M_5 \equiv 0$, $M_6 M_7 < 0$, *then the unperturbed motion is stable;*
VII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv 0$, $M_3 M_6 \not\equiv 0$, *then the unperturbed motion is unstable;*
VIII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv 0$, $M_6 \not\equiv 0$, $M_{10} < 0$, *then the unperturbed motion is unstable;*
IX. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv 0$, $M_6 \not\equiv 0$, $M_{10} > 0$, *then the unperturbed motion is stable;*
X. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv 0$, $M_6 M_{12} > 0$, *then the unperturbed motion is unstable;*
XI. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv 0$, $M_6 M_{12} < 0$, *then the unperturbed motion is stable;*
XII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv M_{12} \equiv 0$, $M_6 \not\equiv 0$, *then the unperturbed motion is stable;*
XIII. $M_1 \equiv M_{15} \equiv M_6 \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series.

The expressions S_1, K_3, M_1, M_3, M_5, M_6, M_7, M_{10}, M_{12}, M_{15}, *are from* (3)-(5).

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Equivalence of some algebras of singular integral operators. Noetherian criteria

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Definition. Two algebras A_1 ($\subset L(B_1)$) and A_2 ($\subset L(B_2)$) will be called equivalent if there are invertible operators $M \in L(B_1, B_2)$ such that the set of operators of the form MAM^{-1} ($A \in A_1$) coincides with the algebra A_2.

Having an algebra A_1 and if we prove that it is equivalent to an algebra A_2, then many properties known for the operators of algebra A_2 can be extended for operators of algebra A_1. For example, if conditions of invertibility of operators from algebra A_2 are known (the left and the right ones), then we make the respective conclusions for operators from algebra A_1.

In this work the equivalence of algebras is used in order to determine Noetherian conditions for some classes of integral operators in various spaces with weights.

Theorem 1. The algebra $\Sigma(\Gamma, \rho; V)$ is equivalent with algebra $\Sigma_2(\widetilde{\Gamma}_0, \widetilde{\rho}_0)$, where $\widetilde{\rho}_0(t) = (1-t)^{\beta} t^{-1/2}$, $\widetilde{\Gamma}_0 = [0,\, 1]$.

Theorem 2. The operator $A \in \Sigma(\Gamma, \rho; V)$ is Noetherian in the space $L_p(\Gamma, \rho)$ if and only if its symbol is different of zero. If this condition is satisfied, then the following is true:

$$Ind\, A = -\frac{1}{2\pi}\Big\{arg\, det\, A(t,\mu)/\, det\, a_{22}(t,0)\Big\}_{0\le t,\, \mu\le 1}.$$

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On Set-Valued Continuous Periodic Functions on Topological Spaces

Dorin Pavel

Abstract

In the present article the almost periodic continuous compact-valued functions are studied.

Keywords: Pompeiu-Hausdorff distance, almost periodicity, set-valued function.

1 Introduction

Let A and B be two non-empty subsets of a metric space (G, d). We define their Pompeiu-Hausdorff distance $d_P(A, B)$ by $d_P(A, B) = max\{sup\{inf\{d(x, y) : y \in B)\} : x \in A\}, sup\{inf\{d(x, y) : x \in A)\} : y \in B\}\}$. Denote by d the Euclidean distance on the space $\mathbb{R}$ of reals and $B(\mathbb{R})$ the space of all non-empty bounded subsets of $\mathbb{R}$ with the Pompeiu-Hausdorff distance $d_P(A, B)$. The space $(B(\mathbb{R}), d_P)$ is a complete metric space. Fix a topological space G. By $T(G)$ denote the family of all single-valued continuous mappings of G into G. Relatively to the operation of composition, the set $T(G)$ is a monoid (a semigroup with unity).

A single-valued $\varphi : G \longrightarrow B(\mathbb{R})$ is called a set-valued function on G. For any two set-valued functions $\varphi, \psi : G \longrightarrow B(\mathbb{R})$ and $t \in \mathbb{R}$ there are determined the distance $\rho(\varphi, \psi) = sup\{d_P(\varphi(x), \psi(x)) : x \in G\}$ and the set-valued functions $\varphi + \psi$, $\varphi \cdot \psi$, $-\varphi$, $\varphi \cup \psi$, where $(\varphi \cup \psi)(x) = (\varphi(x) \cup \psi(x)$, and $t\varphi$. We put $(\varphi \circ f)(x) = \varphi(f(x))$ for all $f \in T(G)$, $\varphi \in SF(G)$ and $x \in G$. The set-valued function $\varphi : G \longrightarrow B(\mathbb{R})$ is lower (upper) semi-continuous if for any open (closed) subset H of $\mathbb{R}$

the set $\varphi^{-1}(H) = \{x \in G : \varphi(x) \cap H \neq \emptyset\}$ is open (closed) in G. A function φ is continuous if it is lower and upper semi-continuous.

Let $SF_c(G)$ be the family of all compact-valued functions on G. The space $(SF_c(G), \rho)$ is a complete metric space. The subspace $CSF_c(G)$ of all continuous compact-valued functions is closed in $SF_c(G)$. The subspace $MF_c(G)$ of all compact-valued and convex-valued functions is closed in $SF_c(G)$ too. We denote $CMF_c(G) = MF_c(G) \cap CSF_c(G)$.

2 Almost periodic set-valued functions

Fix a topological space G. On $SF_c(G)$ we consider the topology generated by the distance ρ. If $\varphi \in SF(G)$ and $f \in T(G)$, then $\varphi_f = \varphi \circ f$ and $\varphi_f(x) = \varphi(f(x))$ for any $x \in G$). Evidently, $\varphi_f \in SF_c(G)$.

Fix a submonoid P of the monoid $T(G)$. We say that P is a monoid of continuous translations of G. The set P is called a transitive set of translations of G if for any two points $x, y \in G$ there exists $f \in P$ such that $f(x) = y$.

For any function $\varphi \in C(G)$ we put $P(\varphi) = \{\varphi_f : f \in P\}$.

Remark 2.1. *The following assertions are true: (1) If $\varphi \in MF_c(G)$, then $cl_{SF_c(G)}P(\varphi) \subseteq MF_c(G)$.*

(2) If $\varphi \in CSF_c(G)$, then $cl_{SF_c(G)}P(\varphi) \subseteq CSF_c(G)$.

(3) If $\varphi \in CMF_c(G)$, then $cl_{SF_c(G)}P(\varphi) \subseteq CMF_c(G)$.

Definition 2.2. *A function $\varphi \in SF_c(G)$ is called an almost P-periodic function on a space G if the closure $\bar{P}(\varphi)$ of the set $P(\varphi)$ in the space $SF_c(G)$ is a compact set.*

Denote by P-$ap_s(G)$ the subspace of all P-periodic set-valued functions on the space G.

Theorem 2.3. *Let $\varphi : G \to R$ be a compact-valued and convex-valued continuous function on G. We put $\varphi_{min}(x) = min(\varphi(x)$ and $\varphi_{max}(x) = max(\varphi(x)$ for any $x \in G$. The following assertions are equivalent:*

1. φ is an almost periodic set-valued function.

2. $\varphi_{max}, \varphi_{min}$ are almost periodic continuous single-valued functions on G.

Proof. We prove the following assertions.
Assertion 1. *If φ is continuous and compact-valued, then $\varphi_{max}, \varphi_{min}$ are continuous single-valued functions.*

Fix $x_0 \in G$, $\varepsilon > 0$ and let $\varphi_{min}(x_0) = y_0$. We put $U_1 = \{t \in \mathbb{R} : |t - y_0| < \varepsilon/2\}$ and $U_2 = \{t \in \mathbb{R} : inf\{|t - y| : y \in \varphi(x_0)\} < \varepsilon/2\}$. The sets U_1 and U_2 are open in $\mathbb{R}$.

Since φ is lower semicontinuous, the set $V_1 = \{x \in G : \varphi(x) \cap U_1 \neq \emptyset\}$ is open in G. Since φ is upper semicontinuous, the set $V_2 = \{x \in G : \varphi(x) \subseteq U_2\}$ is open in G.

By construction $x_0 \in V = V_1 \cap V_2$. Let $x \in V$ and $y_1 = \varphi_{min}(x_1)$. Assume $|y_1 - y_0| \geq \varepsilon$. We have that $|y_1 - y_0| = |\varphi_{min}(x_1) - \varphi_{min}(x_0)| \geq \varepsilon$.

We have two possible cases.
Case 1. $y_1 < y_0$. Since $x_1 \in V_2$ and $y_1 \in \varphi(x_1)$, there exists $t \in \varphi(x_0)$ such that $|t - y_1| < \varepsilon/2$. Hence $y_1 < y_0 \leq t, |y_1 - y_0| \geq \varepsilon$ and $|t - y_1| < \varepsilon/2$, a contradiction. This case is impossible.
Case 2. $y_1 > y_0$. Since $x_1 \in V_1$, there exists $t \in \varphi(x_1)$ such that $|y_0 - t| < \varepsilon/2$. Hence $y_0 < y_1 \leq t, |y_1 - y_0| \geq \varepsilon$ and $|t - y_0| < \varepsilon/2$, a contradiction. This case is impossible too.

Therefore $|\varphi_{min}(x) - \varphi_{min}(x_0)| \geq \varepsilon$ for any $x \in V$. Hence the function φ_{min} is continuous. In the similar way we prove that φ_{max} is continuous.
Assertion 2. *Let $CSF_c(G)$ be the space of all compact-valued continuous functions $C(G)$ the space of all single-valued continuous functions on G in the topology of uniform convergence. Then $C(G) \subseteq CSF_c(G)$ and the mapping $\psi_{min} : CSF_c(G) \to C(G)$ and $\psi_{max} : CSF_c(G) \to C(G)$, where $\psi_{min}(\varphi) = \varphi_{min}$ and $\psi_{max}(\varphi) = \varphi_{max}$ for any $\varphi \in CF(G)$, are continuous mappings.*

If $d_P(\varphi, \psi) < \varepsilon$, then $d_P(\varphi_{min}, \psi_{min}) < \varepsilon$ and $d_P(\varphi_{max}, \psi_{max}) < \varepsilon$. Hence ψ_{min}, ψ_{max} are continuous functions.
Assertion 3. *If φ is almost periodic continuous compact-valued function, then the functions $\varphi_{min}, \varphi_{max}$ are almost periodic.*

The space $(CSF_c(G), d_P)$ and $(C(G), d_P)$ are complete and $C(G)$ is closed subspace of $CSF_c(G)$. If $h \in \mathcal{P}, \psi = \varphi_h$ and $f = \varphi_{min}$, then $f_h = \psi_{min}$. Hence $\psi_{min}(\{\varphi_h : h \in \mathcal{P}\}) = \{\varphi(min, h) : h \in \mathcal{P}\}$. Since ψ_{min} is a continuous mapping and the set $cl_{CSF_c(G)}\{\varphi_h : h \in \mathcal{P}\}$ is compact, the set $cl_{C(G)}\{\varphi_{min,h)} : h \in \mathcal{P}\}$ is compact too. Hence the function φ_{min} is almost periodic. The proof that φ_{max} is almost periodic is similar.

Assertion 4. *For any two functions $f, g \in C(G)$ we put $\varphi_{(f,g)}(x) = [f(x), g(x)]$, if $f(x) \leq g(x)$, and $\varphi_{(f,g)}(x) = [g(x), f(x)]$ if $g(x) \leq f(x)$. Then:*

1. $\varphi_{(f,g)}$ is a compact-valued continuous function.

2.$\varphi : C(G) \times C(G) \to CF(G)$, where $\varphi(f, g) = \varphi_{(f,g)}$, is a continuous mapping.

3. If f, g are almost periodic functions, then $\varphi_{(f,g)}$ is almost periodic.

We can assume that $f(x) \leq g(x)$ for any $x \in G$. We put $\varphi_t(x) = tf(x) + (1-t)g(x)$. Then φ_t is a continuous function and $\varphi_{(f,g)}(x) = \{\varphi_t(x) : t \in [0, 1]\}$. This fact proved the assertion 1 and 2. Assume that f and g are almost periodic. Then there exists a compact subset Φ of $C(G)$ such that $\{f_h : h \in \mathcal{P}\} \cup \{g_h : h \in \mathcal{P}\} \subseteq\subseteq \Phi$. If $\psi = \varphi(f, g)$, the $\{\psi_h : h \in \mathcal{P}\} \subseteq \varphi(\Phi \times \times \Phi$. Assertion 4 is proved.

Assertion 5. *If φ is a compact-valued convex-valued function, then $\varphi = \varphi(\varphi_{min}, \varphi_{max}$.*

Proof. Is obvious.

Theorem follows from the above assertions.

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Convergence estimates for some abstract second order differential equations in Hilbert spaces

Andrei Perjan, Galina Rusu

Abstract

In a real Hilbert space H we consider the following perturbed Cauchy problem

$$\begin{cases} \varepsilon\, u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B(u_{\varepsilon\delta}(t)) = f(t), t \in (0,T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases} \tag{$P_{\varepsilon\delta}$}$$

where $u_0, u_1 \in H$, $f : [0,T] \mapsto H$ and ε, δ are two small parameters, A is a linear self-adjoint operator, B is a locally Lipschitz and monotone operator.

We study the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(P_{\varepsilon\delta})$ in two different cases:

(i) when $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$;

(ii) when $\varepsilon \to 0$ and $\delta \to 0$.

We establish that the solution to the unperturbed problem has a singular behavior, relative to the parameters, in the neighborhood of $t = 0$. We show the boundary layer and boundary layer function in both cases.

Keywords: Singular perturbation; abstract second order Cauchy problem; boundary layer function; a priori estimate.

Let H and V be two real Hilbert spaces endowed with norms $|\cdot|$ and $||\cdot||$, respectively. Denote by $(\cdot,\cdot)$ the scalar product in H.

The framework of our studying is determined by the following conditions:

(H) *$V \subset H$ densely and continuously, i.e.*

$$||u|| \geq \omega_0\, |u|, \quad \forall u \in V, \quad \omega_0 > 0.$$

(HA) *$A : D(A) = V \mapsto H$ is a linear, self-adjoint and positive definite operator, i.e.*

$$(Au, u) \geq \omega\, |u|^2, \quad \forall u \in V, \quad \omega > 0.$$

(HB1) *Operator $B : D(B) \subseteq H \to H$ is $A^{1/2}$ locally Lipschitz, i. e. $D(A^{1/2}) \subset D(B)$ and for every $R > 0$ there exists $L(R) \geq 0$ such that*

$$|B(u_1) - B(u_2)| \leq L(R)|A^{1/2}(u_1 - u_2)|, \forall u_i \in D(A^{1/2}), |A^{1/2}u_i| \leq R;$$

(HB2) *Operator B is the Fréchet derivative of some convex and positive functional $\mathcal{B}$ with $D(A^{1/2}) \subset D(\mathcal{B})$.*
(HB3) *Operator B possesses the Fréchet derivative B' in $D(A^{1/2})$ and there exists constant $L_1(R) \geq 0$ such that*

$$\big|\big(B'(u_1) - B'(u_2)\big)v\big| \leq L_1(R)\,\big|A^{1/2}(u_1 - u_2)\big|\,|A^{1/2}v|, \;\; \forall u_i,\, v \in D(A^{1/2}),$$

$$|A^{1/2}u_i| \leq R, \quad i = 1, 2.$$

The hypothesis **(HB2)** implies, in particular, that operator B is monotone and verifies condition

$$\frac{d}{dt}\mathcal{B}(u(t)) = \big(B(u(t)), u'(t)\big), \quad \forall t \in [a, b] \subset \mathbb{R},$$

in the case when $u \in C([a, b], D(A^{1/2})) \cap C^1([a, b], H)$.

Consider the following perturbed Cauchy problem

$$\begin{cases} \varepsilon\, u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B(u_{\varepsilon\delta}(t)) = f(t), t \in (0, T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases} \qquad (P_{\varepsilon\delta})$$

where $u_0, u_1 \in H$, $f : [0, T] \mapsto H$ and ε, δ are two small parameters.

We study the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(P_{\varepsilon\delta})$ in two different cases:
(i) $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$, relative to the following unperturbed system:

$$\begin{cases} \delta l'_\delta(t) + Al_\delta(t) + B(l_\delta(t)) = f(t), \quad t \in (0, T), \\ l_\delta(0) = u_0, \end{cases} \qquad (P_\delta)$$

(ii) $\varepsilon \to 0$ and $\delta \to 0$, relative to the following unperturbed problem:

$$Av(t) + B(v(t)) = f(t), \quad t \in [0, T). \tag{P_0}$$

The problem $(P_{\varepsilon\delta})$ is the abstract model of singularly perturbed problems of hyperbolic-parabolic type in the case (i) and of hyperbolic-parabolic-elliptic type in the case (ii). Such kind of problems arises in the mathematical modeling of elasto-plasticity phenomena. These abstract results can be applied to singularly perturbed problems of hyperbolic-parabolic-elliptic type with stationary part defined by strongly elliptic operators.

For the case $\delta \geq \delta_0 > 0$, in conditions **(H)**, **(HA)**, **(HB1)**, **(HB2)** and **(HB3)**, using results obtained in [1], we obtain some *a priori* estimates of solutions to the perturbed problem, which are uniform with respect to parameters, and get the relationship between the solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ) emphasised in the following inequalities

$$||u_{\varepsilon\delta} - l_\delta||_{C([0,T];H)} + ||A^{1/2}u_{\varepsilon\delta} - A^{1/2}l_\delta||_{L^2(0,T;H)} \leq C\,\varepsilon^\beta,$$

$$||u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta}e^{-\delta^2 t/\varepsilon}||_{C([0,\,T]\,;H)} + \big|\big|A^{1/2}\big(u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta}e^{-\delta^2 t/\varepsilon}\big)\big|\big|_{L^2(0,T;H)} \leq$$

$$\leq C\,\varepsilon^\beta, \forall \varepsilon \in (0, \varepsilon_0], T > 0, p > 1,$$

where $u_{\varepsilon\delta}$ and l_δ are strong solutions to problems $(P_{\varepsilon\delta})$ and (P_δ) respectively, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega, T, \delta_0, u_0, u_1, f) \in (0,1)$, $\beta = \min\{1/4,\ (p-1)/2p\}$, $C = C(T, p, \delta_0, \omega_0, \omega, u_0, u_1, H_{\varepsilon\delta}, f) > 0$,

$$H_{\varepsilon\delta} = \delta^{-1}f(0) - u_1 - \delta^{-1}Au_0 - \delta^{-1}B(u_0).$$

For the case $\varepsilon \to 0, \delta \to 0$, in conditions **(H)**, **(HA)**, **(HB1)** and **(HB2)**, in [2] we established the relationship between the solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ) emphasised in the following inequality

$$||u_{\varepsilon\delta} - v - h_\delta||_{C([0,T];H)} \leq C\,\Theta(\varepsilon, \delta),\ \forall \varepsilon \in (0, \varepsilon_0],\ \forall \delta \in (0, 1],$$

where $u_{\varepsilon\delta}$ and v are strong solutions to problems $(P_{\varepsilon\delta})$ and (P_0) respectively, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega, T, \delta_0, u_0, u_1, f) \in (0,1)$, $\beta = \min\{1/4,\ (p-1)/2p\}$,

$C = C(T, p, \omega_0, \omega, u_0, u_1, f) > 0$, the function h_δ is the solution to the problem

$$\begin{cases} \delta h'_\delta(t) + A h_\delta(t) + B(l_\delta(t)) - B\big(v(t)\big) = 0, \quad t \in (0,T), \\ h_\delta(0) = u_0 - (A+B)^{-1} f(0), \end{cases}$$

$$\Theta(\varepsilon, \delta) = \frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}} + \sqrt{\delta}.$$

From the last three inequalities we can state that the solution to the unperturbed problem has a singular behavior, relative to the parameters, in the neighborhood of $t = 0$ in both cases.

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On the structure of some LCA groups with commutative π-regular ring of continuous endomorphisms

Valeriu Popa

Abstract

We determine here, for some interesting classes $\mathcal{S}$ of LCA groups, the structure of those groups in $\mathcal{S}$ whose ring of continuous endomorphisms is commutative and π-regular.

Keywords: LCA groups, rings of continuous endomorphisms, commutative rings, π-regular rings.

Let $\mathcal{L}$ be the class of all LCA groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of all continuous endomorphisms of X. One may ask:

For which groups $X \in \mathcal{L}$, the ring $E(X)$ is commutative and π-regular?

In the following, we answer this question for different types of groups in $\mathcal{L}$. Let us fix some notations. Given $X \in \mathcal{L}$, we set $X_\omega = \cap_{n>1}\overline{nX}$ and denote by $t(X)$ the torsion subgroup of X. If $(X_i)_{i\in I}$ is a family of groups in $\mathcal{L}$ and, for each $i \in I$, U_i is a compact open subgroup of X_i, we let $\prod_{i\in I}(X_i; U_i)$ denote the local direct product of the groups X_i with respect to the subgroups U_i. Further, given a prime number p and a positive integer n, we denote by $\mathbb{Z}(p^n)$ the discrete group of integers modulo p^n, by $\mathbb{Z}_p$ the group of p-adic integers, and by $\mathbb{Q}_p$ the group of p-adic numbers, both taken with their usual topologies. We also use the discrete group of rationals $\mathbb{Q}$, the character group $\mathbb{Q}^*$ of $\mathbb{Q}$ and the reals $\mathbb{R}$, both with their usual topologies.

Definition 1. A group $X \in \mathcal{L}$ is called topologically completely decomposable if X is topologically isomorphic to a group of the form $\prod_{i\in I}(X_i; U_i)$, where, for each $i \in I$, X_i is a group in $\mathcal{L}$ of special rank one and U_i is a compact open subgroup of X_i.

Definition 2. Let P_1 and P_2 be properties of LCA groups. A group $X \in \mathcal{L}$ is said to be P_1-by-P_2 if X has a closed subgroup C satisfying P_1 such that the quotient group X/C satisfies P_2.

Theorem 1. Let $X \in \mathcal{L}$ be torsion-free and topologically completely decomposable-by-bounded order. The following statements are equivalent:

(i) $E(X)$ is commutative and π-regular.

(ii) X is topologically isomorphic with one of the groups: $\mathbb{Q}^*$, $\mathbb{Q}^{(\alpha)}$, or $\prod_{p\in S} \mathbb{Q}_p; \mathbb{Z}_p\big)$, where α is a cardinal number and S is a set of prime numbers.

Theorem 2. Let $X \in \mathcal{L}$ be densely divisible and bounded order-by-topologically completely decomposable. The following statements are equivalent:

(i) $E(X)$ is commutative and π-regular.

(ii) X is topologically isomorphic with one of the groups: $\mathbb{Q}$, $(\mathbb{Q}^*)^\alpha$, or $\prod_{p\in S} \mathbb{Q}_p; \mathbb{Z}_p\big)$, where α is a cardinal number and S is a set of prime numbers.

Theorem 3. Let X be a mixed, residual group in $\mathcal{L}$ such that $t(X)$ splits topologically from X. The following statements are equivalent:

(i) $E(X)$ is commutative and π-regular.

(ii) X is topologically isomorphic with $\bigoplus_{p\in S_1} \mathbb{Z}(p^{n_p})\times\prod_{p\in S_2} \mathbb{Q}_p;\mathbb{Z}_p\big)$, where S_1 and S_2 are sets of prime numbers and the n_p's are natural numbers.

Definition 3. A group $X \in \mathcal{L}$ is called

(i) comixed if $\{0\} \neq X_\omega \neq X$.

(ii) residual if the maximal divisible subgroup of X is dense in X and the maximal divisible subgroup of X^*, the character group of X, is dense in X^*

(iii) nonresidual if X is not a residual group.

Theorem 4. Let X be a comixed, residual group in $\mathcal{L}$ such that X/X_ω is compact-by-bounded order. The following statements are equivalent:

(i) X_ω splits topologically from X and $E(X)$ is commutative and π-regular.

(ii) X is topologically isomorphic with $\prod_{p\in S_1} \mathbb{Z}(p^{n_p})\times\prod_{p\in S_2} \mathbb{Q}_p;\mathbb{Z}_p\big)$, where S_1 and S_2 are sets of prime numbers and the n_p's are natural numbers.

Theorem 5. Let X be a mixed, nonresidual group in $\mathcal{L}$ such that $t(X)$ splits topologically from X. The following statements are equivalent:

(i) $E(X)$ is commutative and π-regular.

(ii) X is topologically isomorphic with one of the groups:

$$\mathbb{R}\times\bigoplus_{p\in S_1}\mathbb{Z}(p^{n_p})\times\prod_{p\in S_2}\mathbb{Q}_p;\mathbb{Z}_p\big),$$

$$\mathbb{Q}\times\bigoplus_{p\in S}\mathbb{Z}(p^{n_p}),\quad\text{or}\quad\mathbb{Q}^*\times\bigoplus_{p\in S}\mathbb{Z}(p^{n_p}),$$

where S_1, S_2 and S are sets of prime numbers and the n_p's are natural numbers.

Theorem 6. Let X be a comixed, nonresidual group in $\mathcal{L}$ such that X/X_ω is compact-by-bounded order. The following statements are equivalent:

(i) X_ω splits topologically from X and $E(X)$ is commutative and π-regular.

(ii) X is topologically isomorphic with one of the groups:

$$\mathbb{R} \times \prod_{p \in S_1} \mathbb{Z}(p^{n_p}) \times \prod_{p \in S_2} \mathbb{Q}_p; \mathbb{Z}_p\big),$$

$$\mathbb{Q} \times \prod_{p \in S} \mathbb{Z}(p^{n_p}), \quad \text{or} \quad \mathbb{Q}^* \times \prod_{p \in S} \mathbb{Z}(p^{n_p}),$$

where S_1, S_2 and S are sets of prime numbers and the n_p's are natural numbers.

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Classification of cubic systems with invariant straight lines along one direction of total multiplicity at least seven

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Abstract

The Darboux integrability of a polynomial differential system depends on the multiplicities of its invariant curves. We show that there are 34 canonical forms of cubic systems which have real or complex invariant straight lines of total multiplicity seven along one direction including the straight line at the infinity.

Keywords: invariant straight lines, canonical forms.

1 Introduction

Let's consider a differential system

$$\dot{x} = P(x;y), \;\; \dot{y} = Q(x;y), \tag{1}$$

and its associated vector field $\mathbb{X} = P(x;y)\frac{\partial}{\partial x} + Q(x;y)\frac{\partial}{\partial y}$, where $P;Q \in \mathbb{R}[x,y]$, $\max\{\deg P, \deg Q\} = 3$ and $GCD(P,Q) = 1$.

Definition 1. *An algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ is called invariant algebraic curve for the system (1), if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, such that the following identity holds:*

$$\mathbb{X}(F) = f(x,y)K_f(x,y).$$

According to [1], it is possible to calculate a Darboux first integral for this system, if this system has sufficiently many invariant straight lines considered with their multiplicities. Therefore, it is of high interest to do investigations of polynomial differential systems with invariant straight lines. These studies are done using different types of multiplicities of the invariant straight lines, for example: parallel multiplicity, geometric multiplicity; algebraic multiplicity etc [2]. In this work we will use the notion of algebraic multiplicity of an invariant straight line. In [3] we showed that there are exactly 26 canonical forms of cubic differential systems which possess real invariant straight lines of total multiplicity at least seven (including the invariant straight line at the infinity) along one direction. Considering complex invariant straight lines we'll expand these result. And we'll show that there are 8 more canonical forms.

2 Obtaining the canonical forms

If the system (1) has an invariant straight lines $\alpha x + \beta y + \gamma = 0$, then we can bring this straight line to the form $\overline{x} = 0$ using the affine transformation $\overline{x} = \alpha x + \beta y + \gamma$, $\overline{y} = y$. The invariant straight line $f = 0$ has the algebraic multiplicity k, if k is the greatest positive integer such that f^k divides $\det M_r \equiv P\mathbb{X}(Q) - Q\mathbb{X}(P)$. To study the multiplicity of the invariant straight line at the infinity we carry out the Poincaré transformation $x = 1/\overline{x}$, $y = \overline{y}/\overline{x}$. The multiplicity of the invariant straight line at the infinity is equal with multiplicity of the invariant straight line $\overline{x} = 0$ of the system $\dot{\overline{x}} = \overline{y}\,\overline{x}^3 P\left(\frac{1}{\overline{x}, \frac{\overline{y}}{\overline{x}}}\right) - \overline{x}^3 Q\left(\frac{1}{\overline{x}}, \frac{\overline{y}}{\overline{x}}\right)$, $\dot{\overline{y}} = \overline{x}^4 P\left(\frac{1}{\overline{x}}, \frac{\overline{y}}{\overline{x}}\right)$.

We emphasize that the computations in determining of the canonical forms were quite large, therefore we'll describe how a single canonical form can be obtained. Let's note by $(d_1(m_1)r + d_2(m_2)i + d_3r) + I(M)$ a configuration of invariant straight lines, where d_i is the number of straight lines, m_i – their corresponding multiplicities, r and i suffix shows if the straight lines are real or complex and M is the multiplicity of the invariant straight line at the infinity. If $m_i = 1$, then we will not write it. For example, the notation (1(4)r+2i)+I(2) indicates that

there are three parallel invariant straight lines, where one of them is real with the multiplicity equal to four, the other two are complex with their multiplicities equal to one and the straight line at the infinity has multiplicity equal to 2. The configurations of the invariant straight lines, of which at least one is complex, are the following:
1) $(1(4)r + 2i) + I(1)$; 2) $(1(2)r + 1(2)i + 1(2)i) + I(1)$;
3) $(1(3)r + 2i) + I(2)$; 4) $(1r + 1(2)i + 1(2)i) + I(2)$;
5) $(1(2)r + 2i) + I(3)$; 6) $(1(2)i + 1(2)i) + I(3)$;
7) $(1r + 2i) + I(4)$; 8) $(2i) + I(5)$.
1) Asking for the system (1) that the straight line $x = 0$ to be invariant with the multiplicity equal to 4 system, we obtain the following 4 systems:

$$\dot{x} = a_{30}x^3, \qquad \dot{y} = b_{00} + b_{10}x + b_{20}x^2 + b_{11}xy + b_{30}x^3 + (b_{21} + b_{12})x^2y;$$

$$\dot{x} = x^2(a_{20} + a_{30}x), \quad \dot{y} = b_{00} + b_{10}x + b_{20}x^2 + 2a_{20}xy + b_{30}x^3 + b_{21}x^2y;$$

$$\begin{cases} \dot{x} = x^2\left(a_{20} + \dfrac{2a_{20}^2 + a_{21}b_{10} - a_{20}b_{11}}{b_{01}}x + a_{21}y\right), \\ \dot{y} = \dfrac{a_{20}b_{01}}{a_{21}} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{30}x^3 + b_{21}x^2y + 2{a_{21}}^2xy^2; \end{cases}$$

$$\begin{cases} \dot{x} = x\left(a_{10} + \left(b_{11} - \dfrac{b_{00}b_{12}}{a_{10}}\right)x + b_{21}x^2 + b_{12}xy\right), \\ \dot{y} = b_{00} + \dfrac{a_{10}b_{00}b_{11} - {b_{00}}^2b_{12}}{{a_{10}}^2}x + a_{10}y + \dfrac{b_{00}b_{21}}{a_{10}}x^2 + b_{11}xy + b_{30}x^3 + \\ \qquad + b_{21}x^2y + b_{12}xy^2; \end{cases}$$

From these, only the last system can have the algebraic curve $f \equiv x^2 + 1 = 0$ (the complex straight lines $x = \pm i$) as an invariant curve. After obeying the condition that $x = \pm i$ are invariant, this system has the form $\dot{x} = a_{10}x(x^2 + 1)$, $\dot{y} = b_{00} + a_{10}y + b_{00}x^2 + b_{30}x^3 + a_{10}x^2y$, where b_{00}, a_{10}, b_{30} $(b_{30} \cdot a_{10} \neq 0)$ are some coefficients of the system. Using the transformation $x \to x$, $y \to \frac{a_{10}}{b_{30}}y - \frac{b_{00}a_{10}}{3}$, $t \to a_{10}\tau$, we obtain the system (c1) from Theorem 1.

3 Conclusion

Theorem 1. *Any cubic differential system with real or complex invariant straight lines along one direction with total algebraic multiplicity equal to at least 7 including the invariant straight line at the infinity, by an affine transformation and time rescaling can be brought to one of 34 canonical forms. If the invariant straight lines are real, then the corresponding systems are given in [3] and if any of them are complex, then the corresponding systems have one of the following 8 forms:*

$$(c1)\begin{cases}\dot{x}=x(x^2+1),\\ \dot{y}=y+x^3+x^2y.\end{cases} \qquad (c2)\begin{cases}\dot{x}=x(x^2+1), a\in\mathbb{R}\\ \dot{y}=a+x+y+3x^2y.\end{cases}$$

$$(c3)\begin{cases}\dot{x}=x(x^2+1), a\in\mathbb{R}\\ \dot{y}=y+ax^2+x^3.\end{cases} \qquad (c4)\begin{cases}\dot{x}=x(x^2+1), a\in\mathbb{R}\\ \dot{y}=ax-2y+bx^2+x^3.\end{cases}$$

$$(c5)\begin{cases}\dot{x}=x(x^2+1), a\in\mathbb{R}\\ \dot{y}=ax+y+x^3.\end{cases} \qquad (c6)\begin{cases}\dot{x}=x^2+1, a\in\mathbb{R}\\ \dot{y}=a+2xy+x^3.\end{cases}$$

$$(c7)\begin{cases}\dot{x}=x(x^2+1), a\in\mathbb{R}\\ \dot{y}=-2y+ax^3.\end{cases} \qquad (c8)\begin{cases}\dot{x}=x(x^2+1),\\ \dot{y}=1.\end{cases}$$

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On completeness as to ¬-expressibility in a 4-valued provability logic

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Abstract

The simplest non-trivial 4-valued extension $GL4$ of the propositional provability logic GL is considered together with the notion of ¬-expressibility of formulas in GL. The necessary and sufficient conditions when a system of formulas of $GL4$ is complete relative to ¬-expressibility are found out which is formulated in terms of classes formulas that are pre-complete relative to ¬-expressibility.

Keywords: provability logic, expressibility of formulas, completeness relative to ¬-expressibility.

1 Introduction

We consider the simplest 4-valued non-trivial extension of the propositional provability logic GL [1]. A light modification of the notion of expressibility of formulas considered by A.V. Kuznetsov [2] is investigated. The results obtained are similar to the well-known result of E. Post concerning Boolean functions [3].

2 Preliminaries

2.1 Propositional provability logic GL

The propositional provability logic GL is based on propositional variables, logical connectives $\&, \vee, \supset, \neg, \Delta$ [1]. Axioms of GL are the usual axioms of the classical propositional logic together with the following $((\Delta(p \supset q)) \supset (\Delta p \supset \Delta q))$, $\Delta p \supset \Delta\Delta p$, $(\Delta(\Delta p \supset p) \supset \Delta p)$. Rules of inference of the logic L are:

1. *substitution rule* allowing to pass from any formula F to the result of substitution in F instead of a variable p of F by any formula G (denoted by $F(p/G)$, or by $F(G)$),

2. *modus ponens rule* which permits to pass from two formulas F and $F \supset G$ to the formula G, and

3. *rule of necessity* admitting to go from formula F to ΔF.

As usual, by the logic of the calculus GL we understand the set of all formulas that can be deduced in the claculus GL and it is closed with respect to its rules of inference, and we call it the propositional provability logic of Gödel-Löb, denoted also by GL. They say logic L_2 is an extension of the logic L_1 if $L_1 \subseteq L_2$ (as sets).

2.2 Diagonalizable algebras

A diagonalizable algebra [4] $\mathfrak{D}$ is a boolean algebra $\mathfrak{A} = (A; \&, \vee, \supset, \neg, \mathbb{0}, \mathbb{1})$ with an additional operation Δ satisfying the relations:

$$\begin{array}{rcl} \Delta(x \supset y) & = & (\Delta x \supset \Delta y), \\ \Delta x & = & \Delta\Delta x, \\ \Delta(\Delta x \Delta x) & = & \Delta x, \\ \Delta\mathbb{1} & = & \mathbb{1}, \end{array}$$

where $\mathbb{1}$ is the unit of $\mathfrak{A}$. The diagonalizable algebras serve as algebraic models for propositional provability logic GL [1].

It is known [5, 6] that diagonalizable algebras can serve as algebraic models for propositional provability logic. Interpreting logical connectives of a formula F by corresponding operations on a diagonalizable algebra $\mathfrak{D}$ we can evaluate any formula of GL on any algebra $\mathfrak{D}$. If for any evaluation of variables of F by elements of $\mathfrak{D}$ the resulting value of the formula F on $\mathfrak{D}$ is $\mathbb{1}$, they say F *is valid on* $\mathfrak{D}$. The set of all valid formulas on the given diagonalizable algebra $\mathfrak{D}$ is an extension of GL [7], denoted by $L\mathfrak{D}$. So, if diagonalizable algebra $\mathfrak{D}_2$ is a subalgebra of the diagonalizable algebra $\mathfrak{D}_1$, then $L\mathfrak{D}_1 \subseteq L\mathfrak{D}_2$, i.e. $L\mathfrak{D}_2$ is an extension of $L\mathfrak{D}_1$.

Consider the 4-valued diagonalizable algebra $\mathfrak{B}_2 = (\{\mathbb{0}, \rho, \sigma, \mathbb{1}\}; \&, \vee, \supset, \neg, \Delta)$, where $\Delta\mathbb{0} = \Delta\rho = \sigma, \Delta\sigma = \Delta\mathbb{1} = \mathbb{1}$ and its corresponding 4-valued provability logic $L\mathfrak{B}_2$.

2.3 ¬-expressibility of formulas

They say formula F of the logic L is ¬-expressible (see for example de definition of expressibility in [2]) via a system of formulas Σ in L if F can be obtained from variables and $\Sigma \cup \{\neg p\}$ using finitely many times any of the two rules: weak rule of substitution allowing to pass from any formulas A and B to the result of substitution of any of them in other one instead of any variable, and rule of replacement by equivalent formula (if formulas A and $A \sim B$ are given, then we have also B).

3 Main result

Theorem 1. *There is a relative simple algorithm which allows to determine whether a system of formulas Σ is complete relative to ¬-expressibility of formulas in the provability logic $L\mathfrak{B}_2$.*

4 Conclusion

We can also examine other types of expressibility for formulas, such as: parametric expressibility, existential expressibility, weak expressibility. The case of explicit expressibility was examined earlier in [8].

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About orthogonality of multiary operations

Fedir Sokhatsky

Abstract

In this article orthogonality of multiary quasigroups and Latin hypercubes are under consideration. In particular, criteria of orthogonality of n-ary operations are systematized and a criterion for an operation in a set of orthogonal operations to be invertible is found. Corollaries for ternary case are given.

Keywords: n-ary quasigroup, Latin hypercubes, orthogonal quasigroups, orthogonal n-ary operations.

1 Introduction

The notions of Latin square and its generalization Latin hypercube and also their orthogonal sets are well-known and applicable in various areas including orthogonal and projective geometries, cryptology, functional equations. In this article, we continue their investigation (for example, see [1]–[4]).

2 Preliminaries

An n-ary operation f defined on a set Q is called *invertible* if there are *inverses* ${}^{[i]}f$ of f for every $i = 1, \ldots, n$:

$$ {}^{[i]}f(x_1, \ldots, x_n) = x_{n+1} :\Leftrightarrow f(x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n) = x_i, \tag{1} $$

This is a partial case of a parastrophe ${}^{\sigma}f$ of an invertible operation f:

$$ {}^{\sigma}f(x_1, \ldots, x_n) = x_{n+1} :\Leftrightarrow f(x_{1\sigma}, \ldots, x_{(n)\sigma}) = x_{(n+1)\sigma}, \quad \sigma \in S_{n+1}. \tag{2} $$

The algebra $(Q; f, {}^{[1]}f, \ldots, {}^{[n]}f)$ is called a *quasigroup*.

3 Equivalent definitions of orthogonality

$\alpha : A \to B$ is *complete*, if all pre-images have the same cardinality.

A k-tuple $(f_1, \ldots, f_k)$ of operations defined on Q $(m := |Q| < \infty)$ is called *orthogonal*, if for all $a_1, \ldots, a_k$ in Q the system

$$\begin{cases} f_1(x_1, \ldots, x_n) = a_1, \\ \dots\dots\dots\dots\dots\dots \\ f_k(x_1, \ldots, x_n) = a_k \end{cases} \tag{3}$$

has exactly m^{n-k} solutions. A k-tuple $(f_1, \ldots, f_k)$ of operations is called *embeddable* into an m-tuple $(g_1, \ldots, g_m)$ of operations, if each of the operations $f_1, \ldots, f_k$ is an entry in $(g_1, \ldots, g_m)$, i.e., $g_{i_1} = f_1, \ldots, g_{i_k} = f_k$, for some $i_1, \ldots i_k \in \{1, \ldots, m\}$.

A mapping f from Q^n in Q^k is called a *multioperation* of the *arity* n and the *rank* k or (n, k)-*multioperation*. Every (n, k)-multiopertion f uniquely defines and is uniquely defined by a k-tuple $(f_1, \ldots, f_k)$:

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)).$$

For briefly, $f = (f_1, \ldots, f_k)$. The tuple is *coordinates* of f. Therefore,

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)).$$

Some multioperations are complete. For example, the multioperation

$$\iota_{1,\ldots,k} := (\iota_1, \ldots, \iota_k), \qquad \iota_{1,\ldots,k}(x_1, \ldots, x_n) := (x_1, \ldots, x_k)$$

is complete because preimage of each tuple $(a_1, \ldots, a_k)$ is

$$\iota^{-1}_{1,\ldots,k}(a_1, \ldots, a_k) = \{(a_1, \ldots, a_k, x_{k+1}, \ldots, x_n) \mid x_{k+1}, \ldots, x_n \in Q\}$$

and it has m^{n-k} elements.

Theorem 1. *Let $f = (f_1, \ldots, f_k)$ be an (n, k)-multioperation defined on a finite set Q $(m := |Q|)$ and let $k < n$, then the following assertions are equivalent:*

1. *the multioperation f is complete;*

2. *each preimage under f has m^{n-k} elements;*

3. *the tuple $(f_1, \ldots, f_k)$ of n-ary operations is orthogonal;*

4. *there exists a bijection $\theta : Q^n \to Q^n$ such that $f = \iota_{1,\ldots,k}\theta$;*

5. *the tuple $(f_1, \ldots, f_k)$ of n-ary operations is embeddable into an orthogonal n-tuple of n-ary operations.*

4 Orthogonality of hypercubes

A *hypercube* or *cube* of *dimension* m^n over a set Q ($|Q| = m$) is a table of the dimension m^n whose each cell contains an element from Q called an *entry*. A table of results (i.e., Cayley table) of an n-ary operation f defined on Q is a cube of the dimension m^n with entries from the set Q. The cube is called *Latin* if all entries in each line are pairwise different. Cayley table of a function is Latin if and only if the function is invertible.

Let $C_1, \ldots, C_n$ be n-ary cubes defined over the same set Q. Let us superimpose all of them. As a result, we obtain a cube $C_{1,\ldots,n}$ such that each its cell contains one n-tuple of elements from Q. If all of the tuples are pairwise different, the cubes $C_1, \ldots, C_n$ are called *orthogonal*. It is easy to verify that cubes are orthogonal iff the corresponding functions are orthogonal.

The following question is natural: *When $n-1$ n-ary cubes have a Latin compliment?*

The set of all cells taken exactly one from each line of an n-ary table is called its $(n-1)$-*ary diagonal*.

Lemma 1. *A set d of cells of an n-ary table is its diagonal if and only if there exists an $(n-1)$-ary invertible operation g such that*

$$d = \{(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1})) \mid x_1, \ldots, x_{n-1} \in Q\}. \qquad (4)$$

A *diagonal partition* of a table will be called its partition whose blocks are diagonals of the table. A *natural partition* of a cube will be called its partition whose blocks are sets of cells containing the same element. An $(n-1)$-ary diagonal d of n-ary cubes $C_1, \ldots, C_{n-1}$ will be called their *transversal*, if sub-cubes of these cubes defined by d are orthogonal. A *transversal partition* of $n-1$ n-ary cubes of the same order will be called their diagonal partition, if each block is a transversal of the cubes.

Theorem 2. *n-ary cubes $C_1, \ldots, C_{n-1}$ of the same dimension have a Latin compliment iff they have a transversal partition.*

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Center problem and classification of cubic differential systems with the line at infinity of multiplicity three and an invariant affine straight line of multiplicity two

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Abstract

In this work, the cubic differential systems with a nondegenerate monodronomic critical point, and with the line at infinity of multiplicity three and an invariant affine straight line $\alpha x + \beta y + \gamma = 0$ of multiplicity two are classified. For these systems the problem of the center is solved.

Keywords: cubic differential system, the problem of the center, multiple invariant straight line.

We consider the real cubic system of differential equations

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv p(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv q(x,y), \\ \gcd(p,q) = 1,\ sx^4 + (k+q)x^3y + (m+n)x^2y^2 + (l+p)xy^3 + ry^4 \not\equiv 0. \end{cases} \quad (1)$$

The critical point $(0,0)$ of system (1) is either a focus or a center, i.e. is monodromic. The problem of distinguishing between a center and a focus is called *the problem of the center*.

The straight line $\alpha x + \beta y + \gamma = 0$, $\alpha, \beta, \gamma \in \mathbb{C}$ is called *invariant* for (1) if there exists a polynomial $K \in \mathbb{C}[x,y]$ such that the identity $\alpha p(x,y) + \beta q(x,y) \equiv (\alpha x + \beta y + \gamma) K(x,y)$, $(x,y) \in \mathbb{R}^2$ holds.

The homogeneous system associated to the system (1) has the form

$$\begin{cases} \dot{x} = yZ^2 + (ax^2 + cxy + fy^2)Z + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x, y, Z), \\ \dot{y} = -(xZ^2 + (gx^2 + dxy + by^2)Z + sx^3 + qx^2y + nxy^2 + ly^3) \\ \qquad \equiv Q(x, y, Z). \end{cases}$$

Denote $\mathbb{X} = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}$ and $\mathbb{X}_\infty = P(x, y, Z)\frac{\partial}{\partial x} + Q(x, y, Z)\frac{\partial}{\partial y}$.

We say that the invariant straight line $\alpha x + \beta y + \gamma = 0$ (respectively, the line at infinity $Z = 0$) has *multiplicity* ν (respectively, $\nu + 1$) if ν is the greatest positive integer such that $(\alpha x + \beta y + \gamma)^\nu$ (respectively, Z^ν) divides $E = p \cdot \mathbb{X}(q) - q \cdot \mathbb{X}(p)$ (respectively, $E_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$) [1].

Theorem 1. *The system* (1) *has an invariant affine real straight line* $L = 0$ *of multiplicity two and the line at infinity* $Z = 0$ *of multiplicity three if and only if one of the following ten sets of conditions holds:*

$$\begin{array}{l} a = c = f = k = l = m = p = r = 0,\ s = b^2(1 + b^2B^2), \\ g = -b(2 + b^2B^2),\ n = (1 + b^2B^2)/B^2,\ q = 2b(1 + b^2B^2)/B, \\ d = -2/B. \text{ The invariant straight line is } L = bBx + y - B; \end{array} \tag{2}$$

$$\begin{array}{l} a = c = f = k = m = p = r = 0,\ d = -(b^4 - 16l^2)/(4bl), \\ g = -b,\ n = -(b^4 - 8l^2)/(2b^2),\ q = (b^4 - 32l^2)/(16l),\ s = b^2/4. \\ L = b^2x - 4ly - 2b; \end{array} \tag{3}$$

$$\begin{array}{l} a = (\pm 2q^2s \pm 4s^3 + gq\delta)/(2s\delta), \\ b = q(\pm 2q^2s \pm 8s^3 + gq\delta)/(4s^2\delta), \\ c = q(\pm 2q^2s \pm 4s^3 + gq\delta)/(2s^2\delta), \\ d = (\pm q^2s \pm 4s^3 + gq\delta)/(s\delta), \\ f = q^2(\pm 2q^2s \pm 4s^3 + gq\delta)/(8s^3\delta),\ n = q^2/(4s), \\ k = l = m = p = r = 0,\ \delta = \sqrt{s(q^2 + 4s^2)}. \\ L = 2qsx + q^2y \mp 2\delta; \end{array} \tag{4}$$

$$\begin{array}{l} a = b = c = 0,\ d = -2/\gamma,\ n = 1/\gamma^2, \\ g = k = l = m = p = q = r = s = 0. \\ L = y - \gamma; \end{array} \tag{5}$$

$$a = f = k = l = m = n = p = r = s = 0,\ b = -c,\ g = c,\ q \neq 0. \\ L = cx + 1; \tag{6}$$

$$a = 0,\ b = -p\gamma,\ c = -2p\gamma,\ d = -2/\gamma,\ g = k = 0,\ n = 1/\gamma^2, \\ l = p,\ m = q = 0,\ r = p^2\gamma^2,\ s = 0. \\ L = y - \gamma; \tag{7}$$

$$a = b = 0,\ c = -2/B,\ d = -2Bq,\ f = g = 0, \\ k = l = 0,\ m = 1/B^2,\ n = p = r = s = 0,\ q \neq 0. \\ L = x - B; \tag{8}$$

$$a = 2Br,\ b = 2A(B^2r - 1)/B,\ c = 2(A^2 - 1)Br/A, \\ d = 2(1 - A^2)(B^2r - 1)/B,\ f = -2Br,\ m = (A^2 - 2)r, \\ g = 2A(1 - B^2r)/B,\ k = Ar,\ l = A(1 - B^2r)/B^2, \\ n = 2A^2 - 1)(B^2r - 1)/B^2,\ p = (1 - 2A^2)r/A, \\ q = A(2 - A^2)(B^2r - 1)/B^2,\ s = A^2(1 - B^2r)/B^2. \\ L = Ax - y + B; \tag{9}$$

$$a = -Au(ABu + A^3Bu \pm A^2 \pm 2), \\ b = (ABu \pm 1)(Bu + A^2Bu \mp A)/B, \\ c = 2u(ABu + A^3Bu \pm 1), \\ d = -2(ABu \pm 1)(ABu + A^3Bu \pm 1)/B, \\ f = -u(Bu + A^2Bu \mp A), \\ g = A(ABu \pm 1)(ABu + A^3Bu \pm 2 \pm A^2)/B, \\ k = -A^2(1 + A^2)u(ABu \pm 1)/B, \\ l = -(1 + A^2)u(ABu \pm 1)/B, \\ m = A(1 + A^2)u(3ABu \pm 2)/B, \\ n = (1 + A^2)(ABu \pm 1)(3ABu \pm 1)/B^2, \\ p = -(1 + A^2)u(3ABu \pm 1)/B,\ r = u^2(1 + A^2), \\ q = -A(1 + A^2)(ABu \pm 1)(3ABu \pm 2)/B^2, \\ s = A^2(1 + A^2)(ABu \pm 1)^2/B^2. \\ L = Ax - y + B; \tag{10}$$

$$b = -g = s(1 + Bf)/(Bk),\ d = (Bfs^2 - 2k^2 - Bfk^2)/(Bk^2), \\ c = (Bfs^2 - k^2 - Bfk^2 - s^2)/(Bks),\ m = (2k^2 - s^2)/s, \\ a = -f,\ l = -k,\ n = (k^2 - 2s^2)/s,\ p = k(k^2 - 2s^2)/s^2, \\ q = (2k^2 - s^2)/k,\ r = -k^2/s. \\ L = sx + ky - Bk; \tag{11}$$

$$
\begin{aligned}
&b = (Ak - aABk - A^2s - aBs)/(AB(Ak + s)),\\
&c = 2(Ak + A^3k - aABk - aBs)/(AB(Ak + s)),\\
&d{=}2(aB - 1)/B,\ p = k(s - 2Ak)/(A^2s),\\
&f{=}(aABk{+}aBs{-}2Ak{-}2A^3k)/(A^2B(Ak{+}s)),\\
&g{=}(A^2{-}aA^2B{+}AB^2k{+}B^2s)/(AB),\ l{=}k/A^2,\\
&m{=}k(Ak{-}2s)/(As), n = (s - 2Ak)/A^2,\\
&q = (Ak - 2s)/A,\ r = k^2/(A^2s),\\
&A^2B^2k^2 - A^2s - A^4s + 2AB^2ks + B^2s^2 = 0.\\
&L = Ax - y + B.
\end{aligned}
\tag{12}
$$

Theorem 2. *Any cubic system* (1)*, with the line at infinity of multiplicity three and an invariant affine real straight line of multiplicity two, has a center at the origin* $(0,0)$ *if and only if the first Lyapunov quantity vanishes.*

Theorem 3. *The cubic system* (1)*, with the line at infinity of multiplicity three and an invariant affine real straight line* $\alpha x + \beta y + \gamma = 0$ *of multiplicity two, has a center at the origin* $(0,0)$ *if and only if the system has the integrating factor* $1/(\alpha x + \beta y + \gamma)^2$.

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On total inner mapping groups of middle Bol loops

Parascovia Syrbu, Ion Grecu

Abstract

We consider total inner mapping groups of middle Bol loops in the present work. In particular, it is shown that the usual inner mapping group of a middle Bol loop is a normal subgroup of the total inner mapping group while the right Bol loops do not have this property.

Keywords: middle Bol loop, total multiplication group, total inner mapping group.

1 Introduction

Let $(Q, \cdot)$ be a quasigroup and $x \in Q$. The left, right and middle translations by x, denoted by L_x, R_x, D_x, respectively, are mappings from the symmetric group S_Q, defined as follows: $L_x(y) = x \cdot y$, $R_x(y) = y \cdot x$, $D_x(y) = y \backslash x$, $\forall x, y \in Q$. The groups $Mlt(Q, \cdot) = \langle L_x, R_x | x \in Q \rangle$ and $TMlt(Q, \cdot) = \langle L_x, R_x, D_x | x \in Q \rangle$ are called the multiplication group and the total multiplication group of $(Q, \cdot)$, respectively. If $(Q, \cdot)$ is a loop, then its inner mapping group (the stabilizer of its unit in $Mlt(Q)$) is denoted by $Inn(Q)$ and the stabilizer of the unit in $TMlt(Q)$) is called the total inner mapping group of $(Q, \cdot)$ and is denoted by $TInn(Q)$.

The total multiplication groups and total inner mapping groups have been introduced by Belousov at the end of 60's [1]. Total inner mapping groups of middle Bol loops are considered in the present work. In particular, it is shown that the inner mapping group of a middle Bol loop is a normal subgroup of the total inner mapping group.

2 Inner mappings of middle Bol loops

Let $(Q,\cdot)$ be a loop, $x,y\in Q$, and let consider the mappings: $R_{x,y}=R_{xy}^{-1}R_yR_x$, $T_x=L_x^{-1}R_x$, $L_{x,y}=L_{xy}^{-1}L_xL_y$, $P_{x,y}=R_y^{-1}L_xD_yD_x$, $K_{x,y}=L_x^{-1}R_yD_x^{-1}D_y^{-1}$, $V_x=D_x^{-1}R_x$, $U_x=D_xR_x$. It is clear that $R_{x,y},L_{x,y},T_x,P_{x,y},K_{x,y}$, V_x, $U_x\in TInn(Q)$, $\forall x,y\in Q$. Indeed, let e be the unit of the loop $(Q,\cdot)$, then $V_x(e)=D_x^{-1}R_x(e)=x/x=e$, $U_x(e)=D_xR_x(e)=x\setminus x=e$. For the remaining two mappings we have: $x=e\backslash x\Leftrightarrow x\setminus y=(e\backslash x)\setminus y\Leftrightarrow x\cdot[(e\backslash x)\setminus y]=y\Leftrightarrow L_xD_yD_x(e)=R_y(e)\Leftrightarrow R_y^{-1}L_xD_yD_x(e)=e\Leftrightarrow P_{x,y}(e)=e$; $y=y/e\Leftrightarrow x/y=x/(y/e)\Leftrightarrow[x/(y/e)]\cdot y=x\Leftrightarrow R_yD_x^{-1}D_y^{-1}(e)=L_x(e)\Leftrightarrow L_x^{-1}R_yD_x^{-1}D_y^{-1}(e)=e\Leftrightarrow K_{x,y}(e)=e$.

Theorem 1. *Let $(Q,\cdot)$ be a loop. Then*

$$TInn(Q,\cdot)=\langle R_{x,y},L_{x,y},T_x,P_{x,y},U_x|x,y\in Q\rangle.$$

Proof. It was shown in [2] that

$$TInn(Q,\cdot)=\langle R_{x,y},L_{x,y},T_x,P_{x,y},K_{x,y},V_x,U_x|x,y\in Q\rangle.$$

So, we will prove that $V_x,K_{x,y}\in\langle R_{x,y},L_{x,y},T_x,P_{x,y},U_x\mid x,y\in Q\rangle$. As $K_{x,x}=T_xD_x^{-2}$ and $V_xU_x^{-1}=D_x^{-2}$, we get $D_x^2=K_{x,x}^{-1}T_x=U_xV_x^{-1}$, which implies $V_x=T_x^{-1}K_{x,x}U_x$. Also, $P_{x,x}=T_x^{-1}D_x^2\Rightarrow T_x=D_x^2P_{x,x}^{-1}=U_xV_x^{-1}P_{x,x}^{-1}$, hence

$$V_x=T_x^{-1}K_{x,x}U_x=P_{x,x}^{-1}T_x^{-1}U_x. \tag{1}$$

Now, let consider the equalities $T_{xy}R_{x,y}=L_{xy}^{-1}R_yR_x$ and $T_{xy}^{-1}L_{x,y}=R_{xy}^{-1}L_xL_y$, which imply

$$L_{xy}^{-1}R_y=T_{xy}R_{x,y}R_x^{-1},\quad R_{xy}^{-1}L_x=T_{xy}^{-1}L_{x,y}L_y^{-1}. \tag{2}$$

Using the first equality from (2), we get: $K_{xy,y}=L_{xy}^{-1}R_yD_{xy}^{-1}D_y^{-1}=T_{xy}R_{x,y}R_x^{-1}D_{xy}^{-1}D_y^{-1}=T_{xy}R_{x,y}V_x^{-1}(D_yD_{xy}D_x)^{-1}$, hence

$$K_{xy,y}=T_{xy}R_{x,y}V_x^{-1}(D_yD_{xy}D_x)^{-1}. \tag{3}$$

Analogously, using the second equality from (2), we have:
$P_{x,xy}=R_{xy}^{-1}L_xD_{xy}D_x=T_{xy}^{-1}L_{x,y}L_y^{-1}D_{xy}D_x=T_{xy}^{-1}L_{x,y}T_yR_y^{-1}D_{xy}D_x=$

$= T_{xy}^{-1} L_{x,y} T_y U_y^{-1}(D_y D_{xy} D_x)$, so

$$(D_y D_{xy} D_x)^{-1} = P_{x,xy}^{-1} T_{xy}^{-1} L_{x,y} T_y U_y^{-1}. \tag{4}$$

From (3), (4) and (1), it follows:

$$K_{xy,y} = T_{xy} R_{x,y} U_x^{-1} T_x P_{x,x} P_{x,xy}^{-1} T_{xy}^{-1} L_{x,y} T_y U_y^{-1}.$$

Now, denoting $xy = z$ in the last equality, we finish the proof. □

Corollary 1. *If $(Q, \cdot)$ is a power associative loop, then*

$$TInn(Q, \cdot) = \langle R_{x,y}, L_{x,y}, P_{x,y}, U_x \mid x, y \in Q\rangle.$$

Proof. If $(Q, \cdot)$ is a power associative loop, then ${}^{-1}x = x^{-1}$, which implies $D_e^2 = \varepsilon$, where e is the unit of $(Q, \cdot)$ and ε is the identity mapping on Q. Thus $U_x P_{e,x} U_e P_{x,x}^{-1} = D_x R_x R_x^{-1} D_x D_e^2 D_x^{-2} L_x^{-1} R_x = D_x^2 D_x^{-2} L_x^{-1} R_x = T_x$, i.e. $T_x \in \langle R_{x,y}, L_{x,y}, P_{x,y}, U_x \mid x, y \in Q\rangle$. □

Recall that a loop $(Q, \cdot)$ is called middle Bol if it satisfies the identity

$$x(yz \setminus x) = (x/z)(y \setminus x). \tag{5}$$

It is known that the class of middle Bol loops coincides with the class of loops with universal (i.e. invariant under the isotopy of loops) anti-automorphic inverse property: $(xy)^{-1} = y^{-1}x^{-1}$.

Theorem 2. *If $(Q, \cdot)$ is a middle Bol loop, then*

$$TInn(Q) = \langle R_{x,y}, P_{x,y}, U_x | x, y \in Q\rangle.$$

Proof. Let $(Q, \cdot)$ be a middle Bol loop. As Q is power associative, $TInn(Q) = \langle L_{x,y}, R_{x,y}, P_{x,y}, U_x | x, y \in Q\rangle$, so we will show that $L_{x,y} \in \langle R_{x,y}, P_{x,y}, U_x | x, y \in Q\rangle$. From $x^{-1} = {}^{-1}x$ and $(x^{-1})^{-1} = x$, $\forall x \in Q$, denoting $I : Q \mapsto Q$, $I(x) = x^{-1}$, we obtain that $I^2 = \varepsilon$ – the identical mapping on Q. Also, middle Bol loop satisfies the anti-authomorphic inverse property, so $IL_x I = R_{x^{-1}}$, which implies $IL_{x,y} I = R_{y^{-1},x^{-1}}$, for every $x, y \in Q$, thus $L_{x,y} = IR_{y^{-1},x^{-1}} I = U_e R_{y^{-1},x^{-1}} U_e$, where e is the unit of $(Q, \cdot)$. □

Theorem 3. *If $(Q, \cdot)$ is a middle Bol loop, then $Inn(Q) \trianglelefteq TInn(Q)$.*

Proof. Let $(Q, \cdot)$ be a middle Bol loop. From (5) it follows: $L_x D_x R_z(y) = L_{x/z} D_x(y)$ and $L_x D_x L_y(z) = R_{y\setminus x} D_x^{-1}(z)$, so

$$D_x R_z = L_x^{-1} L_{x/z} D_x, \quad D_x L_y D_x^{-1} = L_x^{-1} R_{y\setminus x} D_x^{-2}, \tag{6}$$

for every $x, y, z \in Q$. Taking $y = e$ in (5), we have $x \cdot (z\backslash x) = (x/z) \cdot x$, i.e. $L_x D_x(z) = R_x D_x^{-1}(z)$, which implies $D_x^{-2} = R_x^{-1} L_x$, $\forall x \in Q$. From the second equality of (6) and the last one we get

$$D_x L_y = L_x^{-1} R_{y\backslash x} R_x^{-1} L_x D_x, \tag{7}$$

for every $x, y \in Q$. According to (6) and (7), for every $\sigma \in TMlt(Q)$ and every $\varphi \in Mlt(Q)$, $\sigma\varphi\sigma^{-1} \in Mlt(Q)$. Moreover, if $\sigma(e) = e = \varphi(e)$, then $\sigma\varphi\sigma^{-1}(e) = e$, so $\sigma\varphi\sigma^{-1} \in Inn(Q)$, for $\forall\sigma \in TInn(Q), \forall\varphi \in Inn(Q)$, i.e. $Inn(Q, \cdot)$ is a normal subgroup of $TInn(Q, \cdot)$. $\square$

Remark. Using GAP System for Computational Discrete Algebra ($https : //www.gap - system.org/$), there were found right Bol loops of order 12 the inner mapping group of which is not a normal subgroup of the total inner mapping group.

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When a commutator of two matrices has rank one?

Victor Ufnarovski

Abstract

Let A be a square matrix. We construct matrix B such that the commutator $C = AB - BA$ has rank 1.

Keywords:rank one, commutator, eigenvectors.

Let A, B be square matrices and $C = AB - BA$ is their commutator. In [1] one can find several different proofs that if C has rank one, then A and B have non-trivial common invariant subspace. In this article we want to study the related question: for which square matrix A there exists a matrix B such that the commutator C has rank one.

An obvious observation is that this never happens if A is a scalar matrix cI because it commutes with any matrix B. Surprisingly this is the only exception if the main field is algebraically closed.

Theorem 1. *For any non-scalar square matrix A over an algebraically closed field there exists a square matrix B such that the commutator $C = AB - BA$ has rank one.*

Proof. We will try to find B in the form $B = A + XY^T$, for some column-vectors X and Y of appropriate size. Then

$$C = AB - BA = AXY^T - XY^TA.$$

If we choose X as an eigenvector for A, say $AX = \lambda X$, we get

$$C = AXY^T - XY^TA = \lambda XY^T - XY^TA = X(\lambda Y^T - Y^TA).$$

This is the matrix of rank one or zero and we need only to avoid the condition

$$\lambda Y^T = Y^T A \Leftrightarrow A^T Y = \lambda Y$$

to get the matrix of rank 1, so Y can be chosen as any vector different from eigenvector of the transposed matrix A^T with eigenvalue λ which is always possible for non-scalar matrix. $\square$

When we get an easy proof we need to understand what and how can be generalized. First of all do we really need an algebraically closed field? We have used it to get an eigenvector, but may be the statement is valid without such restrictions? Unfortunately this is not the case.

Theorem 2. *For any 2×2 real matrix A without real eigenvalues and arbitrary square matrix B the commutator $C = AB - BA$ has either rank zero or rank two.*

Proof. Suppose the opposite, that the rank is one. As it was shown in [1], A and B should have a common invariant subspace which should be one-dimensional and provide a real eigenvector for A.

We can give an alternative proof. Considering A and B over complex numbers we can find a common complex eigenvector X. We have

$$AX = \lambda X \Rightarrow A\overline{X} = \overline{\lambda X}$$

and similarly for B. Thus in the basis $X, \overline{X}$ the matrices A and B can be diagonalized simultaneously and therefore commute. $\square$

Let us go back to the case of the algebraically closed field. We know that A and B should have a common eigenvector when C has rank one. Can we find it in our proof? Almost directly, it is X, because

$$XY^T X = X(Y,X) = (Y,X)X \Rightarrow BX = (\lambda + (Y,X))X,$$

where we have used that the 1×1 matrix $Y^T X$ can be identified with the inner product (Y, X).

What we can find more in our proof? If we take Y as eigenvector for A^T, say $A^T Y = \mu Y \Leftrightarrow Y^T A = \mu Y^T$, then we can still get rank one:

$$C = AXY^T - XY^TA = AXY^T - \mu XY^T = (AX - \mu X)Y^T.$$

Now the demand is that X should be different from any eigenvector with eigenvalue μ.

But we can improve our statement more. The important observation is that we can replace A by any matrix commuting with A, e.g. $p(A)$ for any polynomial $p(x)$. Thus with the same proof as above we can formulate more general statement.

Theorem 3. *For any non-scalar square matrix A and any matrix A' commuting with A consider the square matrix $B = A' + XY^T$, where X, Y are non-zero column-vectors and either X is eigenvector for A or Y is an eigenvector for A^T. If both are eigenvectors, we demand that their eigenvalues should be different. In the both cases the commutator matrix $C = AB - BA$ has rank one.*

As the example consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

We can easily find eigenvectors:

$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$A^T\begin{pmatrix} 2 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad A^T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now we can create many examples, for example:

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$B = 3e^{At} + XY^T = \begin{pmatrix} e^{5t} + 2e^{-t} + 1 & e^{5t} - e^{-t} + 1 \\ 2e^{5t} - 2e^{-t} & 2e^{5t} + e^{-t} \end{pmatrix},$$

$$C = AB - BA = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}.$$

Can the last theorem be reversed, thus is this true that for any square matrices A, B such that the rank of the commutator $AB - BA$ is equal to one, the matrix B can be written as $B = A' + R$, for some matrix A' commuting with A and some matrix R of rank one?

This is true if the size of matrices is not greater than 3, but for $n = 4$ we can construct the following example.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have that

$$AB - BA = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is of rank one, but any matrix commuting with A has the form

$$A' = \begin{pmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Now obviously $B - A'$ cannot have rank one.

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Building BCK-Algebras of higher order using new BCK Extensions

Radu Vasile

Abstract

First of all, we remark that this talk is based on the paper [5]. Starting from some results obtained in the papers [6], [4], we provide some examples of finite bounded commutative BCK-algebras, using the Wajsberg algebra associated to a bounded commutative BCK-algebra. This method is an alternative to the Iseki's construction, since by Iseki's extension some properties of the obtained algebras are lost.

Keywords: Bounded commutative BCK Algebras, Wajsberg Algebras.

1 Introduction

BCK-algebras were first introduced in mathematics in 1966 by Y. Imai and K. Iseki, through the paper [7]. In [8] the reader can find the definition and properties of BCI and BCK algebras. For our purpose we present only the following definition:

Definition 1.1. I)A BCK-algebra $(X, *, \theta)$ is called *commutative* if

$$x * (x * y) = y * (y * x),$$

for all $x, y \in X$
II) If in the BCK-algebra $(X, *, \theta)$ there is an element 1 such that $x \leq 1$, for all $x \in X$, therefore the algebra X is called a *bounded BCK-algebra*. In a bounded BCK-algebra, we denote $1 * x = \overline{x}$.

Also in [8] (Theorem 3.6), the Iseki Extension and its properties are presented. From there we underline the following result:

Remark 1.2. **[8]** The Iseki's extension of a commutative BCK-algebra is not a commutative BCK-algebra.

2 Connections between finite bounded commutative BCK-algebras and Wajsberg algebras

In the following, we will give some examples of finite bounded commutative BCK-algebras. In the finite case, it is very useful to have many examples of such algebras. But, such examples, in general, are not so easy to find. A method for finding bounded BCK algebras can be Iseki's extension. But, from the above, we remark that the Iseki's extension can't be used to obtain examples of finite commutative bounded BCK-algebras, since the commutativity is lost. From this reason, we use other technique to provide examples of such algebras. We use the connections between finite commutative bounded BCK-algebras and Wajsberg algebras and the algorithm and examples given in the papers [4] and [6].

In [1] the mv-algebra definition is presented and in [9] there are defined alternative multiplications for such an algebra.

In [2], Theorem 1.7.1, an equivalence between bounded commutative algebras and MV-algebras is presented.

In [2], Definition 4.2.1 the Wajsberg algebra is defined and in Lemma 4.2.2 and Theorem 4.2.5 an equivalence between Wajsberg Algebras and MV-algebras is presented.

Remark 2.1. From the above, if $(W, \circ, ^{-}, 1)$ is a Wajsberg algebra, therefore $(W, \oplus, \odot, ^{-}, 0, 1)$ is an MV-algebra, with

$$x \ominus y = \overline{(\overline{x} \oplus y)} = \overline{(x \circ y)}. \tag{2.1.}$$

Defining

$$x * y = \overline{(x \circ y)}, \tag{2.2.}$$

we have that $(W, *, \theta, 1)$ is a bounded commutative BCK-algebra.

Using the above remark, starting from some known finite examples of Wajsberg algebras given in the papers [4] and [6], we can obtain examples of finite commutative bounded BCK-algebras, using the following algorithm.

The Algorithm

1) Let n be a natural number, $n \neq 0$ and

$$n = r_1 r_2 ... r_t, r_i \in \mathbb{N}, 1 < r_i < n, i \in \{1, 2, ..., t\},$$

be the decomposition of the number n in factors. The decompositions with the same terms, but with other order of them in the product, will be counted one time. The number of all such decompositions will be denoted with π_n.

2) There are only π_n non isomorphic, as ordered sets, Wajsberg algebras with n elements. We obtain these algebras as a finite product of totally ordered Wajsberg algebras (see [6] Theorem 4.8).

3) Using Remark 2.4 from above, to each Wajsberg algebra a commutative bounded BCK-algebra can be associated.

3 Conclusion

We provided an algorithm for finding examples of finite commutative bounded BCK-algebras, using their connections with Wajsberg algebras. This algorithm allows us to find such examples no matter the order of the algebra. This thing is very useful, since examples of such algebras are very rarely encountered in the specialty books.

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Some remarks regarding Fibonacci and Lucas numbers

Mariana-Geanina Zaharia

Abstract

First of all, we must remark that this presentation is based on the paper [3].
We present new applications of Fibonacci and Lucas numbers.
We generalized Fibonacci and Lucas numbers by using an arbitrary binary relation over the real fields instead of addition of the real numbers and we give properties of the new obtained sequences.

1 Introduction

There have been numerous papers devoted to the study of the properties and applications of Fibonacci and Lucas sequences. Due to this fact, to obtain new results in this direction is not always an easy problem. Our purpose in this talk is to provide new applications of these sequences.

2 Some remarks regarding Fibonacci and Lucas numbers

Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence

$$f_n = f_{n-1} + f_{n-2},\ n \geq 2, f_0 = 0; f_1 = 1,$$

and $(l_n)_{n\geq 0}$ be the Lucas sequence

$$l_n = l_{n-1} + l_{n-2},\ n \geq 2, l_0 = 2; l_1 = 1.$$

First of all, we recall some elementary properties of the Fibonacci and Lucas numbers, properties which will be used in this chapter.
Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence and let $(l_n)_{n\geq 0}$ be the Lucas sequence. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, be two real numbers.

The following formulae are well known:

Binet's formula for Fibonacci sequence

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad n \in \mathbb{N}.$$

Binet's formula for Lucas sequence

$$l_n = \alpha^n + \beta^n, \quad n \in \mathbb{N}.$$

Fibonacci numbers are defined by using the addition operation over the real field. It will be interesting to search what happens when, instead of addition, we use an arbitrary binary relation over $\mathbb{R}$. Therefore, we will consider "$*$" a binary relation over $\mathbb{R}$ and $a, b \in \mathbb{R}$. We define the following sequence

$$\varphi_n = \varphi_{n-1} * \varphi_{n-2}, \varphi_0 = a, \varphi_1 = b.$$

We call this sequence the *left Fibonacci type sequence attached to the binary relation "$*$", generated by a and b.*

The following sequence

$$\varphi_n = \varphi_{n-2} * \varphi_{n-1}, \varphi_0 = a, \varphi_1 = b$$

is called the *right Fibonacci type sequence attached to the binary relation "$*$", generated by a and b.*

Proposition 1. *Let $A, B \in \mathbb{R}$ such that $\Delta = A^2 + 4B > 0$. We define on $\mathbb{R}$ the following binary relation*

$$x * y = Ax + By, x, y \in \mathbb{R}.$$

We consider $(\varphi_n)_{n\in N}$ *the left Fibonacci type sequence attached to the binary relation "$*$". Therefore, we have*

$$\varphi_{n+1} = \frac{1}{\beta - \alpha}\left[(-b + a\beta)\,\alpha^{n+1} + (b - a\alpha)\,\beta^{n+1}\right], \tag{3.1.}$$

where $\alpha = \frac{A+\sqrt{\Delta}}{2}, \beta = \frac{A-\sqrt{\Delta}}{2}$ *and*

$$\mathcal{L} = \lim \frac{\varphi_{n+1}}{\varphi_n} = \max\{\alpha, \beta\}. \tag{3.2.}$$

Remark 2. 1) Similar results can be obtained for the right Fibonacci type sequences.
2) The above formula (3.1) is a Binet's-type formula. Indeed, for $A = B = 1, a = 0, b = 1$, we obtain the Binet's formula for Fibonacci sequence.
3) The limit $\mathcal{L}$, from relation (3.2) is a Golden-ratio type number. Indeed, for $A = B = 1, a = 0, b = 1$, we obtain the Golden-ratio.

Proposition 3. *With the above notations, if* $g_1 * g_0 = g_0 * g_1$, *the following relation holds:*

$$\varphi_n = g_1^{d_n} * g_0^{d_{n-1}}. \tag{3.3.}$$

3 Conclusion

The study of equations in quaternions and octonions algebras, in the context of this paper, refers to the study of particular cases of such equations, the cases in which the coefficients have the form of one of the elements defined before: Fibonacci Quaternion, Generalized Fibonacci Quaternion, Lucas Quaternion. It is possible that solutions to such equations can be found exactly because of the properties and particularities of these relationships of recurrence. The topics discussed by me today also appear in "Some applications of Fibonacci and Lucas numbers", a chapter in the Springer Book: "Algorithms as an approach of Applied Mathematics", Editors Cristina Flaut, Srka Hoskova-Mayerova, F. Maturo.

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On normal 3-isohedral spherical tilings for group series $*nn$, nn, $*22n$ and $n*$

Elizaveta Zamorzaeva

Abstract

Tilings of the sphere with disks that fall into 3 transitivity classes under the group action are studied. For group series $*nn$, nn, $*22n$ and $n*$, all normal 3-isohedral spherical tilings are enumerated.

Keywords: 3-isohedral tilings, sphere, normal tilings, group series

The classification of isohedral, isogonal and isotoxal tilings of the sphere as well as the survey of earlier results can be found in [1]. In [2] the author developed some general methods for finding k-isohedral ($k \geq 2$) tilings of a two-dimensional space of constant curvature (i.e. the Euclidean plane, the sphere and the hyperbolic plane) from the known isohedral tilings. Applying these methods the author obtained a complete classification of 2-isohedral tilings of the sphere. All the normal fundamental 2-isohedral tilings of the sphere were given in [3]. The same methods in terms of Delaney-Dress symbols were implemented [4] in algorithms and computer programs (see also [5]).

Now define basic concepts. Let W be a tiling of the 2-dimensonal sphere with topological disks and G be a discrete isometry group of the sphere.

Definition 1. *The tiling W is called k-isohedral with respect to the group G if G maps W onto itself and the tiles of W fall into exactly k transitivity classes under the group G.*

Definition 2. *Consider all possible pairs (W, G), where the tiling W of the sphere is k-isohedral with respect to the group G. Two pairs*

(W, G) and (W', G') are said to belong to the same Delone class if there exists a homeomorphism φ of the sphere which maps the tiling W onto the tiling W' and the relation $G = \varphi^{-1}G'\varphi$ holds.

In a tiling of the sphere with disks we define a connected component of the intersection of two or more different disks to be a vertex of the tiling if it is a single point or to be an edge of the tiling otherwise. We distinguish between fundamental and non-fundamental Delone classes depending on whether the group G acts one time (or simply) transitively on the set of tiles or not.

We use the Conway's orbifold symbol for discrete isometry groups of the sphere. The symbol characterizes the orbifold which is the quotient of the sphere by the group. Altogether there are 7 countable series and 7 sporadic discrete isometry groups of the sphere.

A tiling of the sphere is called normal if it satisfies the following conditions [1]:

SN1. Each tile is a topological disk.

SN2. The intersection of any set of tiles is a connected (possibly empty) set.

SN3. Each edge of the tiling has two endpoints which are vertices of the tiling.

In a normal tiling of the sphere every tile contains at least three edges on its boundary and the valency of each vertex is at least three.

Our method for finding Delone classes of k-isohedral tilings uses the splitting procedure applied to the known fundamental $(k-1)$-isohedral tilings of the sphere with disks [2]. Earlier the splitting procedure was applied to all fundamental isohedral (1-isohedral) tilings of the sphere with disks resulting in the classification of Delone classes of fundamental 2-isohedral tilings of the sphere with disks [3].

The application of the splitting procedure to fundamental 2-isohedral tilings of the sphere with disks has yielded 41 Delone classes of fundamental 3-isohedral tilings of the sphere with disks for group series $*nn$ ($n = 1, 2, \dots$), 43 Delone classes for group series nn ($n = 1, 2, \dots$), 194 Delone classes for group series $*22n$ ($n = 1, 2, \dots$) and 168 Delone classes for group series $n*$ ($n = 1, 2, \dots$). All the 4 numbers coincide with the numerical results given in [5].

Now we select among the obtained 3-isohedral tilings those which satisfy the above normality conditions SN1 – SN3. We are going to enumerate the obtained Delone classes of 3-isohedral tilings by listing 3 cycles of vertex valencies for each Delone class.

For group series $*nn$ ($n = 1, 2, \dots$), which corresponds to series of 3-dimensional point isometry groups $N \cdot m$, there are 4 series of Delone classes of normal 3-isohedral tilings of the sphere: $[3.4^2; 3.4.2n.4; 3.4.2n.4]$, $[4^2.2n; 4^2.2n; 4^4]$, $[3.4.4n; 3.4.4n; 3.4.4n]$ and $[4^2.6; 4.6.2n; 4.6.2n]$.

For group series nn ($n = 1, 2, \dots$), which corresponds to series of 3-dimensional point isometry groups N, there are 10 series of Delone classes of normal 3-isohedral tilings of the sphere: $[3^4; 3^4.n; 3^4.n]$, $[3^3.n; 3^3.n; 3^6]$, $[3.4.3.4; 3.4.3.n; 3.4.3.n]$, $[3^3; 3^4.n; 3^5.n]$, $[3^2.4; 4.3.4.n; 3^2.4.3.n]$, $[4^2.n; 3^3.n; 3^2.4^2]$, $[3^2.2n; 3^3.2n; 3^4.n]$, $[3.5^2; 5^2.n; 3.5.3.n]$, $[3.4.2n; 3.4.2n; 4.3.4.n]$ and $[4^2.n; 4^2.n; 4^4]$.

For group series $*22n$ ($n = 1, 2, \dots$), which corresponds to series of 3-dimensional point isometry groups $m \cdot N : m$, there are 32 series of Delone classes of normal 3-isohedral tilings of the sphere: $[3.4^3; 3.4^3; 3.4.2n.4]$, $[3.4^2; 3.4^3; 3.4^2.2n.4]$, $[3.4^2; 3.4^3; 3.4^2.2n.4]$, $[3.4^2; 3.4.2n.4; 3.4^4]$, $[4^3; 4^4; 4^3.2n]$, $[3.4.8; 3.4^3; 3.4.2n.8]$, $[3.4.8; 3.4^2.8; 3.4.2n.4]$, $[3.4^2; 3.4^2.8; 3.4.2n.8]$, $[3.4^2; 3.4^2.4n; 3.4^2.4n]$, $[4^2.2n; 4^4; 4^4]$, $[3.4.4n; 3.4^3; 3.4^2.4n]$, $[4^3; 4^3; 4^4.2n]$, $[3.4.8; 3.4.8; 3.4^2.2n.4]$, $[4^3; 4^2.2n; 4^5]$, $[3.4.4n; 3.4.4n; 3.4^4]$, $[4^2.6; 4^2.6; 4^2.6.2n]$, 2 different series $[4^2.6; 4^2.6; 4.6.4.2n]$, $[3.4.8; 3.8^2; 3.4.2n.8]$, $[4^3; 4.8.2n; 4^3.8]$, 2 different series $[4^2.6; 4.6.2n; 4^3.6]$, $[4^2.8; 4^2.2n; 4^3.8]$, $[4^3; 4^2.4n; 4^3.4n]$, $[3.4.8; 3.8.4n; 3.4^2.4n]$, $[3.4.4n; 3.8.4n; 3.4^2.8]$, $[4^2.6; 4.6.8; 6.8.2n]$, $[4.6.8; 4.6.8; 4.6.2n]$, $[4^2.12; 4^2.12; 4.12.2n]$, $[4^2.6; 4.6.4n; 4.6.4n]$, $[3.8^2; 3.8.4n; 3.8.4n]$ and $[4^2.6n; 4^2.6n; 4^2.6n]$.

For group series $n*$ ($n = 1, 2, \dots$), which corresponds to series of 3-dimensional point isometry groups $N : m$, there are 41 series of Delone classes of normal 3-isohedral tilings of the sphere: $[3^4; 3^4.n; 3^4.4^2]$, $[3^3.n; 3^3.4^2; 3^6]$, $[3^2.4^2; 3^3.4^2; 3^4.n]$, $[3.4.3.4; 3.4.3.n; 3.4.3.4^2]$, $[4^3; 3^3.n; 3^3.4^2]$, $[3^3.6; 3^2.6^2; 3^4.n]$, $[3.4^3; 3.4^3; 3.4.n.4]$, $[3^3; 3^4.4^2; 3^5.n]$, $[3^3; 3^4.n;$

$3^5.4^2]$, $[3^2.4; 3.4^4; 3^2.4.3.n]$, $[3^2.6; 3^3.6^2; 3^4.n]$, $[4^2.n; 3^3.4^2; 3^3.4^2]$, $[3^3; 3^2.4^2; 3^2.4^2.3^2.n]$, $[3.4^2; 3^3.n; 3^5.4^2]$, $[3^3; 3^2.6^2; 3^2.6.3^2.n]$, $[3^2.4; 3.4.n.4; 3^2.4.3.4^2]$, $[3.4^2; 3^3.4; 3^2.4^2.3.n]$, $[3.6^2; 3^3.n; 3^5.6]$, $[3^2.2n; 3^3.2n; 3^4.4^2]$, $[4^3; 3.4.3.n; 3.4^4]$, $[3.4.6; 3^2.4.6; 3^2.6.3.n]$, $[3^2.2n; 3^2.4^2; 3.4^2.3.2n]$, $[3.4^2; 3.4^2.2n; 3.4^2.2n]$, $[3.5^2; 4^2.5^2; 3.5.3.n]$, $[3.4.6; 3.4.6^2; 3.4.n.4]$, $[4.6^2; 3.4.6.4; 3.4.3.n]$, $[4^2.n; 4^4; 4^4]$, $[3^2.2n; 3^2.6^2; 3.6.3.2n]$, $[3.4^2; 4^2.n; 3.4^2.3.4^2]$, $[3^2.6; 3.4.6; 3^2.4.6.3.n]$, $[4^3; 4^3; 4^4.n]$, $[3.5^2; 5^2.n; 3.4^2.3.5]$, $[3.6^2; 4^2.n; 3.4^2.3.5]$, $[3^2.8; 3.8^2; 3^2.8.3.n]$, $[3.4.2n; 3.4.2n; 3.4^4]$, $[4^2.5; 5^2.n; 4^2.5^2]$, $[3.4.6; 3.6.2n; 3.4.6.2n]$, $[4^2.6; 4^2.6; 4.6.4.n]$, $[5^2.6; 5.6^2; 5^2.n]$, $[3.8^2; 3.8.2n; 3.8.2n]$ and $[4^2.3n; 4^2.3n; 4^2.3n]$.

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Section 2

Applied Mathematics

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An algorithm based on continuation techniques for minimization problems with non-linear equality constraints

Elisabete Alberdi, Joseba Makazaga, Ander Murua,
Mikel Antoñana

Abstract

We present a technique to solve numerically minimization problems with equality constraints. We focus on problems with a great number of local minima which are hard to obtain by local minimization algorithms with random starting guesses, making it difficult to find the global minimum.

The described technique has been applied to search 10th-order optimized time-symmetric composition integrators. By applying our technique, we are able to obtain 10th order integrators with smaller 1-norm than any other integrator found in the literature up to now.

Keywords: Global optimization, solving polynomial systems, constrained minimization problems, composition methods.

1 Introduction

We present a technique to solve numerically minimization problems with equality constraints. We focus on problems with a great number of local minima which are hard to obtain by local minimization algorithms with random starting guesses, making it difficult to find the global minimum. We are particularly interested in constrained problems that arise in the context of optimized differential equation solvers.

We propose a method to solve constrained minimization problems in which the set of (equality) constraints is highly non-linear (typically polynomial), with a great number of local minima with tiny basins of attraction of typical local minimization algorithms. We aim at computing a large set of local minima that hopefully will contain the global minimum of the target problem (or at least, a local minimum close to it). We formulate the minimization problem in terms of Lagrange multipliers. The proposed algorithm starts computing a set of points that locally minimize the objective function subject to some of the constraints. Then, the algorithm subsequently computes (using continuation techniques) new sets of points that locally minimize the objective function subject to one additional constraint at each step.

2 Results

The described technique has been applied to search 10th-order optimized time-symmetric composition integrators. Such an integrator is considered optimal by several authors if the 1-norm of its coefficients is minimized subject to certain polynomial constraints (order conditions). By applying our technique, we are able to obtain 10th order integrators with smaller 1-norm than any other integrator found in the literature up to now.

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On averaging in multifrequency systems with linearly transformed arguments

Yaroslav Bihun, Roman Petryshyn, Ihor Skutar

Abstract

The research deals with the existence of the solution of the initial problem for hyperbolic equation under the multifrequency disturbances, which are described by the system of ordinary differential equations (ODE) with multipoint and integral conditions. The averaging method over fast variables is grounded, and estimation of accuracy of the method which obviously depends on the small parameter was found.

Keywords: averaging method, multifrequency problem, system, resonance, small parameter, integral conditions.

1 Statement of the problem, the condition of passing through resonance and averaging problem

Multifrequency systems of the form

$$\frac{da}{dt} = \varepsilon X(\tau, a, \varphi), \quad \frac{d\varphi}{dt} = \omega(\tau, a) + \varepsilon Y(\tau, a, \varphi), \tag{1}$$

by means of averaging method were researched in the works [1-2] and in many others. Here $a \in D \subset \mathbb{R}^n$, $\varphi \in \mathbb{R}^m$, $m \geq 2$, ε – a small parameter, $\tau = \varepsilon t$ – slow time, vector-function X and Y 2π-periodic by variables φ_ν, $\nu = \overline{1, m}$.

The question of existence and uniqueness of the solution and grounding of the averaging method for multifrequency systems with stable and changeable delay, the solutions of which meat initial or integral conditions, are solved in [3,4], etc.

In this paper we consider a problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + b(x,\tau)u + f(\tau, u_\Delta, a_\Lambda, \varphi_\Theta, \varepsilon) \tag{2}$$

$$u(z,0) = v(z), \frac{\partial u(z,0)}{\partial t} = w(z), z \in \mathbb{R}. \tag{3}$$

where a, φ – are solutions of the multifrequency system of equations

$$\frac{da}{d\tau} = X(\tau, a_\Lambda, \varphi_\Theta), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau, a_\Lambda, \varphi_\Theta), \tag{4}$$

with the conditions

$$\sum_{\nu=1}^{r} c(t_\nu) a(t_\nu) = d_1, \quad 0 \le t_1 < \ldots t_r \le L, \tag{5}$$

$$\int_0^{\tau_1} \sum_{\nu=1}^{q} \left[g_{1,\nu}(\tau, a_\Lambda(\tau)) \varphi_{\theta_\nu}(\tau) + f_1(\tau, a_\Lambda(\tau), \varphi_\Theta(\tau)) \right] d\tau +$$

$$+ \int_{\tau_2}^{L} \sum_{\nu=1}^{q} \left[g_{2,\nu}(\tau, a_\Lambda(\tau)) \varphi_{\theta_\nu}(\tau) + f_2(\tau, a_\Lambda(\tau), \varphi_\Theta(\tau)) \right] d\tau = d_2. \tag{6}$$

Here $0 \le \tau \le L$, $(0, \varepsilon_0] \ni \varepsilon$ – a small parameter, $0 < \tau_1 < \tau_2 < L$, $a \in D \subset \mathbb{R}^n$, $\varphi \in \mathbb{T}^m$, $m \ge 1$, $\Delta = (\delta_1, \ldots, \delta_\kappa)$, $\Lambda = (\lambda_1, \ldots, \lambda_p)$, $\Theta = (\theta_1, \ldots, \theta_q)$, $\delta_i, \lambda_j, \theta_\nu \in (0,1)$, $a_{\lambda_j}(\tau) = a(\lambda_j \tau)$, $\varphi_{\theta_\nu}(\tau) = \varphi(\theta_\nu \tau)$, $u_{\delta_i}(x,\tau) = u(x, \delta_i \tau)$, $a_\Lambda(\tau) = (a_{\lambda_1}(\tau), \ldots, a_{\lambda_p}(\tau))$.

The main condition of the research of the multifrequency systems is the condition of passage of the system through resonance. For the system (1) the condition of resonance $(k, \omega(\tau, a(\tau, \varepsilon))) \approx 0, \mathbb{Z}^m \ni k \neq 0, (\cdot, \cdot)$ – scalar product. For the system with transformed arguments a condition is found [3,4]:

$$\sum_{\nu=1}^{q} \theta_\nu(k_\nu, \omega(\theta_\nu \tau)) = 0, \quad k_\nu \in \mathbb{R}^m, \quad \|k\| \neq 0.$$

If $V(\tau) \neq 0, \tau \in [0, L]$, where $V(\tau)$ – Wronsky determinant by the system of functions $\{ \omega(\theta_1\tau), \ldots, \omega(\theta_q\tau) \}$, then the system gets out of resonance. For $q = 1$, $\theta_1 = 1$ such a condition is offered in [1,2]. For one-frequency system with one linearly transformed argument the Wronsky determinant is:

$$V(\tau) = \begin{vmatrix} \omega(\tau) & \omega(z) \\ \dfrac{d\omega(\tau)}{d\tau} & \theta\dfrac{d\omega(z)}{dz} \end{vmatrix}, z = \theta\tau, \theta \in (0, 1).$$

In problems (2) - (6) equations as well as integral conditions are averaged over fast variables. Averaging system of equations takes the form of

$$\frac{\partial^2 \bar{u}}{\partial \tau^2} = c^2 \frac{\partial^2 \bar{u}}{\partial x^2} + b(x, \tau)\bar{u} + f_0(\tau, \bar{u}_\Delta, \bar{a}_\Lambda), \tag{7}$$

$$\frac{d\overline{a}}{d\tau} = X_0(\tau, \overline{a}_\Lambda), \quad \frac{d\overline{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y_0(\tau, \overline{a}_\Lambda). \tag{8}$$

For solutions of the system of equations (7), (8) condition (5) has the same form. In condition (6) vector-functions f_1 and f_2 are averaged over fast variables $\varphi_{\theta_\nu}, \nu = \overline{1, q}$

2 Main result

Averaged problem is remained a problem with delay, but it is much more simple, than (2)–(6), because the equations for $\bar{u}$, $\bar{a}$ do not depend on the fast variables $\bar{\varphi}_\Theta$, and also because finding $\bar{\varphi}$ is reduced to the problem of integration, if we know $\bar{a}(\tau)$.

Theorem. Suppose, that conditions are true:

1) functions $\omega_\nu \in C^{sm-1}[0, L], \nu = \overline{1, m}$, vector-functions X, Y, c, f, f_i, $g_i, i = 1, 2$ are twice continuously differentiable over variables τ, a_Λ, u_Δ, ε, and $(mq + 1)$ – is once continuously differentiable over fast variables $\varphi(\Theta)$ in corresponding areas;

2) functions b, v, w – are continuous and limited, $\tau \in [0, L]$, $x \in \mathbb{R}$;

3) Wronsky determinant by the system of functions $\omega_i(\theta_\nu \tau), i = \overline{1, m}$, $\nu = \overline{1, q}$, is not equal to zero on $[0, L]$;

4) a unique solution for the averaged problem exists, while the component of the solution $\bar{a} = \bar{a}(\tau), \bar{a}(0) = y \in D$, lies in the area D together with certain ρ-neighbourhood.

Then for quite small $\varepsilon_0 > 0$ a unique solution for the problem exists (2)–(8), such that for all $(x, \tau, \varepsilon) \in \mathbb{R} \times [0, L] \times (0, \varepsilon_0]$ inequality is satisfied (7) and

$$\|a(\tau, \varepsilon) - \overline{a}(\tau)\| + \|\varphi(\tau, \varepsilon) - \overline{\varphi}(\tau, \varepsilon)\| + |u(x, \tau, \varepsilon) - \overline{u}(x, \tau)| \le c_1 \varepsilon^\alpha,$$

where $\alpha = (mq)^{-1}$, $c_1 > 0$ and not depending on ε.

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On the maximum nontrivial convex cover number of the join of graphs

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Abstract

Given a connected graph G, a set $S \subseteq X(G)$ is convex in G if, for any two vertices $x, y \in S$, all vertices of every shortest path between x and y are in S. If $3 \leq |S| \leq |X(G)| - 1$, then S is a nontrivial set. The greatest $p \geq 2$ for which there is a cover of G by p nontrivial convex sets is the maximum nontrivial convex cover number of G. In this paper, we establish the maximum nontrivial convex cover number of the join of graphs.

Keywords: nontrivial convex cover, join of graphs.

1 Introduction

We denote by G a graph with vertex set $X(G)$ and edge set $U(G)$. The *neighborhood* of $x \in X(G)$ is the set of all vertices $y \in X(G)$, $y \sim x$ (i.e. adjacent to x), and it is denoted by $\Gamma(x)$. A set $S \subset X(G)$ is called *nontrivial* if $3 \leq |S| \leq |X(G)| - 1$. A vertex $x \in X(G)$ is called *universal* if $\Gamma(x) = X(G) \backslash \{x\}$. A *dominating* set of a graph G is a subset D of $X(G)$ such that every vertex not in D is adjacent to at least one member of D. A *minimum dominating set* is a dominating set of the smallest size in a given graph. The *domination number* $\gamma(G)$ is the number of vertices in a minimum dominating set of G. A set $S \subseteq X(G)$ is called *convex* in G if, for any two vertices $x, y \in S$, all vertices of every shortest path between x and y are in S. The *convex hull* of $S \subseteq X(G)$, denoted by $d - conv(S)$, is the smallest convex set containing S. A *nontrivial convex cover* $\mathcal{P}(G)$ of G is a family of sets that satisfies the following conditions: every set of $\mathcal{P}(G)$ is nontrivial and convex in G; $X(G) = \bigcup_{S\in\mathcal{P}(G)} S$; $S \not\subseteq \bigcup_{C\in\mathcal{P}(G),C\neq S} C$ for every $S \in \mathcal{P}(G)$. The greatest $p \geq 2$ for which there is a cover of G by p nontrivial convex sets is the *maximum nontrivial convex cover number* of G, denoted by $\varphi_{cn}^{max}(G)$. The *maximum nontrivial convex cover* of a graph G is a nontrivial convex cover of G that contains exactly $\varphi_{cn}^{max}(G)$ sets, and it is denoted by $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.

Since the nontrivial convex cover problem of a graph is NP-complete [1], it is of interest to determine the maximum nontrivial convex cover number for different classes of graphs, including graphs resulting from some graph operations. In this paper we establish the maximum nontrivial convex cover number of graphs resulting from join of graphs.

2 Main results

Let D be a minimum dominating set of a graph G, where $|X(G)| \geq 2$. We define an *association of minimum dominating set D* as a set of tuples: $\mathcal{A}_G(D) = \{(d_1, a_1), (d_2, a_2), \ldots, (d_{\gamma(G)}, a_{\gamma(G)})\}$, where $d_i \in D$, $a_i \in X(G)\backslash D$, $a_i \sim d_i$, for each i, $1 \leq i \leq \gamma(G)$, and $d_i \neq d_j$, $a_i \neq a_j$, for any two $i, j \in \{1, 2, \ldots, \gamma(G)\}$.

Lemma 1. *Let G be a connected graph of order $n \geq 2$ and D a minimum dominating set of G. There exists an association of minimum dominating set D.*

Proof. Assume that there is no an association of minimum dominating set D. This implies that there are two competing vertices d and d' in D relating to a vertex a of $X(G)\backslash D$. This means that all vertices of $(\Gamma(d) \bigcup \Gamma(d'))\backslash\{a\}$ are dominated by some vertices of $D\backslash\{d, d'\}$ or are in D. Since d and d' are already dominated by vertex a, we define a new minimum dominating set $D' = (D\backslash\{d, d'\}) \bigcup\{a\}$, $|D'| < |D|$. So, we obtain a contradiction. $\square$

The join of graphs G and H is the graph $G + H$ on vertex set $X(G+H) = X(G)\cup X(H)$ and edge set $U(G+H) = U(G)\cup U(H)\cup\{xy : x \in X(G), y \in X(H)\}$, where xy is an edge joining vertices x and y.

Theorem 1. *Let G be a connected graph of order n and K_m the complete graph of order m.*

1. $\varphi_{cn}^{max}(G+K_m) \geq n-\gamma(G)$ *if* $m=1$ *and* $n \geq 3$;
2. $\varphi_{cn}^{max}(G+K_m) = n+m-2$ *if* $m \geq 3$ *and* $n=1$, *or* $m \geq 2$ *and* $n \geq 2$.

Proof. 1) Assume that $m=1$ and $n \geq 3$. Let D be a minimum dominating set of graph G. We define a family of nontrivial convex sets $\mathcal{P}(G+K_1)=\varnothing$ that will cover graph $G+K_1$. By Lemma 1, there exists an association of minimum dominating set $\mathcal{A}_G(D)=\{(d_1,a_1),(d_2,a_2),\dots,(d_{\gamma(G)},a_{\gamma(G)})\}$. First, we add set $\{d_i,a_i,k\}$ to $\mathcal{P}(G+K_1)$ for each $(d_i,a_i) \in \mathcal{A}_G(D)$, $1 \leq i \leq \gamma(G)$, where $X(K_1)=\{k\}$. For each vertex $v \in X(G) \backslash (D \cup \bigcup_{(d,a)\in \mathcal{A}_G(D)} a)$, we add set $\{d,v,k\}$ to $\mathcal{P}(G+K_1)$, where $d \in D$, $d \sim v$. Clearly, $|\mathcal{P}(G+K_1)|=n-\gamma(G)$ and $\varphi_{cn}^{max}(G+K_1) \geq n-\gamma(G)$.

2) Assume that $m \geq 3$ and $n=1$. We define a family of nontrivial convex sets $\mathcal{P}(G+K_m)=\varnothing$ that will cover $G+K_m$, and choose two vertices $x,y \in X(K_m)$, $x \neq y$. Then, for each $z \in X(G+K_m)\backslash\{x,y\}$, we add set $\{x,y,z\}$ to $\mathcal{P}(G+K_m)$. Since every set of $\mathcal{P}(G+K_m)$ consists of three vertices, two of which are x and y, and the third vertex is resident in $\mathcal{P}(G+K_m)$, it follows that $\mathcal{P}(G+K_m)$ is a maximum nontrivial convex cover of graph $G+K_m$ and $\varphi_{cn}^{max}(G+K_m)=n+m-2$. Similarly, if $m \geq 2$ and $n \geq 2$, then $\varphi_{cn}^{max}(G+K_m)=n+m-2$. □

Lemma 2 [2]. *Let G and H be connected noncomplete graphs. Then a proper subset $C=S_1 \cup S_2$ of $X(G+H)$, where $S_1 \subset X(G)$ and $S_2 \subset X(H)$, is convex in $G+H$ if and only if S_1 and S_2 are cliques in G and H respectively.*

Theorem 2. *Let G and H be connected noncomplete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) \geq n+m-4$.*

Proof. We define a family of nontrivial convex sets $\mathcal{P}(G+H)=\varnothing$ that will cover $G+H$. Now, we choose two vertices $g_1,g_2 \in X(G)$, $g_1 \sim g_2$, and two vertices $h_1,h_2 \in X(H)$, $h_1 \sim h_2$. Then, we add set $\{g_1,g_2,h\}$ to $\mathcal{P}(G+H)$ for each $h \in X(H)\backslash\{h_1,h_2\}$, and add set

$\{h_1, h_2, g\}$ to $\mathcal{P}(G+H)$ for each $g \in X(G)\backslash\{g_1, g_2\}$. It can be easily verified that $|\mathcal{P}(G+H)| = n+m-4$. So, $\varphi_{cn}^{max}(G+H) \geq n+m-4$. The theorem is proved. $\square$

Theorem 3. *Let G and H be connected noncomplete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) = n+m-2$ if and only if there are two universal vertices in $G+H$.*

Proof. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ be a maximum nontrivial convex cover of $G+H$ such that $\varphi_{cn}^{max}(G+H) = n+m-2$. This yelds that each set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+K_m)$ satisfies the equality $|S| = 3$ and contains exactly one resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ and two vertices x, y of $X(G+H)$, which are common for all sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. Notice that $x \sim y$, otherwise the connectivity of nontrivial convex sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ implies that $\Gamma(x) = \Gamma(y) = X\backslash\{x, y\}$ and furthermore $d-conv(\{x, y\}) = X(G+H)$. Moreover, from the structure of graph $G+H$ and by Lemma 2, it follows that any vertex of $X(G+H)\backslash\{x, y\}$ is adjacent to x and to y. So, vertices x and y are universal in $G+H$.

Assume that there are two universal vertices x and y in $G+H$. We define a family of nontrivial convex sets $\mathcal{P}(G+H) = \varnothing$ that will cover $G+H$. We add set $\{x, y, z\}$ to $\mathcal{P}(G+H)$ for each $z \in X(H)\backslash\{x, y\}$. It is clear that $|\mathcal{P}(G+H)| = \varphi_{cn}^{max}(G+H) = n+m-2$. $\square$

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Linear stability interval for geometrical parameter of the Newtonian eight bodies problem

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Abstract

We consider the Newtonian restricted eight bodies problem with incomplete symmetry. The linear stability of stationary points of this problem are investigated by some numerical methods. For geometric parameter the intervals of stability and instability are found. All relevant and numerical calculations are done with the computer algebra system Mathematica.

Keywords: Newtonian problem; differential equation of motion; configuration; stationary points; linear stability.

1 Introduction

At present, qualitative studies of dynamical models of space are based on the search for exact particular solutions of differential equations of motion and subsequent analysis of their stability using the latest advances in computer mathematics. For this, it is required, first of all, to develop mathematical methods and algorithms for constructing exact partial solutions, since in the case of the Newtonian many bodies problem, for example, the number of solutions found is very limited.

Most of the exact solutions found for the Newtonian n-bodies problem belong to the class of so-called homographic solutions, the sufficient conditions for their existence were obtained by A. Wintner in the first half of the twentieth century, and the necessary conditions were formulated later by E. A. Grebenikov (see [3]).

The research method is based on the application of the analytic and qualitative theory of differential equations, the stability theory of Lyapunov-Poincaré, and also on the use of the capabilities of modern computer algebra systems for performing numerical calculations, processing symbolic information, and visualizing the obtained results.

2 Description of the configuration

We will study a particular case of the n-bodies problem describing in a non-inertial space P_0xyz the motion of seven bodies P_0, P_1, P_2, P_3, P_4, P_5, P_6, with the masses m_0, m_1, m_2, m_3, m_4, m_5, m_6, which attract each other in accordance with the law of universal attraction. We will investigate the planar dynamic pattern formed by a square in the vertices of which the points P_1, P_2, P_3, P_4, are located, the other two points P_5, P_6, having the masses $m_5 = m_6$ are on the diagonal P_1P_3 of the square at equal distances from point P_0, in around which this configuration rotates with a constant angular velocity ω which is determined from the model parameters. We can assume that $P_1(1,1)$, $P_2(-1,1)$, $P_3(-1,-1)$, $P_4(1,-1)$, $P_5(\alpha,\alpha)$, $P_6(-\alpha,-\alpha)$, $f = 1, m_0 = 1$, $m_5 = m_6$. Then out of the differential equations of the motion we obtain the existence conditions of this configuration:

$$\begin{cases} m_1 = m_3,\, m_2 = m_4 = f_1\left(m_1, \alpha\right), \\ m_5 = m_6 = f_2\left(m_1, \alpha\right),\, \omega^2 = f_3\left(m_1, \alpha\right). \end{cases} \tag{1}$$

Intervals of admissible values for the parameter α are determined by the conditions

$$m_2 = m_4 > 0; m_5 = m_6 > 0; \omega^2 > 0. \tag{2}$$

It is known that this dynamic model generates a new problem – the restricted problem of eight bodies. It will be studied the motion of the body P with a infinitely small mass (the so-called passive gravitational body) in the gravitational field by the given seven bodies.

Differential equations that describe motion of the body $P(x;y;z)$ which gravitates passively in the field of the other seven bodies in the rotating space have the form (see [2]):

$$\begin{cases} \dfrac{d^2X}{dt^2} - 2\omega\dfrac{dY}{dt} = \omega^2 X - \dfrac{fm_0X}{r^3} + \dfrac{\partial R}{\partial X}, \\ \dfrac{d^2Y}{dt^2} + 2\omega\dfrac{dX}{dt} = \omega^2 Y - \dfrac{fm_0Y}{r^3} + \dfrac{\partial R}{\partial Y}, \\ \dfrac{d^2Z}{dt^2} = -\dfrac{fm_0Z}{r^3} + \dfrac{\partial R}{\partial Z}, \end{cases} \tag{3}$$

where

$$\begin{cases} R = f\sum\limits_{j=1}^{6} m_j \left(\dfrac{1}{\Delta_{kj}} - \dfrac{XX_j + YY_j + ZZ_j}{r_j^3}\right), \\ \Delta_j^2 = (X_j - X)^2 + (Y_j - Y)^2 + (Z_j - Z)^2, \\ r_j^2 = X_j^2 + Y_j^2 + Z_j^2,\ r^2 = X^2 + Y^2 + Z^2, \\ j = 1, 2, ..., 6, \end{cases} \tag{4}$$

$(X_j; Y_j; Z_j = 0)$ are the respective coordinates of the bodies P_1, P_2, P_3, P_4, P_5, P_6 and are determined by the conditions of existence of the studied configuration.

3 Determination of stationary points

Using the graphical possibilities of the system Mathematica, the stationary points of the system (3) were determined. For this we use an algorithm similar to the algorithms from [1] (see [3] for other configurations). For concrete values of m_1 and α we obtain concrete stationary points. Their linear stability is studied.

Theorem. *There are values of the parameters m_1 and α for which the bisectorial stationary points in the restricted eight bodies problem are stable in the first approximation.*

4 Determination of the admissible variations intervals for the parameters

Moreover, we obtain that only for $0.85812 < \alpha < 0.85854$ and $m_1 = 0.01$ there are stationary points in the research problem that are linearly stable.

5 Concluding remarks

We have determined sufficient existence conditions of configuration describing the restricted Newtonian eight bodies problem. We have used some built in functions of the Mathematica programming environment in order to determine the stationary points. Their linear stability has been studied. It has been demonstrated that there are values of the parameters m_1 and α for which the bisectorial stationary points are stable in the first approximation. Intervals of stability and instability for geometric parameter are found.

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Parallel algorithm for solving 2D block-cyclic distributed bimatriceal games

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Abstract

The ways for solving 2D block-cyclic distributed bimatriceal games are discussed. Also the parallel algorithm for clusters parallel systems with shared and distributed memory to determine the Nash solutions in the bimatrix games are presented.

Keywords: non cooperative game, payoff function, set of strategies, Nash equilibrium, parallel algorithms, distributed memory clusters.

1 Processing grid and block-cyclic distribution of the matrices

The P processes of an abstract parallel computer are often represented as a one-dimensional linear array of processes labeled $0, 1, \ldots, P$. It is often more convenient to map this one-dimensional array of processes into a two-dimensional rectangular grid, or process grid by using row-major order. This grid will have $l_{\max}$ process rows and $c_{\max}$ process columns, where $l_{\max}+c_{\max} = P$. A process can now be referenced by its row and column coordinates, (l, c), within the grid, where $0 \leq l \leq l_{\max}$, and $0 \leq c \leq c_{\max}$.

The choice of an appropriate data distribution heavily depends on the characteristics or flow of the computation in the algorithm. For dense matrix computations, we assume the data to be distributed according to the two-dimensional block-cyclic data layout scheme (2D-cyclic algorithm) [1]. The block-cyclic data layout has been selected for the dense algorithms implemented in DMM parallel systems principally because of its scalability, load balance, and efficient use of computation routines (data locality). The block-partitioned computation proceeds in consecutive order just like a conventional serial algorithm. According to the two-dimensional block cyclic data distribution scheme, an m by n dense matrix is first decomposed into m_A rows by n_A columns blocks starting at its upper left corner. These blocks are then uniformly distributed in each dimension of the process grid. Thus, every process owns a collection of blocks, which are locally and contiguously stored in a two-dimensional column major array.

2 Nash equilibrium profiles in the block-cyclic distributed bimatriceal games

We consider the bimatrix game in the following strategic form: $\Gamma = \langle I, J, A, B\rangle$, where $I = \{1, 2, .., n\}$ is the set of strategies of the player 1, $J = \{1, 2, .., m\}$ the set of strategies of the player 2 and $A = ||a_{ij}||_{i\in I}^{j\in J}$, $B = ||b_{ij}||_{i\in I}^{j\in J}$ are the payoff matrices of player 1 and player 2, respectively. All players know exactly the payoff matrices and the sets of strategies. So the game is in complete and imperfect information. Players maximize their payoffs. The matrices A and B are called global matrices. Denote by $NE[\Gamma]$ the set of all Nash equilibrium profiles in the game Γ.

Suppose that the global matrix A and B are distributed on the $L \times C$ process grid according to the two-dimensional block cyclic data distribution scheme. And so, the process can now be referenced by its row and column coordinates, (c, l), within the grid $L \times C$, where $L = \{1, .., l, ..., l_{\max}\}$ is the number of rows process and $C = \{1, .., c, ..., c_{\max}\}$ is the number of column process. Denote by $A_{(c,l)} = \left\| a_{i_{(l,c)} j_{(l,c)}} \right\|_{i_{(l,c)}=\overline{1,|I_{(l,c)}|}}^{j_{(l,c)}=\overline{1,|J_{(l,c)}|}}$ and $B_{(c,l)} = \left\| b_{i_{(l,c)} j_{(l,c)}} \right\|_{i_{(l,c)}=\overline{1,|I_{(l,c)}|}}^{j_{(l,c)}=\overline{1,|J_{(l,c)}|}}$ submatrices, from global matrices A and B, that are distributed to the $(c, l) \in L \times C$ process. So we obtain the series of bimatriceal subgames in complete and perfect information in the following strategic form $\Gamma_{(c,l)} = \left\langle I_{(c,l)}, J_{(c,l)}, A_{(c,l)}, B_{(c,l)} \right\rangle$. Denote by $NE\left[\Gamma_{(c,l)}\right]$ the set of all Nash equilibrium profiles in the game $\Gamma_{(c,l)}$.

After the distribution of global matrices A, B that takes place on the process grid based on the 2D-cyclic algorithm, each process $(l,c) \in L \times C$ determines the equilibrium profile $\left(i^*_{(l,c)}, j^*_{(l,c)}\right) \in NE\left[\Gamma_{(c,l)}\right]$ based on the following algorithm.

1. for each fixed column $j_{(l,c)} \in J_{(l,c)}$, respectively line $i_{(l,c)} \in I_{(l,c)}$, there is determined $i^*_{(l,c)}(j_{(l,c)}) = \arg \max\limits_{i_{(l,c)} \in I_{(l,c)}} a_{i_{(l,c)} j_{(l,c)}}$, and respectively $j^*_{(l,c)}(i_{(l,c)}) = \arg \max\limits_{j_{(l,c)} \in J_{(l,c)}} b_{i_{(l,c)} j_{(l,c)}}$.
2. The graphics $gr\_i^*_{(l,c)}$, $gr\_j^*_{(l,c)}$ of the point-to-set mappings $i^*_{(l,c)}(\cdot)$, $j^*_{(l,c)}(\cdot)$ are built and $NE\left[\Gamma_{(c,l)}\right] = gr\_i^*_{(l,c)} \cap gr\_j^*_{(l,c)}$.

We will introduce the following applications that compute the global row or column index of a distributed matrix $A_{(l,c)}$, $B_{(l,c)}$ entry pointed to by the local index, namely $\varphi_{(l,c)} : I_{(l,c)} \to I$, $\psi_{(l,c)} : J_{(l,c)} \to J$ for which the following conditions are fulfilled:

$$\forall i \in I, \exists l = \overline{1, l}_{\max}, c = \overline{1, c}_{\max}, i_{(l,c)} \in I_{(l,c)}, i = \varphi_{(l,c)}\left(i_{(l,c)}\right), \quad (1)$$
$$\forall j \in J, \exists l = \overline{1, l}_{\max}, c = \overline{1, c}_{\max}, j_{(l,c)} \in J_{(l,c)}, j = \psi_{(l,c)}\left(j_{(l,c)}\right) \quad (2)$$

It can easily be shown that for the 2D-cyclic algorithm the functions $\varphi_{(l,c)}$ and $\psi_{(l,c)}$ in addition to the conditions (1-2), may also fulfil the following conditions:

a) for all fixed $l = \overline{1, l}_{\max}$, $\varphi_{(l,\overline{c})}\left(i_{(l,\overline{c})}\right) = \varphi_{(l,\widetilde{c})}\left(i_{(l,\widetilde{c})}\right)$ for all $\overline{c} = \overline{1, c}_{\max}, \widetilde{c} = \overline{1, c}_{\max}$ and $\overline{c} \neq \widetilde{c}$.
b) for all fixed $c = \overline{1, c}_{\max}$, $\psi_{(\overline{l},c)}\left(j_{(\overline{l},c)}\right) = \psi_{(\widetilde{l},c)}\left(j_{(\widetilde{l},c)}\right)$ for all $\overline{l} = \overline{1, l}_{\max}, \widetilde{l} = \overline{1, l}_{\max}$ and $\overline{l} \neq \widetilde{l}$.

Using the above, we can prove the following theorem.

Theorem. *Let there is a process $(l,c) \in L \times C$ for which $\left(i^*_{(l,c)}, j^*_{(l,c)}\right) \in NE\left[\left(A_{(l,c)}, B_{(l,c)}\right)\right]$. Then if*

1. *for all process* $(\widetilde{l}, c)$, $\widetilde{l} \neq l$ *from the column* c *of the process grid* $L \times C$ *for which it exists* $\left(i^*_{(\widetilde{l},c)}, j^*_{(\widetilde{l},c)}\right) \in NE\left[\left(A_{(\widetilde{l},c)}, B_{(\widetilde{l},c)}\right)\right]$, *the following condition is fulfilled* $a_{i_{(l,c)}, j_{(l,c)}} \geq a_{i^*_{(\widetilde{l},c)}, j^*_{(\widetilde{l},c)}}$.
2. *for all process* $(l, \widetilde{c})$, $\widetilde{c} \neq c$ *from the line* l *of the process grid* $L \times C$ *for which it exists* $\left(i^*_{(l,\widetilde{c})}, j^*_{(l,\widetilde{c})}\right) \in NE\left[\left(A_{(l,\widetilde{c})}, B_{(l,\widetilde{c})}\right)\right]$ *the following condition is* fulfilled $b_{i_{(l,c)}, j_{(l,c)}} \geq b_{i^*_{(l,\widetilde{c})}, j^*_{(l,\widetilde{c})}}$.

Then $\left(\varphi_{(l,c)}\left(i^*_{(l,c)}\right), \psi_{(l,c)}\left(j^*_{(l,c)}\right)\right) \in NE[\Gamma]$.

3 Conclusion

One of the main problem in constructing and soft implementation of parallel algorithms on the DMM clusters systems is the data parallelizations. In this article we use the two-dimensional block cyclic data distribution scheme for data parallelizations. For the described above algorithm there has been developed a C++ program using MPI functions and ScaLAPACK routines. Program has been testing on the control examples on the Moldova State University HPC cluster. The test results were consistent with theoretical results.

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Green's function for the telegraph equation on the unit sphere in $\mathbb{R}^3$

Alexander D. Kolesnik

Abstract

We consider the telegraph equation on the surface of unit sphere in the Euclidean space $\mathbb{R}^3$. The explicit formula for the Green's function of the respective Cauchy problem, is given.

Keywords: Telegraph equation, Cauchy problem, Green's function, unit sphere

Consider the following hyperbolic diffusion equation, also known as telegraph equation (see [1]) or relativistic diffusion equation on sphere:

$$\frac{1}{c^2}\frac{\partial^2 \tilde{p}(\mathbf{x},t)}{\partial t^2} + \frac{1}{D}\frac{\partial \tilde{p}(\mathbf{x},t)}{\partial t} = k^2 \Delta_{S_2}\, \tilde{p}(\mathbf{x},t), \tag{1}$$

with the initial conditions:

$$\tilde{p}(\mathbf{x},t)|_{t=0} = \delta(\mathbf{x}), \qquad \left.\frac{\partial \tilde{p}(\mathbf{x},t)}{\partial t}\right|_{t=0} = 0, \tag{2}$$

$$\mathbf{x} = (x_1, x_2, x_3) \in S_2, \quad t \geq 0,$$

where $c > 0,\ D > 0,\ k,$ are some constants, Δ_{S_2} is the Laplace operator on the unit sphere S_2 and $\delta(\mathbf{x})$ is the Dirac delta-function.

By passing to unit spherical coordinates, equation (1) becomes:

$$\frac{1}{c^2}\frac{\partial^2 p(\theta,\varphi,t)}{\partial t^2} + \frac{1}{D}\frac{\partial p(\theta,\varphi,t)}{\partial t} = k^2 \Delta_{(\theta,\varphi)}\, p(\theta,\varphi,t), \tag{3}$$

$$\theta \in [0,\pi),\ \varphi \in [0,2\pi),\ t > 0,$$

where

$$\Delta_{(\theta,\varphi)} = \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2} \tag{4}$$

is the Laplace-Beltrami operator on sphere.

It is known (see, i.e., [2, p.72]) that the eigenvalue problem for Laplace operator on sphere has the following exact solution:

$$\Delta_{S_2}\, Y_{lm}(\mathbf{x}) = -l(l+1) Y_{lm}(\mathbf{x}), \qquad l = 0, 1, 2, \dots, \quad m = -l, \dots, 0, \dots, l,$$

where $\{Y_{lm}(\mathbf{x})\}$ is the system of spherical harmonics. Therefore, it is natural to seek a solution of Cauchy problem (1)-(2) in the form:

$$\tilde{p}(\mathbf{x}, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm}(t)\, Y_{lm}(\mathbf{x}), \tag{5}$$

where

$$b_{lm}(t) = \int_{S_2} \tilde{p}(\mathbf{x}, t)\, Y_{lm}^*(\mathbf{x})\, \tilde{\sigma}(d\mathbf{x}), \qquad \tilde{\sigma}(d\mathbf{x}) = \sin\theta d\theta d\varphi \tag{6}$$

and $Y_{lm}^*(\mathbf{x})$ means the complex conjugation of $Y_{lm}(\mathbf{x})$.

By substituting function (5) into equation (1), we obtain:

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[\frac{1}{c^2}\frac{\partial^2 b_{lm}(t)}{\partial t^2} + \frac{1}{D}\frac{\partial b_{lm}(t)}{\partial t} + l(l+1)k^2 b_{lm}(t)\right] Y_{lm}(\mathbf{x}) = 0. \tag{7}$$

To find particular solutions of (7), we need to solve the equation

$$\frac{1}{c^2}\frac{\partial^2 b_{lm}(t)}{\partial t^2} + \frac{1}{D}\frac{\partial b_{lm}(t)}{\partial t} + l(l+1)k^2 b_{lm}(t) = 0. \tag{8}$$

The initial conditions for equation (8) determined from (6) and (2) are:

$$b_{lm}(t)|_{t=0} = 4\pi Y_{lm}^*(\mathbf{0}), \qquad \left.\frac{\partial b_{lm}(t)}{\partial t}\right|_{t=0} = 0. \tag{9}$$

Characteristic equation for (8) reads:

$$\frac{1}{c^2}z^2 + \frac{1}{D}z + l(l+1)k^2 = 0, \tag{10}$$

whose roots are: $z_{1,2} = -\frac{c^2}{2D} \pm \sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}$.

Then the general solution of equation (8) is: $b_{lm}(t) = K_1 e^{z_1 t} + K_2 e^{z_2 t}$, where K_1, K_2 are some constants. From initial conditions (9) we obtain the system of linear equations for determining these coefficients: $K_1 + K_2 = 4\pi Y^*_{lm}(\mathbf{0})$, $z_1 K_1 + z_2 K_2 = 0$, whose solution is: $K_{1,2} = 4\pi\left(\frac{1}{2} \pm \frac{c^2}{4D\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}}\right) Y^*_{lm}(\mathbf{0})$.

Thus, the solution of Cauchy problem (8)-(9) has the form:

$$\begin{aligned} b_{lm}(t) = 4\pi &\left(\frac{1}{2} + \frac{c^2}{4D\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}}\right) Y^*_{lm}(\mathbf{0}) \\ &\times \exp\left[-t\left(\frac{c^2}{2D} - \sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}\right)\right] \\ + 4\pi &\left(\frac{1}{2} - \frac{c^2}{4D\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}}\right) Y^*_{lm}(\mathbf{0}) \\ &\times \exp\left[-t\left(\frac{c^2}{2D} + \sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}\right)\right]. \end{aligned}$$

Returning now to (5), we obtain the solution of Cauchy problem (1)-(2) in the form:

$$\tilde{p}(\mathbf{x},t) = 4\pi\ \exp\left(-\frac{c^2 t}{2D}\right)\sum_{l=0}^{\infty} Q_l(\mathbf{x}) \\ \times \Biggl\{ \left(\frac{1}{2} + \frac{c^2}{4D\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}}\right) e^{\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}} \\ + \left(\frac{1}{2} - \frac{c^2}{4D\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}}\right) e^{-\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}} \Biggr\}, \tag{11}$$

where $Q_l(\mathbf{x}) = \sum_{m=-l}^{l} Y_{lm}^*(\mathbf{0})\ Y_{lm}(\mathbf{x})$.

Some simple transformations in (11) yield the final formula for the Green's function of Cauchy problem (1)-(2):

$$\tilde{p}(\mathbf{x},t) = 4\pi\ \exp\left(-\frac{c^2 t}{2D}\right)\sum_{l=0}^{\infty} Q_l(\mathbf{x}) \Biggl\{ \cosh\left(t\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}\right) \\ + \frac{c^2}{2D\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}}\ \sinh\left(t\sqrt{\frac{c^4}{4D^2} - c^2 l(l+1)k^2}\right) \Biggr\}.$$

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Optimal control of a stochastic system related to the Kermack-McKendrick model

Mario Lefebvre

Abstract

A stochastic optimal control problem for a two-dimensional system of differential equations that is related to the Kermack-McKendrick model for the spread of epidemics is considered. The aim is to maximize the expected value of the time during which the epidemic is under control, taking the quadratic control costs into account. An exact and explicit solution is found in a particular case.

Keywords: Dynamic programming, Brownian motion, first-passage time, partial differential equations, error function.

1 Introduction and theoretical result

Let $x(t)$ represent the percentage of individuals susceptible of being infected with a given virus in a certain population, and let $y(t)$ be the percentage of infected carriers. We consider the following controlled two-dimensional system of differential equations to model the spread of epidemics:

$$\begin{aligned}
\dot{x}(t) &= -k_1 x(t) y(t), \\
\dot{y}(t) &= k_1 x(t) y(t) - k_2[x(t) + y(t)] + b[x(t), y(t)]\, u(t) \\
&\quad + \{v[x(t), y(t)]\}^{1/2}\, \dot{B}(t),
\end{aligned}$$

where k_1 and k_2 are positive constants, $u(t)$ is the control variable, the function $v[x(t), y(t)]$ is positive and $B(t)$ is a standard Brownian motion.

If $u(t)$ and $v[x(t), y(t)]$ are identical to zero, then the above system is a slight modification of the classic two-dimensional Kermack-McKendrick model for the spread of epidemics that they proposed in their paper published in 1927.

Let $x(0) = x$ and $y(0) = y$ be such that $0 < x + y < d$, where d is a value for which the epidemic is considered to be under control, and define the *first-passage time* $T(x, y) = \inf\{t \geq 0 : x(t) + y(t) = 0 \text{ or } d\}$. Our aim is to find the value u^* of the control variable that minimizes the expected value of the cost criterion

$$J(x, y) = \int_0^T \left\{ \tfrac{1}{2} q[x(t), y(t)]\, u^2(t) + \lambda \right\} \mathrm{d}t,$$

where $q[x(t), y(t)]$ is a positive function and λ is a *negative* constant. Hence, the aim is to maximize the expected time during which the epidemic is under control, taking the quadratic control costs into account.

This type of optimal control problem, for which the final time is a random variable, has been termed *LQG homing* by Whittle (1982). Such problems are generally very difficult to solve explicitly, especially in two or more dimensions. LQG homing problems have been considered, in particular, by the author and Zitouni (2012 and 2014). Makasu (2013) solved explicitly a two-dimensional LQG homing problem.

Let $F(x, y) = \inf_{u(t), 0 \leq t \leq T} E[J(x, y)]$. Making use of dynamic programming, we find that *the value function* F satisfies the partial differential equation

$$\tfrac{1}{2} q u^2 + \lambda - k_1 x y F_x + [k_1 x y - k_2(x + y) + b u] F_y + \tfrac{1}{2} v F_{yy} = 0, \tag{1}$$

where $u = u(0)$ and all the functions are evaluated at $t = 0$. The boundary conditions are $F(x, y) = 0$ if $x + y = 0$ or d.

We deduce from Eq. (1) that the optimal control u^* can be expressed in terms of the function F as follows: $u^* = -\frac{b}{q} F_y$. Substituting this expression into Eq. (1), we obtain that the function F satisfies the non-linear partial differential equation

$$-\frac{b^2}{2q} F_y^2 + \lambda - k_1 x y F_x + [k_1 x y - k_2(x + y)] F_y + \tfrac{1}{2} v F_{yy} = 0. \tag{2}$$

We will try to solve the above equation by making use of the *method of similarity solutions*. More precisely, based on the boundary conditions, we look for a solution of the form $F(x, y) = H(w)$, where $w := x + y$ is the *similarity variable*. We find that Eq. (2) is transformed into the non-linear *ordinary* differential equation

$$-\frac{b^2}{2q}[H'(w)]^2 + \lambda - k_2 w H'(w) + \tfrac{1}{2} v H''(w) = 0. \tag{3}$$

The new boundary conditions are

$$H(w) = 0 \quad \text{if } w = 0 \text{ or } d. \tag{4}$$

Proposition 1.1. *If the ratio* b^2/q *and the function* v *can be expressed in terms of the similarity variable* w*, then the optimal control can be deduced from the solution of* (3), (4)*.*

2 A particular case

Let us choose the following particular values for the various terms that appear in the problem set up in Section 1: $\lambda = -1$, $b \equiv 1$, $q \equiv 1$, $v \equiv 1$ and $k_2 = 1$. Thus, we assume that the functions b, q and v are all identical to 1. Notice that Proposition 1.1 then applies, because the ratio b^2/q and v are both constant functions. Moreover, we take $d = 1$.

Next, the differential equation that we must solve becomes

$$-\tfrac{1}{2}[H'(w)]^2 - 1 - wH'(w) + \tfrac{1}{2}H''(w) = 0.$$

Making use of the mathematical software Maple, we find that the solution of the above equation that satisfies the boundary conditions $H(0) = H(1) = 0$ is the following:

$$H(w) = -\frac{1}{2}\ln\left(\frac{\left(w\sqrt{\pi}(1-e) + iw\pi\left(\operatorname{erf}(iw) - \operatorname{erf}(i)\right) + \sqrt{\pi}e^{w^2}\right)^2}{\pi}\right),$$

where erf is the error function. From $H(w)$, we obtain the value function $F(x, y)$, and hence the optimal control $u^* = -F_y$ explicitly.

3 Concluding remarks

In this note, we were able to obtain an explicit solution to a two-dimensional LQG homing problem by making use of the method of similarity solutions. It would be interesting to obtain at least approximate solutions to problems for which we cannot make use of this method. We could also try to solve the appropriate partial differential equation numerically. Finally, we could try to find suboptimal controls, either by making some approximations, or by choosing the form of the control variable (for instance, a linear control).

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Pure and Mixed Stationary Strategies for Stochastic Positional Games

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Abstract

A class of stochastic positional games with average and discounted payoffs that generalizes the deterministic positional games is formulated and studied. Conditions for the existence of Nash equilibria in pure and mixed stationary strategies for the considered class of games are derived.

Keywords: stochastic positional games, average payoffs, discounted payoffs, pure and mixed stationary strategies, Nash equilibrium.

1 Introduction and Problem Formulation

We study a class of stochastic positional games that generalizes the deterministic positional games [1,2]. The considered class of stochastic games we formulate and study by applying the concept of positional games to Markov decision processes with average and discounted optimization criteria. An m-player stochastic positional game consists of the following elements:

- a state space X (which we assume to be finite);
- a partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$, where X_i represents the position set of player $i \in \{1, 2, \ldots, m\}$;
- a finite set $A(x)$ of actions in each state $x \in X$;
- a step reward $f^i(x, a)$ with respect to each player $i \in \{1, 2, \ldots, m\}$, each state $x \in X$ and for an arbitrary action $a \in A(x)$;
- a transition probability function $p : X \times \prod_{x \in X} A(x) \times X \rightarrow [0, 1]$

that gives the probability transitions $p^a_{x,y}$ from an arbitrary $x \in X$ to an arbitrary $y \in X$ for a fixed action $a \in A(x)$, where $\sum\limits_{y \in X} p^a_{x,y} = 1, \quad \forall x \in X, \ a \in A(x)$;

- a starting state $x_0 \in X$.

The game starts in the state x_0 at the moment of time $t = 0$. The player $i \in \{1, 2, \dots, m\}$ who is the owner of the state position x_0 ($x_0 \in X_i$) chooses an action $a_0 \in A(x_0)$ and determines the corresponding rewards $f^1(x_0, a_0), f^2(x_0, a_0), \dots, f^m(x_0, a_0)$ for players $1, 2, \dots, m$. After that the game passes to a state $y = x_1 \in X$ according to probability distribution $\{p^{a_0}_{x_0,y}\}$. At the moment of time $t = 1$ the player $k \in \{1, 2, \dots, m\}$ who is the owner of the state position x_1 ($x_1 \in X_k$) chooses an action $a_1 \in A(x_1)$ and players $1, 2, \dots, m$ receive the corresponding rewards $f^1(x_1, a_1), f^2(x_1, a_1), \dots, f^m(x_1, a_1)$. Then the game passes to a state $y = x_2 \in X$ according to probability distribution $\{p^{a_1}_{x_1,y}\}$ and so on indefinitely. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \dots, x_t, a_t, \dots$ that defines the stage rewards $f^1(x_t, a_t), f^2(x_t, a_t), \dots, f^m(x_t, a_t), \quad t = 0, 1, 2, \dots$. *The average stochastic positional game* is the game with payoffs

$$\omega^i_{x_0} = \lim_{t \to \infty} \inf \ \mathsf{E}\left(\frac{1}{t}\sum_{\tau=0}^{t-1} f^i(x_\tau, a_\tau)\right), \quad i = 1, 2, \dots, m$$

A *discounted stochastic positional game* is the game with payoffs

$$\sigma^i_{x_0} = \mathsf{E}\left(\sum_{\tau=0}^{\infty} \gamma^\tau f^i(x_\tau, a_\tau)\right), \quad i = 1, 2, \dots, m,$$

where γ is a given discount factor ($0 < \gamma < 1$).

We study the problem of the existence Nash equilibria for the considered games when players use pure and mixed stationary strategies of selection the action in the states.

2 Pure and Mixed Stationary Strategies

A strategy of player $i \in \{1, 2, \ldots, m\}$ in a stochastic positional game is a mapping s^i that provides for every state $x_t \in X_i$ a probability distribution over the set of actions $A(x_t)$. If these probabilities take only values 0 and 1, then s^i is called *a pure strategy*, otherwise s^i is called *a mixed strategy*. If these probabilities depend only on the state $x_t = x \in X_i$ (i. e. s^i do not depend on t), then s^i is called *a stationary strategy*. Thus we can identify the set of mixed stationary strategies $\mathbf{S}^i$ of player i with the set of solutions of the system

$$\left\{ \begin{array}{lll} \sum\limits_{a \in A(x)} s^i_{x,a} = 1, & \forall x \in X_i; & \\ s^i_{x,a} \geq 0, & \forall x \in X_i, \ \ \forall a \in A(x) & \end{array} \right. , \qquad (1)$$

where $s^i_{x,a}$ expresses the probability that player i chooses action $a \in A(x)$ in $x \in X_i$. Each basic solution of this system corresponds to a pure stationary strategy.

3 The main results

Let $\mathbf{s} = (s^1, s^2, \ldots, s^m)$ be a profile of stationary strategies of the players. Then $\mathbf{s}$ induces the probability transition matrix $P^{\mathbf{s}} = (p^{\mathbf{s}}_{x,y})$, where

$$p^{\mathbf{s}}_{x,y} = \sum_{a \in A(x)} s^i_{x,a} p^a_{x,y} \quad \text{for} \quad x \in X_i, \ \ i = 1, 2, \ldots, m. \qquad (2)$$

Therefore if $Q^{\mathbf{s}} = (q^{\mathbf{s}}_{x,y})$ is the limiting probability matrix of $P^{\mathbf{s}}$, then the average payoffs per transition $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}), \ldots, \omega^m_{x_0}(\mathbf{s})$ for the players are determined as follows

$$\omega^i_{x_0}(\mathbf{s}) = \sum_{k=1}^{m} \sum_{y \in X_k} q^{\mathbf{s}}_{x_0,y} f^i(y, s^k), \quad i = 1, 2, \ldots, m, \qquad (3)$$

where

$$f^i(y, s^k) = \sum_{a \in A(y)} s^k_{y,a} f^i(y, a), \ \text{ for } \ y \in X_k, \ k \in \{1, 2, \ldots, m\} \qquad (4)$$

expresses the step reward of player i in the state $y \in X_k$ when player k uses the strategy s^k. The functions $\omega^1_{x_0}(\mathbf{s})$, $\omega^2_{x_0}(\mathbf{s})$, $\ldots$, $\omega^m_{x_0}(\mathbf{s})$ on $\mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \cdots \times \mathbf{S}^m$, defined according to (3), (4), determine a game in normal form that we denote $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$. We have shown that this game corresponds to the *average stochastic positional game in stationary strategies* and the following theorem holds.

Theorem 1. *An arbitrary average stochastic positional game possesses a Nash equilibrium in mixed stationary strategies.*

A discounted stochastic game also can be defined in normal form as $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\sigma^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$, where

$$\sigma^i_{x_0}(\mathbf{s}) = \sum_{k=1}^{m} \sum_{y \in X_k} w^{\mathbf{s}}_{x_0,y} f^i(y, s^k), \quad i = 1, 2, \ldots, m, \tag{5}$$

and $w_{x,y}$ represent the elements of matrix $W^{\mathbf{s}} = (I - \gamma P^{\mathbf{s}})^{-1}$. For such a game we proved the following result.

Theorem 2. *An arbitrary discounted stochastic positional game possesses a Nash equilibrium in pure stationary strategies.*

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On a Sequent Form of the Inverse Method

Alexander Lyaletski

Abstract

The Maslov inverse method for establishing of deducibility of special form formulas in classical first-order logic is considered. Its treatment as a special sequent calculus is given.

Keywords: first-order classical logic, inference search, inverse method, sequent calculus.

A sequent interpretation of the Maslov inverse method (MIM) [1] for establishing of deducibility in classical first-order logic is given below. It has the form of a specific calculus of so-called "favorable" sequents consisting of formulas of a special form.

Note that for reasoning on satisfiability, the most adequate interpretation of MIM was independently given in [2] and [3]. The paper [4] contains an interpretation of MIM in special sequent terms, but that study was made for a partial case of sequents being "input" for MIM.

What follows below can be considered as a "symbiosis" of these treatments having the following singularities. First, all reasonings are made on deducibility; at that, as in the case of [1], a starting point for our consideration is sequents of a special form. Second, a certain "transition" from such a sequent to the construction of a special sequent calculus is made, which, on the author's opinion, gives the most adequate treatment of MIM.

1. Classical first-order sequent logic without equality is considered.

If E is an expression and σ a substitution [5], then $E\sigma$ is the result of the application of σ to E.

As usual, a literal is an atomic formula or its negation. If L is a literal, then $\tilde{L}$ denotes its *complement* [5].

If $L_1, \ldots, L_m$ are literals, then the expression $L_1 \vee \ldots \vee L_m$ ($L_1 \wedge \ldots \wedge L_m$) is called a *disjunct* (a *conjunct*). The disjunct not containing literals is denoted by $\Box$. (A disjunct is the same as a *clause* in [5]).

A *disjunctive conjunct*, or *d-conjunct* (cf. [2]) is an expression of the form $D_1 \wedge \ldots \wedge D_r$, where $D_1, \ldots, D_r$ are disjuncts.

Draw your attention to the fact that *conjuncts, disjuncts, and d-conjuncts are considered up to the order of writing their elements*.

It is known that using skolemization and equivalent logical transformations, the establishing of the deducibility of any sequent of the form $\rightarrow F$ can be reduced to the establishing of the deducibility of a *starting sequent* of the form $\rightarrow C_1, \ldots, C_n$, where $C_1, \ldots, C_n$ are d-conjuncts being existential prenex formulas with missed quantifiers and $C_1, \ldots, C_n$ have no common variables in pairs.

Let D be a disjunct, $x_1, \ldots, x_k$ the list of all different variables from D written in the order of looking through D from left to right, and P_D a new predicate symbol of the arity k. Then $P_D(x_1, \ldots, x_k)$ is called an *abbreviation* for D.

2. As in the case of the original inverse method in [1], our sequent treatment of MIM has the form of a so-called favorable sequent calculus containing two inference rules denoted by **A** and **B**.

*Rule **A*** (for generating an *initial favorable sequent*). Let S be a starting sequent $\rightarrow D_{1,1} \wedge \ldots \wedge D_{1,r_1}, \ldots, D_{n,1} \wedge \ldots \wedge D_{n,r_n}$, where $D_{i,j}$ is a disjunct. Then an *initial favorable sequent S_f, deducible by the **A** rule*, is a sequent $D'_1, \ldots, D'_m \rightarrow P_{1,1} \wedge \ldots \wedge P_{1,r_1}, \ldots, P_{n,r_n} \wedge \ldots \wedge P_{n,r_n}$, where $P_{1,1}, \ldots, P_{n,r_n}$ are abbreviations of the disjuncts $D_{1,1}, \ldots, D_{n,r_n}$ respectively and for each $i = 1, \ldots, m$, there are j, k, j' and k' such that D'_i is a variant of $P_{j,k}\sigma \vee P_{j',k'}\sigma$ satisfying the following condition: in $D_{j,k}$ and $D_{j',k'}$, there exist such literals $L_{j,k}$ and $E_{j',k'}$ respectively, that σ is the most general unifier [5] of the set $\{L_{j,k}, \tilde{E}_{j',k'}\}$.

Remark. It is required that S_f always contains only all possible clauses D'_i, not being variants of each other. As a result, S_f contains *only the finite number of two-literals disjuncts in its antecedent and the finite number of conjuncts in its succedent*. Also, it is clear that *there are no occurrences of the negation sign in all the formulas of S_f*.

Rule ***B***. Let an already generated favorable sequent S of the form $D'_1, \ldots, D'_m \rightarrow P_{1,1} \wedge \ldots \wedge P_{1,r_1}, \ldots, P_{n,1} \wedge \ldots \wedge P_{n,r_n}$ contain such a conjunct $P_{i,1} \wedge \ldots \wedge P_{i,r_i}$ in its succedent and such disjuncts $D_{i,1} \vee A_{i,1}, \ldots, D_{i,r_i} \vee A_{i,r_i}$ in its antecedent, where $D'_1, \ldots, D'_m, D_{i,1}, \ldots, D_{i,r_i}$ are disjuncts, $P_{1,1}, \ldots, P_{n,r_n}$ abbreviations, and $A_1, \ldots, A_q$ atomic formulas, that there exists the most general simultaneous unifier σ [5] of the sets $\{A_{i,1}, P_{i,1}\} \ldots, \{A_{i,r_i}, P_{i,r_i}\}$. Then for any variant D of $D_{i,1}\sigma \vee \ldots \vee D_{i,r_i}\sigma$, the *sequent* $D'_1, \ldots, D'_m, D \rightarrow P_{1,1} \wedge \ldots \wedge P_{1,r_1}, \ldots, P_{n,r_n} \wedge \ldots \wedge P_{n,r_n}$ *is said to be deducible from* S *by the* ***B*** *rule.*

Inference search in the favorable sequent calculus begins with the construction by the **A** rule the initial favorable sequent S_f for a starting sequent S under consideration. After this, we try to deduce by only **B** rule applications a so-called *final favorable sequent*, that is a sequent, containing the empty disjunct $\Box$ in its antecedent. (Note that during deduction no formula is added to the succedents of favorable sequents.)

3. The following result is a sequent modification of the theorem on the soundness and completeness of the Maslov inverse method.

Theorem. *A starting sequent S is deducible in the Gentzen sequent calculus* ***LK*** *if and only if a final favorable sequent is deducible from S_f in the favorable sequent calculus.*

Corollary. *A set of d-conjuncts $\{C_1, \ldots, C_n\}$ is valid if and only if a final favorable sequent is deducible from S_f in the favorable sequent calculus, where S_f is an initial favorable sequent for $\rightarrow C_1, \ldots, C_n$.*

Example. Let S_0 denote $\rightarrow Q(y) \wedge (R(y,x) \vee \tilde{Q}(f(y)))$, $\tilde{Q}(f(u)) \vee \tilde{R}(u,u)$ being a starting sequent, where Q and R are predicate symbols, f is a functional symbol, and x, y, u are variables.

Let $P_1(y)$ be an abbreviation of $Q(y)$, $P_2(y,x)$ an abbreviation of $R(y,x) \vee \tilde{Q}(f(x))$, and $P_3(u)$ an abbreviation of $\tilde{Q}(f(u)) \vee \tilde{R}(u,u)$.

We can construct the following inference in our calculus:

(1)$P_1(f(u')) \vee P_3(u')$, (2)$P_2(u'',u'') \vee P_3(u'')$, (3)$P_1(f(y')) \vee P_2(y',x') \rightarrow$ (4)$P_1(x) \wedge P_2(y,x)$, (5)$P_3(u)$ (by **A**);

(1)$P_1(f(u)) \vee P_3(u)$, (2)$P_2(u',u') \vee P_3(u')$, (3)$P_1(f(y')) \vee P_2(y',x')$, (6)$P_3(z) \vee P_3(f(z)) \rightarrow$ (4)$P_1(x) \wedge P_2(y,x)$, (5)$P_3(u)$ (from (1),(2),(4) by **B**);

$(1)P_1(f(u)) \vee P_3(u)$, $(2)P_2(u', u') \vee P_3(u')$, $(3)P_1(f(y')) \vee P_2(y', x')$, $(6)P_3(z) \vee P_3(f(z))$, $(7)P_3(z') \to P_1(x) \wedge P_2(y, x)$, $P_3(u)$ (from (6),(5) by **B**);

$(1)P_1(f(u)) \vee P_3(u)$, $(2)P_2(u', u') \vee P_3(u')$, $(3)P_1(f(y')) \vee P_2(y', x')$, $(6)P_3(z) \vee P_3(f(z))$, $(7)P_3(z')$ $(8)\Box \to P_1(x) \wedge P_2(y, x)$, $P_3(u)$ (from (7),(5) by **B**).

Since the last sequent contains $\Box$, then it is final and S_0 is deduced in **LK** according to Theorem. The set $\{Q(y) \wedge (R(y, x) \vee \tilde{Q}(f(y)))$, $\tilde{Q}(f(u)) \vee \tilde{R}(u, u)\}$ is valid according to Corollary.

Finally note that the given treatment of MIM as a specific sequent calculus allows us to give a transparent description of the MIM actions scheme. Such an approach not only outlines a place of MIM among existing machine methods for inference/refutation search, but also can give the possibility to develop an equality handling technique for it.

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On the stability of approximate solutions of the inverse synthesis problem

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Abstract

The inverse problems of synthesis are investigated by solutions which do not use information about the exact solutions of problems. The stability of approximate solutions of this kind of inverse problems to small changes in the initial data is proved.

Keywords: inverse problems of synthesis, approximate solutions, stability.

1 Introduction

Let's consider the linear inverse problem in the standard form [1]

$$\int_0^t K(t,\tau)z(\tau)\,d\tau = u_\delta(t), or\, A_p z = u_\delta, t \in [0,T], \tag{1}$$

where z is unknown function, $z \in Z, u_\delta \in U; U, Z$ are some normalized functional spaces.

The function u_δ is defined by experimental data with a predetermined error, unknown function z

$$\| u_\delta - u^{ex} \|_U \leq \delta, \tag{2}$$

where u^{ex} is exact right part in Eq. (1).

Let us define the inverse synthesis problem: it is necessary to define unknown function z on segment $[0,T]$ using given from experiment function u_δ on segment $[0,T]$ and fixed operator A_p in such a way that the inequality Eq. 2 is valid. Such inverse problems have not yet become widespread, but they have important meaning for problems of mathematical modeling [2], as well as for problems of physical processes prediction [3],[4] and control problems [5].

The operator A_p for inverse problems is a compact operator for typical cases of functional spaces Z, U choice.

Let us consider the set of possible solutions $Q_{\delta,p}$ of equation Eq. 1:

$$Q_{\delta,p} = \{z \in Z : \| A_p z - u_\delta \|_U \leq \delta\}. \tag{3}$$

The sets $Q_{\delta,p}$ are not bounded at any δ while operators A_p are compact operators. So inverse problems of type Eq. 1 belong to a class of incorrect problems [1] and special methods have to be used for their solution [1], [2].

In addition these inverse problems have the following features:

- the approximate solution can considerably differ from the exact solution z^{ex} as $z^{ex} \notin Q_{\delta,p}$;
- the size of an error of the approximate solution in relation to the exact solution z^{ex} has no importance for further use of the approximate solution;
- the exact solution z^{ex} of an inverse problem in aggregate with initial mathematical model A_p can give worse results of mathematical modeling than the approximate solution as $z^{ex} \notin Q_{\delta,p}$;
- the error of the operator A_p to the exact operator A_p^{ex} is possible not to take into account, as the initial inexact mathematical model of physical process will be used at mathematical modeling further.

The solution of such problems can be interpreted only as a good model for purposes of mathematical modeling.

In work [1] it is assumed that exact solution of equation Eq. 1 z^{ex} belongs to the set of possible solutions $Q_{\delta,p}$. This property was used by construction of regularized algorithms. So in this case ($z^{ex} \notin Q_{\delta,p}$) it is necessary to choose another approaches.

In inverse problems of synthesis there is no need in such regularized algorithms property, but it is sufficient to find any function from the set of possible solutions $Q_{\delta,p}$. Therefore, it makes sense to choose from the set $Q_{\delta,p}$ not an arbitrary element, but an element with additional properties which are suitable for further research. For example, we can choose as the solution of inverse problem the most "convenient" element from the set $Q_{\delta,p}$ for the mathematical modeling purposes. It is possible to choose from $Q_{\delta,p}$ the simplest element [2] that is the most stable to small changes of the initial data, the best element for prediction purposes [3] and so on.

2 On the stability of solutions of the inverse synthesis problem

We will assume that an approximate solution z_δ of the inverse synthesis problem is obtained by the variational method i.e. as a solution to the following extreme problem:

$$\Omega[z_\delta] = \inf_{z \in Q_{\delta,p}} \Omega[z], \tag{4}$$

The stability of such solutions is proved under the assumption that the exact solution to the equation belongs to the set $Q_{\delta,p}$ for any $\delta > 0$.

We consider the case when the set of possible solutions of the inverse problem is not empty by any δ.

Let's investigate the question of the stability of approximate solutions of inverse synthesis problems to small changes in the initial data. In this case it is assumed that the initial function u_δ has the given accuracy δ under a fixed function of the exact data u^{ex}.

The following theorem holds.

Theorem. Suppose that Z is a Gilbert functional space, that the functional $\Omega[z]$ is strongly convex and lower semi continuous on $Q_{\delta,p}$, that the Lebesgue set $M(v)$ is compact in Z for any function from

$v \in Q_{\delta,p}$: $M(v) = \{z \in Q_{\delta,p} : \Omega[z] \leq \Omega[v]\}$. Then the solution of the extreme problem Eq. (4) exists, unique, belongs to $Q_{\delta,p}$ and stable to small change of initial data u_δ.

3 Conclusion

The question of the stability of an approximate solution of the inverse synthesis problem which is obtained by the variational regularization method, is considered. Moreover, the hypothesis that the exact solution z^{ex} belongs to the set of possible solutions is not used.

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Modeling of busy period in polling models with semi-Markov switching of states

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Abstract

Performance characteristics for exhaustive polling models with semi-Markov switching, such as distribution of busy period, probability of states and other auxiliary characteristics are presented. Numerical algorithms for these characteristics are elaborated and a numerical example is presented.

Keywords: polling system, busy period, auxiliary characteristics, Laplace-Stieltjes transform, numerical results.

1 Introduction

Polling model is a system with multiple queues with a single server which visits the queues according to the Polling table and serves customers of these queues. In addition, the Polling models have applications in situations where many users are competing for access to a shared resource which is available at a given time, such as communication systems, traffic and transportation systems [1], manufacturing systems, etc.

- Classic applications of Polling models represent **time-sharing computer systems**; these consist of a number of terminals connected via multi-drop lines to a central computer.

- Polling models can be used to analyse the performance schemes of crossing token in **local area networks** (LANs), where a token is the transmitting right and it is transmitted through different users.

- In the designing of **mobile ad-hoc networks** the Polling systems, which consist of both fixed wireless terminals and mobile, also find place.

It is well known that queues with a single server give some very good prospects even to complex queues with more servers if they are properly approximated. So, queues with a single server are very important in Queueing Theory and are much spread.

2 The busy period – one of performance key characteristics

2.1 Busy period for polling model with semi-Markov switching

We consider a queueing system of polling type with semi-Markov delays. Handling mechanism for this system is given by polling table $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, r\}$, where the function shows that at the stage j, $j = \overline{1, n}$, user number k, $k = \overline{1, r}$, $r \leq n$ is served (see [1]). The items (messages) of the user k, according to Poisson distribution with parameter λ_k arrive. The service time for the items of class k is a random variable B_k with distribution function $B_k(x) = P\{B_k < x\}$. Duration of the orientation from one user to user k is a random variable C_k with distribution function $C_k(x) = P\{C_k < x\}$. Thus C_k can be interpreted as a loss of time in preparing the service process for user of class k [2].

Denote by $\Pi_k^\delta(x)$ distribution function of the k-busy period, and by $\pi_k^\delta(s)$ it's Laplace-Stieltjes transform.

Theorem 1. *Function $\pi_k^\delta(s)$ is determined from equation*

$$\pi_k^\delta(s) = c_k(s + \lambda_k - \lambda_k \pi_k(s))\pi_k(s), \tag{1}$$

where

$$\pi_k(s) = \beta_k(s + \lambda_k - \lambda_k \pi_k(s)) \tag{2}$$

and with $c_k(s)$ and $\beta_k(s)$ denoting the Laplace-Stieltjes transforms of distribution functions $C_k(x)$ and $B_k(x)$.

Denote by $P_{B_k}(x), P_{C_k}(x)$ and $P_0(x)$ the probabilities that at instant x the system is busy by service of k-messages, switching to k-messages and system is free, respectively.

Theorem 2. *The Laplace-Stieltjes transforms of $P_{B_k}(x)$, $P_{C_k}(x)$ and $P_0(x)$ are determined from*

$$p_{B_k}(s) = \frac{\lambda_k[1-\pi_k(s)]}{s[s+\lambda_k-\lambda_k\pi_k^\delta(s)]}, \tag{3}$$

$$p_{C_k}(s) = \frac{\lambda_k[1-c_k(s)]}{s[s+\lambda_k-\lambda_k\pi_k^\delta(s)]}, \tag{4}$$

$$p_0(s) = \frac{1}{s} - [p_{B_k}(s)+p_{C_k}(s)]), \tag{5}$$

where $\pi_k^\delta(s)$ and $\pi_k(s)$ are determined from (1) and (2).

2.2 Busy period for polling model with DD priority discipline

Let's consider a queueing system $M_r|G_r|1|\infty$ with DD (Discretionary Discipline) priority: if the service time of a_k–request is less than set value $\theta_k, (k = 2, \ldots, r)$, then the arrived request with priority higher than k (σ_{k-1}–request) achieves absolute priority, otherwise – relative. The durations of service a_k–requests are independent random variables B_k with distribution function $B_k(x)$, $(k = 1, \ldots, r)$. The durations of switching $(\to i)$ are random variables C_i with distribution function $C_i(x)$, $(i = 1, \ldots, r)$. The variables B_k and C_i are independent. An arbitrary switching $(\to k)$ also may be interrupted by arriving σ_{k-1}-request.

We'll introduce notations following [3, 4]. We'll denote by $\Pi(x)$, $\overline{\Pi}_k(x)$, $\overline{\Pi}_{kk}(x)$, $H_k(x)$, $\overline{\Pi}_{kk}^{(n)}(x)$, $N_k(x)$, $\Pi_k(x)$, $\Pi_{kk}(x)$ the distribution function of busy period, k–period, kk–period, k–service cycle, kkn–period, k–cycle of switching, Π_k–period, Π_{kk}–period and by

$\pi(s) \dots \pi_{kk}(s)$ – their Laplace-Stieltjes transforms (the definition of these see [4]). Let's consider also $\sigma_k = a_1 + \cdots + a_k$, where a_k – the parameter of Poisson flow of k–th priority.

Theorem 3. *For all schemes*

$$\sigma_k \overline{\pi}_k(s) = \sigma_{k-1} \overline{\pi}_{k-1}(s + a_k - a_k \overline{\pi}_{kk}(s)) + a_k \pi_{kk}(s), \tag{6}$$

$$\overline{\pi}_{kk}(s) = h_k(s + a_k - a_k \overline{\pi}_{kk}(s)), \tag{7}$$

$$\sigma_k \pi_k(s) = \sigma_{k-1}\pi_{k-1}(s + a_k) + \sigma_{k-1}\{\pi_{k-1}(s + a_k[1 - \overline{\pi}_{kk}(s)])-$$

$$-\pi_{k-1}(s + a_k)\}\nu_k(s + a_k[1 - \overline{\pi}_{kk}(s)]) + a_k \pi_{kk}(s), \tag{8}$$

$$\pi_{kk}(s) = \nu_k(s + a_k[1 - \overline{\pi}_{kk}(s)])\overline{\pi}_{kk}(s), \tag{9}$$

where $h_k(s + a_k - a_k\overline{\pi}_{kk}(s))$ *and* $\nu_k(s + a_k - a_k\overline{\pi}_{kk}(s))$, *for each of the schemes I.J, are determined from certain relations respectively [4], for* $s = s + a_k - a_k\overline{\pi}_{kk}(s)$.

The analytical results formulated above, although they are of interest from fundamental theoretical point of view, are quite complicated for numerical modelling. Indeed, for example, $\pi_k(s)$ given by the Theorem 3, occurs in most of the performance characteristics. But for determining this function it is necessary to solve the functional equation (7), which does not have the exact analytical solution, but which effectively can be solved numerically [3, 5]. As an example, we will present same results of elaborated numerical algorithms of successive approximations for determining $\pi_k(s), \nu_k(s)$ and $h_k(s)$.

3 Numerical example

Example 1 *It is considered a generalized queueing system with DD priority, for the scheme 1.2 ("repeat", "resume"), which is formed from k queues,* $k = \overline{1, 10}$. *The requests arrive in the queues according to Poisson flow with the parameters* $a_k = \{0.2, 0.1, 0.9, 0.6, 0.8, 0.2, 0.1, 0.3\}$ $\{0.2, 0.9\}$*for each queue* $L_k, k = 1, 2, ..., 10$. *The service time of requests* *of class k is a random variable with Erlang distribution function $B_k(x)$, with the parameters $b_k = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.2, 0.3, 0.2, 0.1, 0.1\}$, $p_k = \{0.5, 0.2, 8, 0.4, 4, 0.1, 0.3, 0.5, 0.9, 0.23\}$ and the switching time from a queue to k queue, it is considered a random variable with Exponential distribution function $C_k(x)$, with the parameters $c_k = \{0.4, 0.9, 0.6, 0.9, 0.1, 0.2, 0.2, 0.1, 0.9, 0.7\}$. The set value $\theta_k = 0.4$, $s = 0.5$, $\varepsilon = 10^{-5}$.*

Table 1. Numerical results of k-busy period and auxiliary characteristics for DD priority, "repeat", "resume" discipline

k	$h_k(s)$	$\nu_k(s)$	$\pi_k(s)$
1	0.778370	0.642857	0.489282
2	1.918259e-4	0.521253	0.025329
3	0.537282	0.378999	0.075940
4	8.396915e-4	0.000000	0.135735
5	0.837261	0.000000	0.292063
6	0.522701	0.444444	0.140271
7	0.534522	0.019607	0.007274
8	0.138574	0.530092	0.025610
9	0.576717	0.414889	0.117760
10	0.127903	0.127903	0.058166

4 Conclusion

As it is mentioned above, the busy period is present in the analytical expressions for most of the performance characteristics. On the other hand, it is known that functional equations by which this characteristic is expressed have no analytical solution, so the busy period cannot be found exactly. But it can be determined with the required accuracy, applying numerical methods and developing calculation algorithms. This is achieved, and this makes it possible also to model other performance characteristics in the body for which the busy period is found.

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Solving the non-linear 4-index transportation problem

Tatiana Pașa, Valeriu Ungureanu

Abstract

We formulate the 4-index transportation problem with the concave piecewise cost function. We describe an algorithm for solving the non-linear problem and bring a practical example to describe the implementation of the algorithm.

Keywords: transportation problem, non-linear programming.

1 Introduction

The 4-index transportation problem (4ITP) is a mathematical model that describes very well the activity of an enterprise, which attracts attention of researchers, because the problem assumes that the capacities of the sources, the demands of the destinations to be supplied, the types and quantities of products and the types and capacities of the transports are known.

The problem is a part of the 4ITP group which was studied by R. Zitouni, who proposes an original algorithm [1] and compares it with other methods [2]. A modification of the classical algorithm was described by A. Djamel et. al. [3]. In [4] the nonlinear 4-index transportation problem (N4ITP) was formulated and an algorithm was proposed that simplifies the problem to a linear problem for which the simplex algorithm can be applied.

A well-known tool for solving this kind of problems is the Mathematica System, in which the proposed algorithm was implemented and tested.

2 Problem formulation

In the following, we are given:

- m sources $A_1, \ldots, A_m$ with the respective supply $\alpha_1, \ldots, \alpha_m$;
- n destinations $B_1, \ldots, B_n$ with the respective demand $\beta_1, \ldots, \beta_n$;
- p different types of products $P_1, \ldots, P_p$ of respective quantities $\gamma_1, \ldots, \gamma_p$;
- q different types of transport $T_1, \ldots, T_p$ with the respective transportation capacities $\delta_1, \ldots, \delta_q$.

The N4ITP needs to determine a flow that minimizes the function

$$F(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} \varphi_{ijkl}(x_{ijkl}), \tag{1}$$

where $\varphi_{ijkl}(x_{ijkl})$ are concave piecewise non-decreasing cost functions.

We must solve the non-linear problem:

$$F(x^*) = \min_{x \in X} F(x) \tag{2}$$

$$\begin{cases} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \alpha_i, & \forall i = 1, \ldots, m \\ \sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{l=1}^{q} x_{ijkl} = \beta_j, & \forall j = 1, \ldots, n \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{q} x_{ijkl} = \gamma_k, & \forall k = 1, \ldots, p \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijkl} = \delta_l, & \forall l = 1, \ldots, q \\ x_{ijkl} \geq 0, & \forall (i, j, k, l) \end{cases} \tag{3}$$

To respect constraints on positive values, the following variables $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_n$, $\gamma_1, \ldots, \gamma_p$ and $\delta_1, \ldots, \delta_q$ must be positive.

3 Main results

The proposed genetic algorithm for solving N4ITP with concave piecewise cost functions consists of the following steps:

Step 1. *Initialization* — generate an initial population of $4nm$ chromosomes, each described by a list of the form:

$$P = \{\{p(i,j), i = 1, ..., n, j = 1, ..., m\} \|$$
$$\| \{p(1), \ldots, p(p)\} \| \{p(1), \ldots, p(q)\}\} .$$

Step 2. *De-codification* — associate each chromosome with an admissible solution. *Evaluation* — calculate the cost for each arc on which the flow is being pushed and the value of the objective function for each solution.

Step 3. In order to *select* the parent chromosomes that will participate in the creation of the next population, they are sorted in increasing order according to the value of the objective functions associated with the chromosomes and the first half will be transferred to the next population and selected as parent chromosomes.

Step 4. *Crossover* — perform between selected pairs. The parent chromosomes are cut in the same random place. The first descendant obtains the sequences of arcs up to the cut and the order of distribution of the products from the mother chromosome, the order of the transports and the other sequences of arcs are taken from the father chromosome. The second descendant obtains the sequences of arcs up to the cut and the order of distribution of the products from the father chromosome, the order of the transports and the other sequences of arcs are taken from the mother chromosome.

Step 5. Perform mutation at a rate ϵ and implies:

- randomly the section of the chromosome is chosen, products or transports;

- for the selected section two elements are selected randomly and swapped.

Step 6. Check the stop condition — the algorithm stops after k iterations. The solution to the problem corresponds to the chromosome with the minimum objective function. If the ending condition is not verified, proceed with Step 2.

4 Conclusion

The experimental results demonstrate the correctness of the described algorithm and if we perform a sufficient number of iterations, the optimal solution of the problem will be obtained. The algorithm exits bottlenecks and tends to local solutions using the operations of selection, crossing and mutation.

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On determining coefficients matrices for transmission line equations

Vladimir Paţiuc, Galina Rîbacova

Abstract

The numerical technique for computing the matrix of potential coefficients for multi-wire electrical lines is proposed. The obtained results are compared with existing methods for calculating these coefficients. The applicability limits of the proposed numerical method are studied depending on the initial parameters of a multi-wire power line.

Keywords: transmission line, finite volume method, potential coefficients matrix.

1 Mathematical model

When studying electromagnetic processes in a multi-wire long power lines, the telegrapher's equations (or just telegraph equations) are often used [1] and [2]. The telegraph equations are a pair of coupled, linear differential equations that describe the voltage and current on an electrical transmission line with distance and time. These equations with respect to voltage vector $\boldsymbol{u}(x,t)$ and current vector $\boldsymbol{i}(x,t)$ have the form

$$\boldsymbol{L}\frac{\partial \boldsymbol{i}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial x} + \boldsymbol{R}\boldsymbol{i} = 0 \tag{1}$$

$$\boldsymbol{C}\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{i}}{\partial x} + \boldsymbol{G}\boldsymbol{u} = 0 \tag{2}$$

In an n-wire line the vector functions $\boldsymbol{u}(x,t)$ and $\boldsymbol{i}(x,t)$ have n components each, but $\boldsymbol{L}$, $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{G}$ in Eqs. (1) and (2) are symmetrical matrices $(n \times n)$ of per-unit-length inductances, capacitances, wire resistances and conductivities of insulation.

When solving a problem for telegraph equations Eqs. (1) and (2), it is usually assumed that the matrices $\boldsymbol{L}$, $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{G}$ are known. Most often, the values of the coefficients of these matrices are taken from reference books or specialty literature, where these values obtained experimentally are represented. Here we present an algorithm that allows computing the potential coefficients matrix numerically.

The transmission line equations contain the matrix of the coefficients of electrostatic induction which allows expressing the charge vector through the vector of potentials in each conductor. The method for calculating the exact values of its coefficients is proposed in [1]. For this purpose it is necessary to solve n (number of wires) Dirichlet problems with known boundary conditions. At the same time it is proposed numerical method for calculating the potential coefficients matrix, which allows to express the vector of potentials through the charge vector in each conductor. In order to obtain the elements of this matrix we obtain the problem that differs from the classical Dirichlet problem for the Laplace equation. The difference consists in replacing the Dirichlet condition by some special boundary condition, which contains integrals over the boundary of the domain from the values of the unknown function. Such problems are called problems with nonlocal boundary conditions.

2 Potential coefficients matrix for two-wire power line

The detailed description and the theoretical foundation of the formulated problem with nonlocal boundary conditions are presented in [4]. The existence and uniqueness of the solution of such a problem are proved in [4] as well. In order to solve numerically the foregoing problem we propose to apply the ideas of the finite volume method. So, the corresponding algorithm has been developed and implemented using Matlab numerical computing environment.

At the same time, elements of the potential coefficients matrix under certain assumptions can be calculated by the formulas given in [3]. To obtain these formulas, one wire is selected and the presence of an electric charge is assumed only on it, but on all other wires the charge is considered zero. It is assumed also that the field of the charged wire will be the same as the field of a single wire stretched above the earth (since the distortion of the field due to the existence of other wires can be neglected due to the smallness of their cross sections and the absence of the electrical charges on them).

The calculations performed by the proposed numerical algorithm for the test problem are in good agreement with the data obtained using the technique described in [3], which allows us to recommend the proposed method for calculating the parameters of objects with complex geometry.

On the other hand, the calculations demonstrated that for wires arrangement, that is used in practice, the values of capacitive coefficients can be calculated with sufficient accuracy using simple approximate formulas for potential coefficients. To confirm this statement we have solved numerically several problems for two-wire line with distinct initial data. So, consider the problem for two wires with the centers at the points $(-d/2, H)$, $(d/2, H)$. Here d, R, H represent the distance between wires, the radius of each wire and the height of their location above the earth, respectively. We have chosen for the study the following main parameters: the ratio between the radius of the wires and the distance between their centers, the ratio between the height of wire location above the earth and the radius. All the obtained numerical data have been compared with the corresponding calculations performed in accordance with [3]. As an example, let us present the results of calculating the potential coefficients matrix α according to the proposed method and the matrix α_{anal} obtained by the formulas from [3]. Considering the case when $d = 0.036$, $R = 0.012$, $H = 0.024$ or $d/R = 3, H/R = 2$ (i.e. the values of the height of wires location H, of the distance between their centers d and of the radii R are comparable), we obtain:

$$\alpha = \begin{pmatrix} 2.284 & 0.803 \\ 0.803 & 2.284 \end{pmatrix} \cdot 10^{10} \frac{1}{F}, \tag{3}$$

$$\alpha_{anal} = \begin{pmatrix} 2.367 & 0.918 \\ 0.918 & 2.367 \end{pmatrix} \cdot 10^{10} \frac{1}{F}. \tag{4}$$

In order to prove the convergence of the proposed method the numerical experiments have been fulfilled using grid thickening.

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Modeling of the formation of protostellar cores in the clouds-clouds collision

Boris Rybakin, Sergey Moiseenko, Grigore Secrieru

Abstract

The paper presents the results of numerical modeling of the process of collision of two molecular clouds (MC).

Keywords: CFD, simulation, cloud-cloud collision (CCC), KH instabilities.

1 Introduction

The paper presents the results of mathematical modeling, which allows one to study the results of turbulization of the MC substance and the effect of instabilities on the resulting dense clumps and filament structures. This process is accompanied by Kelvin-Helmholtz (KN) instability and a violation of the gas density above the perturbed surface layers of the clouds.

The calculation results showed that the mutual collision of the MC leads to the formation of a superdense region on the surface of the collision, in which ever denser gas clumps are constantly formed. These dense clumps (lumps) are gravitationally unstable according to Jeans and are the precursors of new stars and star clusters [1]. A thermal pressure in the MC of 10–20 K is not enough to prevent gravitational collapse.

2 The mathematical model and simulations

Three-dimensional motion of the gaseous medium arising upon the collision of molecular clouds will be described by the system of Euler equations written in a conservative form.

$$\frac{\partial U}{\partial t}+\nabla\cdot T=0\,,\; U=\begin{pmatrix}\rho\\ \rho\mathbf{u}\\ e\end{pmatrix},\; T=\begin{pmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+pI\\ (e+p)\mathbf{u}\end{pmatrix}^{T},\; e=\frac{p}{\gamma-1}+\frac{|\mathbf{u}|^2}{2},$$

Here, ρ - density, $\mathbf{u}=(u,v,w)$ is the velocity vector, e is the energy, p is the pressure. The ongoing processes will be considered adiabatic, with adiabatic index $\gamma=5/3$. For the solution of the basic equations the TVD scheme of second order of accuracy was applied. Numerical simulations of different cases of MS collisions were carried out: a frontal collision with the mutual penetration of two molecular clouds of an initially spherical shape, moving in the opposite direction at different speeds, and a moving cloud collision. The main physical parameters and initial assumptions for modeling CCC are given in [2].

3 Numerical results

The calculation results showed that the mutual collision of the MC leads to the formation of a superdense region on the surface of the collision, in which ever denser gas clots are constantly formed. These dense clumps (lumps) are gravitationally unstable according to Jeans and are the precursors of new stars and star clusters [1]. Figure 1 shows

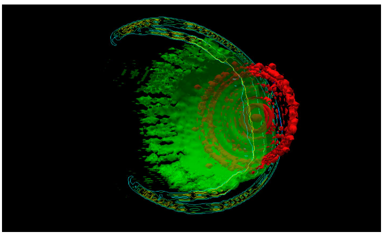

Figure 1. Stage of passage by cloud MC1 of cloud MC2 and the formation of superdense clumps

how a core is formed from fragmented gas condensations. An image is given normalized to the maximum value of the isosurface abs(grad X) – a value characterizing a sharply variable rate of change of gas density. The distribution of fragmented gas condensations corresponds to a contrast X of the order 8000. The analysis shows that the condensations in the form of "clumps" are breaking up into fragments, distributed sporadically along the arcs of concentric circles, with their radius dynamically changing.

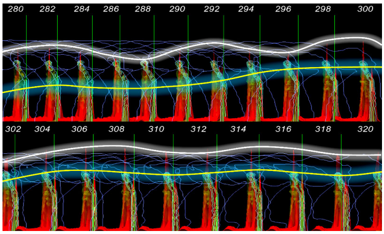

Figure 2. Illustration of pulsation changes in the core for the contrast ratio X = 500/100

Figure 2 shows the temporal evolution of the field of rapidly changing density contrast in the core (red color) and in the shell (green color). The maximum density contrast X in a collision at relatively low speeds increases by two orders of magnitude compared to the initial contrast X = 500. A change in the density, shape, and position of the contact surface leads to the appearance of vibrations that arise on the mid lines (equators) of the MS, and the appearance of vibrations in the interstellar environment. With further interaction of the MC, the shock-compressed lenticular layer forms superdense regions (clumps).

4 Conclusion

The process of formation of filaments, superdense, gravitationally connected objects and the destruction of molecular clouds was studied. The formation of Kelvin-Helmholtz and Richtmeyer-Meshkov instabilities is studied. Supersonic turbulence was developed, which can lead to vibrations, the formation of filaments and superdense regions (clumps), as well as the formation of gravitationally bound regions and the possible appearance of new stars.

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Computer simulation of impulsive loading on buried projectile

Boris Rybakin, Grigore Secrieru, Elena Gutuleac

Abstract

The purpose of the paper is mathematical modelling and analysis of impulsive loading on buried projectile. Numerical calculations for various values of ground parameters, initial pressure and projectile shell material have carried out.

Keywords: elastic-plastic model, projectile, detonation wave.

1 Introduction

Currently Moldova has ammunition depots, which pose a serious danger to nearby civilian objects and the population in the event of strong impact or earthquake. Field experiments are dangerous and expensive, whereas the results of computer simulation can be of practical use to ensure the safety of military facilities. Also the numerical calculations can be used in different industrial sectors with scheduled blasting.

2 The mathematical model

For the detailed study of the processes, occurring in the ground and buried structure, the model of elastic plastic body has chosen. This mathematical model includes main equations, expressing the laws of preservation of mass, the quantity of motion and energy in case of two spatial variables in the Lagrange coordinate system.

The governing system is complemented by the Prandtl-Reuss equations for the elastic plastic condition with von Mises yield criterion.

It closes with the equation of state relating state variables such as pressure, volume, internal energy [1]. After the analysis of the known equations of state the following equations have been selected:

- Mie-Gruneisen equation for modelling the material projectile [2];
- the mathematical model based on experimental data, published by Lyakhov G. and Pokrovsky G. for the description of water-saturated ground [3];
- Theta law for explosives in the solid phase and Landau-Stanyukovich polytropic rule for detonation products.

3 Numerical method

The numerical solution is based on the finite difference method. At the initial moment of time the computational domain is a rectangular in plane or cylindrical coordinate system. The mathematical base of numerical calculations is homogeneous finite-difference based on the modified Wilkins scheme [2], has the second order of accuracy in space and time. A modification of the Wilkins method consists in the possibility of creating a complex computational domain, where:

- construction may have several layers made of various materials;
- the construction may be filled with liquid, explosive or other substance;
- the construction may be surrounded by ground, liquid or other substance.

4 Numerical results

The buried projectile is modelled as a steel structure filled with explosive and having two fuses (in the center – fuse 2, in the end – fuse 1), whose sensitivity to detonation is 3% higher of the filler explosive substance. The projectile is surrounded by saturated ground environment. Impulsive loading is modelled in two ways: with pressure function as a boundary condition on the left boundary of the computational domain; with explosion of shellless charge at time t = 0 μs.

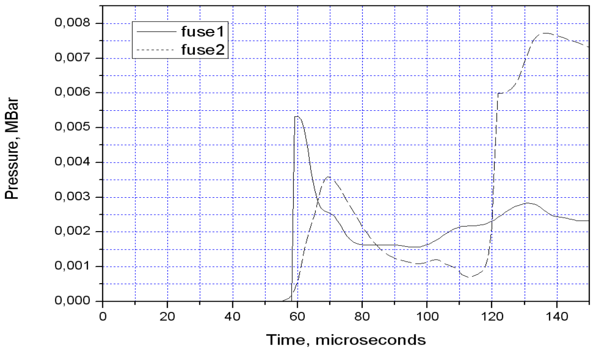

Figure 1. Pressure distribution in control points

Fig.1 shows pressure distribution at the control points located in both fuses. The shock wave reaches the first fuse (fuse 1) and second fuse (fuse 2) almost instantly. The critical pressure in the first fuse is exceeded and initiates detonation there. The second fuse don't detonate in the same time. The detonation wave propagates through the projectile filler, reaching the second fuse at time t = 121 μs. For the given initial and boundary conditions the relationships between ground parameters, initial pressure values and the character of projectile detonation have found. Also the critical pressure for initiating detonation has determined.

5 Conclusion

Computer simulation allows conducting numerical experiments in a wide range of parameters, tracking in real time all the stages of the detonation process inside the projectile, followed by deformation of the projectile shell. Also of interest is the mathematical modelling of the initiation of an explosion inside the explosive shell, followed by the penetration of a cumulative jet into a combined obstacle [4].

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Chebyshev parameters applied to modeling nonlinear processes in semiconductor devices

Galina Sprincean

Abstract

The problem investigated relates to the use of Chebyshev parameters for numerical modeling of nonlinear processes in semiconductor devices. The mathematical model of the problem represents a system of nonlinear differential equations in the unknowns φ – electrostatic potential, φ_n, φ_p – quasi Fermy's levels for electrons and holes, respectively. The problem is further complicated by the fact that the border conditions are of two types: the Dirichlet conditions and the Neumman conditions, which act on different portions of the border. The subproblems that were solved in this research are: discretization of nonlinear differential equations, separation of obtained algebraic systems and linearization of system equations. The obtained linear algebraic systems have symmetric, positively defined and rare (five diagonal) matrices. To solve them many iterative methods can be applied, in this research the Chebyshev two-layer parameter method will be applied.

Keywords: Nonlinear processes, semiconductor devices, doping, discrediting, linearization, Cebyshev parameters.

1 Introduction

Let's consider a semiconductor diode model (fig.1). The diode consists of two regions with different types of doping: the hole zone (p-type zone), with a dominant hole concentration and the electron zone (n-type), with a dominant electron concentration. The anode electrode is connected to the p-type region and the cathode is connected to the n-type region. The impurities, added to the semiconductor material, made of silicon, determine the type of conductivity of each zone. In semiconductor physics the concentration of impurities is noted with N. The function $N(x_1, x_2) = N_D^+(x_1, x_2) - N_D^-(x_1, x_2)m^{-3}$ defines the

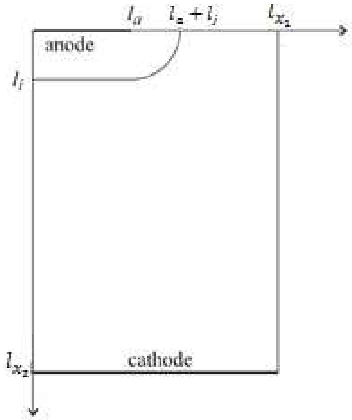

Fig.-1

profile of impurities and is expressed by the concentration of ionized donors and acceptors, noted by N_D^+ and N_D^+, respectively. The negative value of the impurity concentration is determined by the dominance of boron acceptor atoms (p-type semiconductor), and a positive value is determined by the dominance of phosphorus donor atoms (n-type semiconductor).

2 Mathematical formulation of the problem based on the drift diffusion model (DDM)

The problem solving area is a rectangular surface $\Omega = \{(x_1, x_2) : 0 \leq x_1 \leq l_{x_1}, 0 \leq x_2 \leq l_{x_2})$, (fig. 1). With Γ_D note the border of this field, marked with a thick line and the words "cathode / anode". We define on this part of the border the values of unknown functions, thus obtaining Dirichlet's conditions. On the remaining portion of the border, denoted by Γ_N, we equate the values

of the partial derivatives of the unknown functions with zero (the normal flux at the border), thus obtaining Neumann's conditions. Suppose that
$\partial\Omega = \Gamma_D \bigcup \Gamma_N, \quad \Gamma_D \bigcap \Gamma_N = \emptyset.$

$$\nabla \cdot (\epsilon \nabla \varphi) = q(p - n + N) \qquad (Poisson's\ equation) \tag{1.1}$$
$$-\nabla \cdot (J_n) = -q(R_{SRH} + R_{AUG}) \ (The\ continuity\ equation\ for\ electrons) \tag{1.2}$$
$$\nabla \cdot (J_p) = -q(R_{SRH} + R_{AUG}) \ (The\ continuity\ equation\ for\ holes) \tag{1.3}$$
$$J_n = -qn\mu_n \nabla \varphi_n \qquad (Current\ density\ for\ electrons) \tag{1.4}$$
$$J_p = -qp\mu_p \nabla \varphi_p \qquad (Current\ density\ for\ holes) \tag{1.5}$$
$$\varphi_n = \varphi - \varphi_T \cdot ln(n/n_i) \qquad (Cuasi\ Fermi's\ level\ for\ electrons) \tag{1.6}$$
$$\varphi_p = \varphi + \varphi_T \cdot ln(p/n_i) \qquad (Cuasi\ Fermi's\ level\ for\ holes) \tag{1.7}$$

Here $\epsilon = \epsilon_r \epsilon_0$ – absolute permittivity (dielectric constant); ϵ_r – relative permittivity ($\epsilon_r = 11,8$); ϵ_0 – electric constant ($\epsilon_0 = 8,854187817*10^{-12} F \cdot m^{-1}$); q – positive electron charge ($q = 1,60217656510_{-19}C$); N – concentration of impurities; J_n şi J_p – density of electron currents and holes, respectively; R_{SRH}, R_{AUG} – speed of recombination Shockley-Hall.

Border conditions

Consider that the cathode is grounded ($V_c = 0$) and an external voltage is applied to the anode $V_a > 0$. Then on the border Γ_D, i.e. at the anode and cathode, functions φ, n, p satisfy relations:
$\varphi(\bar{x}) = V_a + \varphi_T \cdot ln((N + \sqrt{N^2 + 4n_i^2})/(2n_i);$
$n(\bar{x}) = (N + \sqrt{N^2 + 4n_i^2})/2; p(\bar{x}) = (-N + \sqrt{N^2 + 4n_i^2}/2, x \in \Gamma_D.$
(1.8)
On the portion Γ_N the border conditions are in the form of Neumann's conditions: $\vec{n} \cdot \nabla\varphi = \partial\varphi/\partial n = 0; \vec{n} \cdot J_n = 0; \vec{n} \cdot J_p = 0, \vec{x} \in \Gamma_N$, (1.9) where the point means the scalar product of the vectors.

Initial conditions

In the absence of external voltage on the anode ($V_a = 0$) the solution of the problem on the whole surface Ω is (1.8). This solution can be used as an initial approximation of the sistem solution (1.1)–(1.7) at its resolution by the iterative method, with the gradual increase of the external voltage on the anode V_a.

3 Solving the problem in two-dimensional space

Based on the chosen network of nodes, following the discredit of the differential equations of the system (1.1) – (1.3) three algebraic systems of nonlinear

equations are obtained. The boundary conditions are also discredited by finite differences, on the chosen network, and by solving the algebraic systems obtained we establish the initial values of the unknown functions, noted by $\bar{\varphi}^0, \bar{\varphi}_n^0, \bar{\varphi}_p^0$ on Ω_h. For the equation (1.1), after discrediting on the chosen network, we obtain the following linear algebraic system, the number of equations whose number coincides with the number of internal nodes (NNxMM):
$2\epsilon(\frac{1}{h_{x_1}^2}+\frac{1}{h_{x_1}^2})\varphi_{ij}^{(k+1)} - \frac{\epsilon}{h_{x_1}^2}(\varphi_{i+1,j}^{(k+1)}+\varphi_{i-1,j}^{(k+1)}) - \frac{1}{h_{x_1}^2}(\varphi_{i,j+1}^{(k+1)}+\varphi_{i,j-1}^{(k+1)}) = q(p_{ij}^{(k)} - n_{ij}^{(k)} + N_{ij}^{(k)},\ pe\ \ \Omega_h,$
We apply the Chebyshev parameter method for the two-level scheme:
$B\ \frac{\varphi^{(k+1)}-\varphi^{(k)}}{\tau_{k+1}} + A\varphi^{(k)} = f^{(k)},\ \ B=E.$
$\tau_{k+1} = \frac{\tau_0}{1+\rho_0 t_k},\ k=1,2,\ldots,n,\ n_0(error) = \frac{ln(2/error)}{ln(1/\rho_1)},\ n=[n_0]+1$
$\tau_0 = 2/(\lambda_{min}(A)+\lambda_{max}(A)),\ \rho_0 = (1-\sqrt{\xi})/(1+\sqrt{\xi}),\ \xi = \lambda_{min}(A)/\lambda_{max}(A)),$
$t_k = \frac{(2k-1)n}{2n},\ k=0,1,\ldots,n-1,\ \lambda_{min}(A),\ \lambda_{max}(A)$– the eigenvalues of the algebraic system operator.
Because determining our own values, in some cases, is a rather complex problem, in practice we have done so:
a) We estimate the maximum eigenvalue, which of the observed ones is a constant value and depends only on the domain of definition of unknown functions.:
$\lambda_{max}(A) = 4/(h_{x_1}^2)cos^2((\pi h_{x_1})/(2l_{x_1})) + 4/(h_{x_2}^2))cos^2((\pi h_{x_2})/(2l_{x_2}))$
b) We are looking for it's minimum value in the form:
$\lambda_{min}(A) = \lambda_{min}(A)^{(teoretic)} * \alpha = \epsilon * (\frac{\pi^2}{(l_{x_1}^2} + \frac{\pi^2}{l_{x_2}^2}) * \alpha$
The parameter $\alpha = 0.4$ has been determined experimentally, so that the number of iterations coincides with the number of iterations theoretically n, for the chosen network. If we note by $\aleph_n$ the set of roots of the Chebyshev polynomials : $\aleph_n = \frac{-cos(2i-1)}{2n} * \pi, i = 1,2,\ldots,n,$ then the iterative parameters of Cebashev will be calculated according to the formula: $\tau_k = \tau_0/(1+\rho_0\mu_k), \mu_k \in \aleph_n, k=1,2,\ldots,n.$ The successiveness of the roots, based on which the parameters of Cebashev are calculated, is of great importance. The speed of convergence depends on the successiveness of the roots, and in the cases of the successive successions, phenomena such as the "downtime" of the machine may occur, as a result of the increase of the intermediate iterations, or the loss of the accuracy of the solution, due to the accumulation of the error following the rounding. A way to construct optimal consecutive of the parameters of the method θ_n is:
Whether $n \in N^*$ – number of iterations required.
Step 1: We represent this number as the sum of the powers under 2 as follows:

$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}, k_j \leq k_{j-1} - 1, j = 2, 3, \ldots, s.$
Step 2: We calculate the following value: $n_j = \sum_{i=1}^{j} 2^{k_i - k_j}, j = 1, 2, \ldots, sn_{s+1} = 2n + 1$.
Step 3: We build the following sets:
$\theta_{n_j} = \{\theta_i^{(n_j)} = \theta_i^{(n_{j-1})},\ \theta_{n_j}^{(n_j)} = n_j,\ i = 1, 2, \ldots, n_j - 1\}$. For $j = 1$ we choose $\theta_1 = \{1\}$. Then we build the crowd: $\theta_{2m} = \{\theta_{2i}^{(2m)} = 4m - \theta_i^{(m)}, \theta_{2i-1}^{(2m)} = \theta_i^{(m)}, i = 1, 2, \ldots, m\} for\ m = n_j, 2n_j, 4n_j, \ldots, [\frac{n_j - 1}{4}]$. If $[(n_j - 1)/4] < n_j$, then the calculations according to Step 3 are not performed and proceed to the next step.
Step 4 If $j = s$, then the crowd θ_n is built. In case of contract we calculate $m = (n_{j+1} - 1)/2$ and we build the crowd: $\theta_{2m} = \{\theta_{2i}^{(2m)} = 4m + 2 - \theta_i^{(m)}, \theta_{2i-1}^{(2m)} = \theta_i^{(m)}, i = 1, 2, \ldots, m\}$.
Step 5 Then j increases its value by one unit and the calculations start again with step 2. Finally, when the set θ_n will be constructed, and based on this set, the set of parameters Cebashev $\aleph_n$ will be ordered, thus we obtain the optimum consecutivity of Chebyshev's parameters $\aleph_n^{(optimal)}$.

4 Conclusion

The numerical solution for algebraic systems was obtained by discretizing the nonlinear differential equations (1.1)–(1.3). The case of dynamic equilibrium $V = 0$ is represented in figure 2. Figure 3 represents the surfaces of unknown functions $\varphi, \varphi_n\ and\ \varphi_p$, when applying to anode, from outside, the voltage $V_a = 0.1V$ and with a concentration of impurities $N \approx 10^{18}$. Figure 4 illustrates the distribution of functions $\varphi, \varphi_n\ and\ \varphi_p$, when applying to anode, from outside, the voltage $V_a = 0.2V$ and with a concentration of impurities $N \approx 10^{18}$.

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Figure 2 The surfaces of unknown functions φ, φ_n and φ_p, when applying anode, from outside, the voltage $V_a = 0\,V$ and with a concentration of impurities $N \approx 10^{12}$ (the case of dynamic equilibrium):

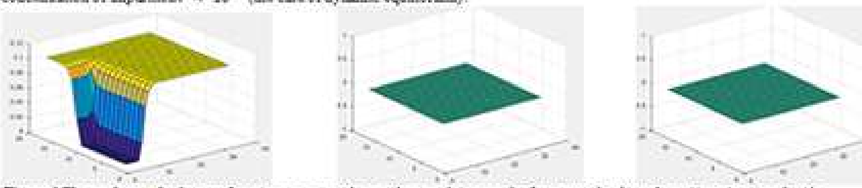


Figure 3 The surfaces of unknown functions φ, φ_n and φ_p, when applying anode, from outside, the voltage $V_a = 0.1\,V$ and with a concentration of impurities $N \approx 10^{13}$

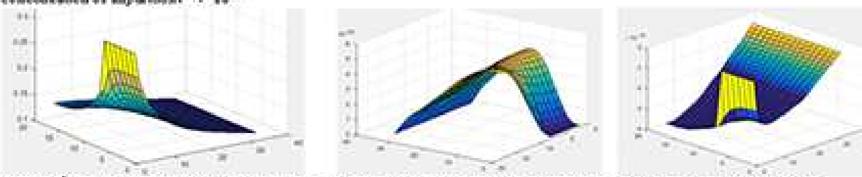
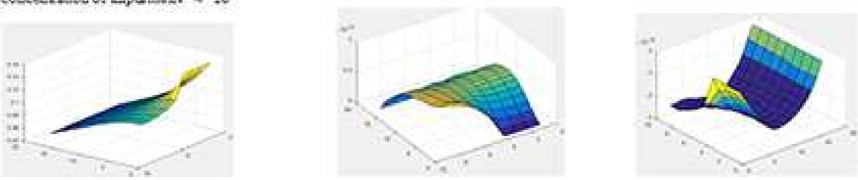


Figure 4 The surfaces of unknown functions φ, φ_n and φ_p, when applying anode, from outside, the voltage $V_a = 0.2\,V$ and with a concentration of impurities $N \approx 10^{13}$



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Investigation of Some Cryptographic Properties of the 8x8 S-boxes Created by Quasigroups

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Abstract

We investigate several cryptographic properties in 8-bit S-boxes obtained by quasigroups of order 4 and 16 by different methods. The best produced S-boxes so far are regular and have algebraic degree 7, nonlinearity 98 (linearity 60), differential uniformity 8, and autocorrelation 88.

Keywords: Nonlinearity, differential uniformity.

1 Introduction

The main building blocks for obtaining confusion in all modern block ciphers are so called substitution boxes, or S-boxes. Designers of block ciphers very often choose S-boxes with special cryptographic properties, which means high nonlinearity (or low linearity), low differential uniformity, high algebraic degree, low autocorrelation and regularity (balance). The well known fact is that the bijective S-boxes are always regular. The AES S-box is the example of the best found 8x8 S-boxes, which is optimal with respect to most of the cryptographic properties (with algebraic degree 7, nonlinearity 112 (or linearity 32), differential uniformity 4, and autocorrelation 32).

Let $\mathbb{F}_2$ denote the Galois field with two elements, and let $\mathbb{F}_2^n$ denote the vector space of binary n-tuples over $\mathbb{F}_2$ with respect to addition $\oplus$ and scalar multiplication. An $n-ary$ *Boolean function* is a function $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$. A *Boolean map* is a map $S : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$, $(m \geq 1)$. Every Boolean map S can be represented as: $S(x_1, \ldots, x_n) =$

$(f_1(x_1,\ldots,x_n), f_2(x_1,\ldots x_n),\ldots,f_m(x_1,\ldots,x_n))$. Each f_i can be represented in ANF as $f_i(x_1,x_2,\ldots,x_n) = \bigoplus_{I\subseteq\{1,2,\ldots,n\}} \alpha_I(\prod_{i\in I} x_i)$, where $\alpha_I \in \mathbb{F}_2$.

For all $\mathbf{x} \in \mathbb{F}_2^n$, the Walsh-Hadamard transform $W_f : \mathbb{F}_2^n \to \mathbb{R}$ of f is $W_f(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{F}_2^n} (-1)^{f(\mathbf{a})\oplus\mathbf{a}\cdot\mathbf{x}}$, where $W_f(\mathbf{x}) \in [-2^n, 2^n]$ is known as a spectral Walsh coefficient, while the Autocorrelation transform of f is $ACT_f(\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{F}_2^n} (-1)^{f(\mathbf{a})\oplus f(\mathbf{a}\oplus\mathbf{x})}$, where $ACT_f(\mathbf{x}) \in [-2^n, 2^n]$ is known as a spectral autocorrelation coefficient. The *autocorrelation (absolute indicator)* of f is $AC(f) = \max\limits_{\mathbf{x}\in\mathbb{F}_2^n\setminus\mathbf{0}} |ACT_f(\mathbf{x})|$. The *nonlinearity* of a Boolean function f is defined as $NL(f) = 2^{n-1} - \frac{1}{2}\max\limits_{\mathbf{x}\in\mathbb{F}_2^n} |W_f(\mathbf{x})|$, while the *linearity* of f is defined as $L(f) = \max\limits_{\mathbf{x}\in\mathbb{F}_2^n} |W_f(\mathbf{x})|$. They are related by the equation $L(f) + 2NL(f) = 2^n$.

For Boolean map S we have the following definitions [2, 3]:

- Algebraic degree: $deg(S) = \max\limits_{i\in\{1,2,\ldots,m\}} \{deg(f_i)\}$
- Nonlinearity: $NL(S) = \min\limits_{\mathbf{v}\in\mathbb{F}_2^m\setminus\{\mathbf{0}\}} NL(\mathbf{v}\cdot S)$
- Linarity: $L(S) = \max\limits_{\mathbf{v}\in\mathbb{F}_2^m\setminus\{\mathbf{0}\}} L(\mathbf{v}\cdot S)$
- Autocorrelation: $AC(S) = \max\limits_{\mathbf{v}\in\mathbb{F}_2^m\setminus\{\mathbf{0}\}} AC(\mathbf{v}\cdot S)$
- Differential uniformity: $\Delta(S) = \max\limits_{\mathbf{u}\in\mathbb{F}_2^n\setminus\{\mathbf{0}\},\mathbf{v}\in\mathbb{F}_2^m} |\{\mathbf{x}\in\mathbb{F}_2^n | S(\mathbf{x}) \oplus S(\mathbf{x}\oplus\mathbf{u}) = \mathbf{v}\}|$

2 Main Results

Mihajloska and Gligoroski [1] constructed optimal 4x4 S-boxes from quasigroups of order 4, by using four e quasigroup transformations, alternating in normal and reverse mode (in a sense that they apply the string in reverse order – *oe*). We investigate several cryptographic properties of the 8x8 S-boxes obtained by similar constructions with quasigroups of order 4 and 16. In some of the constructions we combine quasigroup transformations with the addition of 2-, 4-, or 8-bit constants.

Method 1 – alternate use of *e* and *oe* transformations generated by quasigroups of order 4, like in [1]. Part of the results are given in Table 1, where *n*eoe type means that there are total of *n* quasigroup transformations.

Table 1. Method 1 – part of the results

Type	NL(S)	L(S)	$\Delta(S)$	AC(S)	deg(S)	No. of S
4eoe	64	128	24	256	7	192
8eoe	98	60	10	88,96,64	7	3360
10eoe	98	60	10	88	7	27392
12eoe	98	60	8 10	96 88	7	$\geq$ 714 $\geq$ 84281

Method 2 – combination of *e* and *oe* transformations, with addition of 2-bit, 4-bit or 8-bit constants (some results in Table 2).

Table 2. Method 2 – part of the results

Type	NL(S)	L(S)	$\Delta(S)$	AC(S)	deg(S)	No. of S
1e_add2	0	256	256	256	4	4608
1e_add4	0	256	128	256	4	2816
1e_add8	4 32	248 192	132 164	256	7	6144 1536
1oe_add8	0	256	132	256	7	6144
2e_add2_oe_add2	0	256	128	256	6	98304
2e_add4_oe_add4	64	128	96	256	6	3072
4eoe_add2	64	128	24	256	7	768
4eoe_add8	88	80	24	160	7	16

Method 3 – as Method 1 and 2, but with one randomly generated shapeless quasigroup of order 16 (Fig. 1).

The best produced 8x8 S-box is obtained by 6 *e* quasigroup transformations, alternating in normal and reverse mode, from the quasigroup of order 16, with consecutive leaders (0, 3, 5, 3, 0, 0).

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14	8	4	9	1	13	2	6	5	0	12	3	15	7	10	11
6	11	7	3	13	10	14	2	0	12	8	15	4	9	5	1
10	4	15	8	2	9	13	12	11	7	5	1	6	0	3	14
0	15	13	10	5	4	1	14	9	2	7	8	3	12	11	6
15	10	2	11	6	8	5	4	12	3	14	9	0	1	13	7
9	12	5	1	7	11	4	0	14	8	2	6	10	3	15	13
2	6	12	0	10	15	7	9	1	14	3	13	11	5	4	8
3	14	8	15	0	6	12	11	13	1	9	5	7	10	2	4
13	3	6	5	14	7	8	1	10	11	4	0	12	2	9	15
1	9	11	7	12	2	6	13	3	4	0	10	8	15	14	5
4	13	9	6	3	14	15	10	2	5	11	12	1	8	7	0
12	0	10	13	15	5	3	7	6	9	1	11	14	4	8	2
5	7	0	12	11	1	9	3	8	15	13	4	2	14	6	10
11	2	1	14	8	3	0	15	4	13	10	7	5	6	12	9
7	5	3	2	4	0	11	8	15	10	6	14	9	13	1	12

Figure 1. Shapeless quasigroup of order 16.

Table 3. Method 3 – part of the results

Type	NL(S)	L(S)	$\Delta(S)$	AC(S)	deg(S)	No. of S
1e_add2	32	192	34	256	6	64
1e_add4	32 64	192 128	34 44	256	6	64 8
1e_add8	64	128	26	232-248	7	20
2e_add4_oe_add4	96 98	64 60	10 12,14	88 96-104	7	100 65536
2e_add8_oe	98	60	10	88	7	11
4eoe	98 94-90	60 68-76	10-12 8	88 96-112	7	15 5
6eoe	98	60	8	88 104	7	1 2

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Four color theorem – map solver

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Abstract

Today graph theory has grown into a significant area of mathematical research, with applications almost everywhere (in chemistry, operations research, ..). In this paper, the problem of map coloring with four colors with theory of graphs is considered. We considered the map of metropolitan area of Skopje. For coloring, the software "Four color theorem – map solver" is used.

Keywords: graph, graph theory, maps, map coloring, algorithms.

1 Introduction

Graphs are very practical mathematical model and they are used in various areas of science and everyday life. We are considering the problem of map coloring with four colors. The problem consisted of the possibility of coloring the world map with only four colors, but two neighboring countries must not be colored with the same color [1].

This problem can be considered with the theory of graphs. Graphs are very closely related to maps, as every map can be treated as a graph. A graph $G(V, E)$ consists of a finite non-empty set V whose elements are called nodes, vertices, or points and a finite set E of unordered pairs of distinct nodes, i.e. links of G [4], [5], [6].

2 Chromatic number on graph

A nodes coloring (or simply coloring) of a graph G, is a labelling $f : V(G) \to \{1, 2, ...\}$. The labels are called colors, such that no two adjacent nodes get the same color and each node gets one color. A k - coloring of a graph G consists of k different colors and G is then called k - colorable. It follows from the definition that the k - coloring of a graph $G(V, E)$ partitions the set of nodes V into k independent sets $V_1, V_2, ..., V_k$ such that $V = V_1 \cup V_2 \cup ... \cup V_k$. The independent sets $V_1, V_2, ..., V_k$ are called the color classes and the function $f : V(G) \to \{1, 2, ..., k\}$ such that $f(v) = i$, where $v \in V_i$, $1 \leq i \leq k$, is called the color function [2], [3].

For map coloring, we will use the following greedy algorithm. The algorithm is as follows:

Algorithm 1

1. Define Graph G.
2. Sort the nodes in descending order by degrees.
3. Assign color B_1 to the first node, and then to all nodes that are not neighbors to the previous node, but are not neighbors to each other.
4. Repeat the step 2, with color B_2 for the next uncolored node.
5. Repeat the step 3 while there are nodes without colors.
6. Finish.

3 Four color theorem - map solver

"Four color theorem – map solver" is a software which paints map with 4-coloring algorithm. In the Skopje there will be taken the following Municipalities: Novo Selo, Lepenec, Bardovci, Vizbegovo, Butel, Staro Skopje, Gazi Baba, Chair, Gorce Petrov, Centar 1, Karposh, Zlocucani, Vlae, Gorce Petrov 6, Mirce Acev, Sisevo, Saraj, Gorce Petrov 3. To establish a relationship with the original definition of a 4-color graph coloring algorithm, here the nodes of the graph will be the cadastral municipalities themselves, and the links between the nodes, in fact will be the borders by which they are separated (Fig.1.).

Regarding the color grading algorithm in 4 colors, to paint the metropolitan area of Skopje, 4 colors are needed. The steps of the

Figure 1. Map of the municipalities of Skopje, properly painted with 4 colors.

algorithm in the concrete case are the following (Fig.1.):

Step 1: Novo Selo has 3 neighbors (Gorce p.3, Mirce A., Lepenec); Lepenec has 5 neighbors (New V., Gorce p.6, Vlae, Zlocucani, Bardovci); Bardovci has 3 neighbors (Lepenec, Zlocucani, Vizbegovo); Vizbegovo has 4 neighbors (Bardovci, Zlocucani, Chair, Butel); Butel has 4 neighbors (Vizbegovo, Chair, Gazi B., Staro S.); Staro Skopje has 2 neighbors (Butel, Gazi B.); Gazi Baba has 4 neighbors (Staro S., Butel, Chair, Centar 1); Centar 1 has 3 neighbors (Karpos, Cair, Gazi B.,); Chair has 6 neighbors (Vizbegovo, Butel, Gazi B., Centar 1, Karposh, Zlocucani); Karposh has 5 neighbors (Centar 1, Chair, Zlocucani, Vlae, Gorce p.6); Zlocucani has 6 neighbors (Karposh, Chair, Vizbegovo, Bardovci, Lepenec, Vlae); Vlae has 4 neighbors (Gorce p.6, Karposh, Zlocucani, Lepenec); Gorce Petrov 6 has 3 neighbors (Vlae, Lepenec, Mirce A.); Mirce Acev has 4 neighbors (Gorce p.6, New V., Gorcep.3, Sisevo); Sisevo has 3 neighbors (Mirce A., Gorce p.3, Saraj); Saraj has 2 neighbors (Sisevo, Gorce p.3); Gorce Petrov 3 has 4 neighbors (Saraj, Sisevo, Mirce A., New V.)

Step 2 : We sort the municipalities in descending order by the number of neighboring municipalities they have.

Step 3 and Step 4: Chair and Zlocucani have the same number of neighbors, so we choose one of them. Chair is painted with one color (red) and all of its non-neighboring municipalities (Staro S., Lepenec, Gorce P.3) will be painted with the same color. After that, we paint Zlocucani with another color (blue) and all of its non-neighboring municipalities (Centar 1, Butel, New V., Sisevo, Gorce p.6) will be painted with the same color. Because Lepenec is already painted, we take the next municipality Karposh, and we paint it with a third color (green) and all of its non-neighboring municipalities (Gazi B., Vizbegovo, Saraj, Mirce A.) will be painted with the same color. Because Vizbegovo, Butel and Gazi Baba are already colored, we take the next municipality of the list, and it is Vlae; Vlae is painted with a fourth color (yellow) and it is only one non-neighboring, no painted municipality and it is Bardovci, so Bardovci will be painted with the same color as Vlae. In the end, four colors are needed for coloring the map of Skopje. Like as example of municipality Shtip, also for this case we can find lower and upper bounds of chromatic number of the graph (Fig. 7). This graph is neither complete nor regular, so from corollary 1, we have that $3 \leq \chi(G)$. From the other side, we have that $\Delta(G) = max\{2, 3, 4, 5, 6\} = 6, \chi(G) \leq \Delta(G) = 6$, and $\chi(G) \in \{3, 4, 5, 6\}$. Using the Algorithm 1 we will find that chromatic number of this graph is 4.

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Method for solving fuzzy fractional multi-criteria transportation problem of "bottleneck" type

Alexandra Tkacenko

Abstract

In this work the development of an iterative fuzzy programming algorithm for solving fuzzy fractional multi-objective transportation problem of "bottleneck" type is described. The algorithm is iterative for each value of optimistic parameter of cost coefficient variation. Finally, for any value of this coefficient we will have the set of all the efficient solutions of the model, corresponding to the value level of time. The algorithm was tested on several examples and proved to be quite effective.

Keywords: fuzzy programming, fuzzy cost, fractional transportation problem, efficient solution.

1 Introduction

Many of economical decision problems lead to the fractional optimization models, because a lot of important characteristics of these may be evaluated really using only some ratio relations. The time-constraining criterion is, obviously, one of conditions so much important for major optimization problems. There are many efficient algorithms that solve such models with deterministic data. Since in real life often some parameters are of fuzzy type [1], in this paper the case of fuzzy cost coefficients is investigated.

2 Problem formulation

The mathematical model of the multi-criteria fractional transportation problem of "bottleneck" type with fuzzy cost coefficients is the following:

$$\min Z_1 = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{1}x_{ij}}{\max\limits_{i,j}\{t_{ij}|x_{ij}>0\}} \quad \min Z_2 = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{2}x_{ij}}{\max\limits_{i,j}\{t_{ij}|x_{ij}>0\}}$$

$$........$$

$$\min Z_r = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{c}_{ij}^{r}x_{ij}}{\max\limits_{i,j}\{t_{ij}|x_{ij}>0\}} \quad \min Z_{r+1} = \max_{i,j}\{\,t_{i,j}|\,x_{i,j}>0\} \qquad (1)$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad \forall i=\overline{1,m}, \quad \sum_{i=1}^{m} x_{ij} = b_j, \quad \forall j=\overline{1,n},$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \quad x_{ij} \geq 0 \; for\; all\;\; i\; and\; j,$$

where : $\tilde{c}_{ij}^{k}$, k=1,2...r, i=1,2,...m, j=1,2,...n are costs or other amounts of fuzzy type, t_{ij} – necessary unit transportation time from source i to destination j, a_i – disposal at source i, b_j – requirement of destination j, x_{ij} – amount transported from source i to destination j.

In the model there may also exist the criteria of maximum, which however does not complicate it.

3 Theoretical benchmarks

For the reasons that the time characteristic for the model (1) is discrete, it follows that the set of efficient solutions for every level of its allowable time coincides with the set of efficient solutions for multi-criteria linear model, including separate "bottleneck" criterion [2]. Since the cost coefficients of transportation multi-criteria models have real practical significance such as unit prices, unit costs and many other, all of them are interconnected with the same parameters of variation. Often these can be calculated by applying various statistical methods. We propose to calculate them using the following formula:

$$p_{ij}^{k} = \frac{c_{ij}^{k} - \underline{c}_{ij}^{k}}{\bar{c}_{ij}^{k} - \underline{c}_{ij}^{k}}, \tag{2}$$

where: $\underline{c}_{ij}^{k}$, $\bar{c}_{ij}^{k}$ – are the limit values of variation interval for each cost coefficient c_{ij}^{k}, where: $i = \overline{1,m},\ j = \overline{1,n},\ k = \overline{1,r}$.

Agreeing to the formula (2), the parameters $\{p_{ij}^{k}\}$ can be considered as the probabilistic parameters of belonging to every value of coefficients $\{c_{ij}^{k}\}$ from their respective variation intervals. This makes it possible to reduce the model (1) to a set of deterministic models that can be solved by applying fuzzy techniques [1].

4 Theoretical justifications and algorithms

Analyzing model (1), we find that it is of a multicriteria type. Usually such models are solved on the set of efficient solutions, also called Pareto-optimal.

Let us suppose that: $\left(\bar{\mathrm{X}}, \bar{\mathrm{T}}\right)$ is one basic solution for the model (1), where: $\bar{\mathrm{T}} = \max\limits_{i,j}\{\bar{t}_{ij}/\bar{x}_{ij} > 0\}$ *and* $\overline{X} = \{\overline{x_{ij}}\}, i = \overline{1,m}, j = \overline{1,n}$ *is one basic solution for the first* $r-$ *criteria model* (1).

Definition 1. *The basic solution* $\left(\bar{\mathrm{X}}, \bar{\mathrm{T}}\right)$ *of the model (1) is a basic efficient one if and only if for any other basic solution* $(X,T) \neq \left(\bar{X}, \bar{T}\right)$ *for which there exists at least one index* $j_1 \in (1, \ldots r)$ *for which the relation* $Z_{j_1}(X) \leq Z_{j_1}\left(\bar{X}\right)$ *is true, there immediately exists another, at least one index* $\exists j_2 \in (1, \ldots. r)$*, where* $j_2 \neq j_1$*, for which at least one of the both relations* $Z_{j_2}\left(\bar{X}\right) < Z_{j_2}(X)$ *or* $\bar{T} < T$ *is true. If all of these three inequalities are verified simultaneously with the equal sign, it means that the solution is not unique.*

Definition 2. *The basic solution* $\left(\bar{\mathrm{X}}, \bar{\mathrm{T}}\right)$ *of the model (1) is one of the optimal (best) compromise solutions for a certain time* $\bar{T}$*, if the solution* $\bar{\mathrm{X}}$ *is located closest to the optimal solutions of each criterion.*

In order to solve the deterministic model, corresponding to (1) we can use the ***fuzzy technique*** [2], iteratively solving the model (3) for the best – L_k and the worst U_k values of k-criterion.

Max λ *in the same availability conditions as in* (1) *and:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij} + \lambda \cdot (U_k - L_k) \leq U_k, \quad k = \overline{1,r}, \tag{3}$$

Applying iteratively the fuzzy technique for each increasing time level, we could get the set of all its optimal compromise solutions.

5 Conclusions

By applying the hypothesis about the interconnection and similarly variation of the model's objective functions coefficients, we reduce initially the model (1) to several linear models of deterministic type, each of which may be solved using fuzzy technique.

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The Inverse of A Fourth Degree Permutation Polynomial

Lucian Trifina, Daniela Tarniceriu, Ana-Mirela Rotopanescu, Emilian Ursu

Abstract

In this paper we prove that a true fourth degree permutation polynomial (4-PP) modulo a positive integer of the form 16Ψ, with Ψ a product of positive integers greater than three, has an inverse true 4-PP, under some constraints on the coefficients.

Keywords: permutation polynomial, fourth degree, inverse permutation polynomial.

1 Introduction

Permutation polynomials (PPs) have been studied for many years ago [1]. They have various applications such as cryptography, sequences' generation or interleavers for turbo codes.

It is known that the permutation induced by a PP, $\pi(x)$, modulo a positive integer L, has an inverse permutation induced also by a PP, $\rho(x)$, modulo L. The PP $\rho(x)$ that generates the inverse permutation, is named the inverse PP modulo L of PP $\pi(x)$. In this paper we prove that a true fourth degree PP (4-PP) modulo a positive integer of the form 16Ψ, with Ψ a product of positive integers greater than three, have an inverse true 4-PP. Some constraints for 4-PP's coefficients are assumed when for a prime p_i from the factorization of Ψ we have $3 \nmid (p_i - 1)$.

2 Preliminaries

Definition 2.1. *A 4-PP modulo L is a fourth degree polynomial*

$$\pi(x) = (f_1x + f_2x^2 + f_3x^3 + f_4x^4) \pmod{L}, \tag{1}$$

so that for $x \in \{0, 1, \ldots, L-1\}$, values $\pi(x) \pmod{L}$ perform a permutation of the set $\{0, 1, \cdots, L-1\}$.

Definition 2.2. *A 4-PP is true if the permutation it performs cannot be performed by a PP of degree smaller than four.*

Definition 2.3. *Two 4-PPs with different coefficients are different if they lead to different permutations.*

Conditions on coefficients f_1, f_2, f_3, and f_4 so that the fourth degree polynomial in (1) is a 4-PP modulo L have been obtained in [2]. We are interested in positive integers of the form

$$L = 2^4 \cdot \prod_{i=1}^{N_p} p_i = 16 \cdot \Psi, p_i > 3, i = 1, 2, \ldots, N_p, p_1 < p_2 < \cdots < p_{N_p}. \tag{2}$$

If $2^{n_{L,2}}$, with $n_{L,2} > 1$, is a factor of L, then the following conditions must be fulfilled [2]

$$f_1 \neq 0, (f_2 + f_4) = 0, f_3 = 0 \pmod{2}. \tag{3}$$

For p_i a prime so that $3 \nmid (p_i - 1)$, $i \in \{1, 2, \ldots, N_p\}$, we will consider only the 4-PPs with coefficients fulfilling conditions

$$f_1 \neq 0, f_2 = 0, f_3 = 0, f_4 = 0 \pmod{p_i}. \tag{4}$$

A 4-PP modulo L

$$\rho(x) = (\rho_1x + \rho_2x^2 + \rho_3x^3 + \rho_4x^4) \pmod{L}, \tag{5}$$

is an inverse of the 4-PP in (1) if

$$\pi(\rho(x)) = x \pmod{L}, \forall x \in \{0, 1, \cdots, L-1\}. \tag{6}$$

3 Main Result

Lemma 3.1. *Let a positive integer be of the form given in* (2)*. Then all the true different 4-PPs fulfilling conditions* (4) *when* $3 \nmid (p_i - 1)$*, have the following possible values for coefficients* f_4*,* f_3 *and* f_2*:*

$$f_4 = \Psi, f_3 = k_{3,f} \cdot 2\Psi, k_{3,f} \in \{0, 1, 2, 3\}, f_2 = k_{2,f} \cdot \Psi, k_{2,f} \in \{1, 3, 5, 7\}. \tag{7}$$

Coefficient f_1 *has to fulfill the following necessary, but not sufficient, condition:* $f_1 \pmod 8 \in \{1, 3, 5, 7\}$.

Proof. For $L = 16\Psi$, a true 4-PP is equivalent to a 4-PP for which $f_2 < L/2 = 8\Psi$, $f_3 < L/2 = 8\Psi$, and $f_4 < L/8 = 2\Psi$. Taking into account the coefficient conditions for a 4-PP given in (3) and (4) and that Ψ is odd, coefficients f_2, f_3, and f_4 from (7) follow.

We note that when $L = 16\Psi$, from condition (3) f_1 results odd. Thus, we can have only $f_1 \pmod 8 \in \{1, 3, 5, 7\}$. □

Lemma 3.2. *Let a positive integer be of the form given in* (2)*. Then, a true 4-PP* $\pi(x)$*, fulfilling conditions* (4) *when* $3 \nmid (p_i - 1)$*, has an inverse true 4-PP* $\rho(x)$*, with*

$$\rho_4 = f_4, \rho_3 = k_{3,\rho} \cdot 2\Psi, \rho_2 = k_{2,\rho} \cdot \Psi. \tag{8}$$

ρ_1 *is the unique modulo* L *solution of the congruence*

$$f_1 \rho_1 = \Psi \cdot k + 1 \pmod{16\Psi}, \text{ with } k \in \{0, 1, 2, \ldots, 15\}, \tag{9}$$

and k*,* $k_{3,\rho} \in \{0, 1, 2, 3\}$*, and* $k_{2,\rho} \in \{1, 3, 5, 7\}$ *are given by equation* (11)*, according to the values of* $k_{3,f}$*,* $k_{2,f}$*, and* $f_1 \pmod 8$*.*

Proof. Because of the limited space of the paper we give here only a sketch of proof.

We have to specify that if for a 4-PP $\pi(x)$, an inverse PP $\rho(x)$ has coefficients as in (8), it means that the inverse PP is a true 4-PP. Thus the 4-PP $\pi(x)$ does not admit an inverse PP of degree smaller than four. From Lemma 3.1 each true 4-PP has an equivalent 4-PP with coefficients as in (7). We will give the equation from which the coefficients of the inverse 4-PP as in the theorem can be got and we will show how this equation is obtained. Because the inverse permutation is unique and the permutation induced by a true 4-PP can not be induced by a smaller degree PP this equation is necessary and sufficient for the proof.

For $L = 16\Psi$, $\rho_4 = f_4 = \Psi$, ρ_3 and ρ_2 as in (8), f_3 and f_2 as in (7), (6) is equivalent to

$$(f_1\rho_1 - 1) \cdot x + \Psi \cdot \Theta(x) = 0 \ (\mathrm{mod}\ 16\Psi), \forall x \in \{0, 1, \ldots, 16\Psi - 1\}, \quad (10)$$

where $\Theta(x)$ is a polynomial of degree 16 depending on $k_{3,\rho}$, $k_{3,f}$, $k_{2,\rho}$, $k_{2,f}$, ρ_1, f_1, and Ψ.

Because $\Psi \mid L$, from (10) we obtain equation (9). Because $\gcd(f_1, 16 \cdot \Psi) = 1$, from Theorem 57 in [3] it results that equation (9) has a unique solution ρ_1 (mod 16Ψ).

With (9), (10) is fulfilled if and only if

$$k \cdot x + \Theta_\Psi(x) = 0 \ (\mathrm{mod}\ 16), \forall x \in \{0, 1, \ldots, 15\}, \quad (11)$$

where

$$\Theta_\Psi(x) = \Theta(x)/\Psi \ (\mathrm{mod}\ 16) = (f_1 k_{2,\rho} + k_{2,f}\rho_1^2) \cdot x^2 +$$
$$+2 \cdot (f_1 k_{3,\rho} + k_{2,f} k_{2,\rho} k_\Psi \rho_1 + k_{3,f}\rho_1^3) \cdot x^3 +$$
$$+(\rho_1^4 + 6k_{3,f}k_{3,\rho}k_\Psi\rho_1^2 + 4k_{2,f}k_{3,\rho}\rho_1 + k_{2,f}k_{2,\rho}^2 k_\Psi^2 + f_1) \cdot x^4 +$$
$$+2k_\Psi \cdot (2k_{2,\rho}\rho_1^3 + 6k_{3,f}k_{3,\rho}\rho_1^2 + 3k_{3,f}k_{2,\rho}^2 k_\Psi \rho_1 + k_{2,f}\rho_1 + 2k_{2,f}k_{2,\rho}k_{3,\rho}k_\Psi) \cdot x^5 +$$
$$+2k_\Psi \cdot (4k_{3,\rho}\rho_1^3 + 3k_{2,\rho}^2 k_\Psi \rho_1^2 + 3k_{3,f}\rho_1^2 + 4k_{3,f}k_{2,\rho}k_{3,\rho}k_\Psi\rho_1 +$$
$$+k_{3,f}k_{2,\rho}^3 k_\Psi^2 + k_{2,f}k_{2,\rho}k_\Psi + 2k_{2,f}k_{3,\rho}^2 k_\Psi) \cdot x^6 +$$
$$+4k_\Psi \cdot (\rho_1^3 + 2k_{2,\rho}k_{3,\rho}k_\Psi\rho_1^2 + k_{2,\rho}^3 k_\Psi^2 \rho_1 + 3k_{3,f}k_{2,\rho}k_\Psi\rho_1 + 2k_{3,f}k_{3,\rho}^2 k_\Psi \rho_1 +$$
$$+3k_{3,f}k_{2,\rho}^2 k_{3,\rho} k_\Psi^2 + k_{2,f}k_{3,\rho}k_\Psi) \cdot x^7 +$$
$$+k_\Psi^2 \cdot (12k_{2,\rho}\rho_1^2 + 8k_{3,\rho}^2\rho_1^2 + 8k_{2,\rho}^2 k_{3,\rho}k_\Psi\rho_1 + 8k_{3,f}k_{3,\rho}\rho_1 + k_{2,\rho}^4 k_\Psi^2 +$$

$$+6k_{3,f}k_{2,\rho}^2k_\Psi + 8k_{3,f}k_{2,\rho}k_{3,\rho}^2k_\Psi + k_{2,f}) \cdot x^8 +$$

$$+2k_\Psi^2 \cdot (4k_{3,\rho}\rho_1^2 + 6k_{2,\rho}^2k_\Psi\rho_1 + 3k_{3,f}\rho_1 + 4k_{2,\rho}^3k_{3,\rho}k_\Psi^2 + 4k_{3,f}k_{2,\rho}k_{3,\rho}k_\Psi) \cdot x^9 +$$

$$+2k_\Psi^2 \cdot (3\rho_1^2 + 2k_{2,\rho}^3k_\Psi^2 + 4k_{2,\rho}^2k_{3,\rho}^2k_\Psi^2 + 3k_{3,f}k_{2,\rho}k_\Psi + 4k_{3,f}k_{3,\rho}^2k_\Psi) \cdot x^{10} +$$

$$+4k_\Psi^3 \cdot (2k_{2,\rho}^2k_{3,\rho}k_\Psi + 3k_{2,\rho}\rho_1 + 3k_{3,f}k_{3,\rho}) \cdot x^{11} +$$

$$+2k_\Psi^3 \cdot (3k_{2,\rho}^2k_\Psi + 4k_{3,\rho}\rho_1 + k_{3,f}) \cdot x^{12} +$$

$$+4k_\Psi^3 \cdot (2k_{2,\rho}k_{3,\rho}k_\Psi + \rho_1) \cdot x^{13} + 4k_\Psi^4 \cdot (2k_{3,\rho}^2 + k_{2,\rho}) \cdot x^{14} +$$

$$+(8k_{3,\rho}k_\Psi^4) \cdot x^{15} + k_\Psi^4 \cdot x^{16} = 0 \pmod{16}, \forall x \in \{0, 1, \ldots, 15\}, \quad (12)$$

and $k_\Psi = \Psi \pmod{16}$.

Because Ψ is odd, we can have $k_\Psi \in \{1, 3, 5, \ldots, 15\}$. Solutions of equation (12) in variables $f_1 \pmod{16}, \rho_1 \pmod{16} \in \{1, 3, 5, \ldots, 15\}$, $k_{3,\rho}, k_{3,f} \in \{0, 1, 2, 3\}$, $k_{2,\rho}, k_{2,f} \in \{1, 3, 5, 7\}$, k, and k_Ψ, were found by means of Matlab program by exhaustive search. Solution of (12) in variables $k_{3,\rho}$, $k_{2,\rho}$, $\rho_1 \pmod{16}$ and k is unique for each combination of the other previous four variables. The solutions in variables k, $f_1 \pmod{16}, \rho_1 \pmod{16}$, $k_{3,\rho}, k_{3,f}$, and $k_{2,\rho}, k_{2,f}$ are the same $\forall k_\Psi \in \{1, 5, 9, 13\}$, and also the solutions in the previously mentioned variables are the same $\forall k_\Psi \in \{3, 7, 11, 15\}$. For every $k_\Psi \in \{1, 3, 5, \ldots, 15\}$, the solutions of equation (12) in variables k, $\rho_1 \pmod{16}$, $k_{3,\rho}, k_{3,f}$, and $k_{2,\rho}, k_{2,f}$ are the same $\forall f_1 \pmod{16} \in \{1, 9\}$, or $\forall f_1 \pmod{16} \in \{3, 11\}$, or $\forall f_1 \pmod{16} \in \{5, 13\}$, or $\forall f_1 \pmod{16} \in \{7, 15\}$. Thus, we can summarize the solutions of (12) for every $k_\Psi \pmod{4} = k_{\Psi,4} \in \{1, 3\}$ and for every $f_1 \pmod{8} \in \{1, 3, 5, 7\}$.

□

Example 3.1. *Let there be* $L = 496 = 16 \cdot 31$ *and the 4-PP* $\pi(x) = x + 31x^2 + 0x^3 + 31x^4 \pmod{496}$*. It means that* $k_\Psi = k_{31} = 15$, $k_{\Psi,4} = k_{31,4} = 3$, $k_{3,f} = 0$, $k_{2,f} = 1$, *and* $f_1 \pmod{8} = 1$. *From the start we have* $\rho_4 = f_4 = 31$. *For previous values of* $k_{3,f}$, $k_{2,f}$, $f_1 \pmod{8}$*, and* k_Ψ*, the solution of equation* (12) *in the four remaining variables is* $k_{3,\rho} = 0$, $k_{2,\rho} = 1$, $\rho_1 \pmod{8} = 5$ *and* $k = 4$. *Then* ρ_1 *can be found by the congruence* $1 \cdot \rho_1 = 31 \cdot 4 + 1 = 125 \pmod{496}$, *i.e.*

$\rho_1 = 125$, *which fulfills the previous solution* $\rho_1 \pmod 8 = 5$. *Thus the inverse 4-PP is* $\rho(x) = 125x + 31x^2 + 0x^3 + 31x^4 \pmod{496}$.

Example 3.2. *Now we consider again* $L = 496 = 16 \cdot 31$, *but the 4-PP* $\pi(x) = 127x + 465x^2 + 248x^3 + 465x^4 \pmod{496}$. *Adding the null polynomial* $124x + 310x^2 + 124x^3 + 434x^4 \pmod{496}$ *to* $\pi(x)$, *it results that* $\pi(x)$ *is equivalent to 4-PP* $\pi_{eq}(x) = 3x + 155x^2 + 124x^3 + 31x^4 \pmod{496}$, *which has the coefficients as in Lemma 3.1. It means that* $k_\Psi = k_{31} = 15$, $k_{\Psi,4} = k_{31,4} = 3$, $k_{3,f} = 2$, $k_{2,f} = 5$, $f_1 \pmod 8 = 3$, *and* $\rho_4 = f_4 = 31$. *For previous values of* $k_{3,f}$, $k_{2,f}$, $f_1 \pmod 8$, *and* k_Ψ, *the solution of equation* (12) *is* $k_{3,\rho} = 0$, $k_{2,\rho} = 1$, $\rho_1 \pmod 8 = 3$ *and* $k = 0$. ρ_1 *is found by the congruence* $3 \cdot \rho_1 = 31 \cdot 0 + 1 = 1 \pmod{496}$, *i.e.* $\rho_1 = 331$, *which fulfills the solution* $\rho_1 \pmod 8 = 3$. *Thus, the inverse 4-PP is* $\rho(x) = 331x + 31x^2 + 0x^3 + 31x^3 \pmod{496}$.

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Triangle Finding

Frank Vega

Abstract

If we assume that we can do the Binary AND Operation in constant time, then Triangle Finding can be solved in quadratic time.

Keywords: algorithm, undirected graphs, lower bound, matrix.

1 Problem Statement

Given an undirected graph $G = (V, E)$, the problem Triangle Finding consists in determining whether there exist three vertices a, b and c in V such that $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$ are edges in E.

2 Open Problem

Triangle Finding can be solved in time $O(n^w)$, where $n = |V|$, and $w < 2.373$ denotes the matrix multiplication constant [2]. However, it is currently unknown if Triangle Finding can be solved in time $O(n^2)$ [2].

3 Overview

Algorithms such as finding the maximum into an array of integers assume that the operation of comparisons between two numbers take constant time [1]. That's why, we can affirm the running time of finding maximum is $O(n)$, where n would be the array length [1].

Operations such as the Binary AND Operation might be more complex than the comparison operations, but if we assume that we can do this operation in constant time, then Triangle Finding can be solved in time $O(n^2)$.

4 The Algorithm

Given an undirected graph of n vertices represented by a Boolean adjacency matrix, then this should be a valid adjacency matrix for an undirected graph: for the symmetry (equals to its transpose) and the main diagonal is filled only with zeroes.

1. First, fill the main diagonal with ones.
2. Next, set to zero all the one values in the entries of the matrix which are above the main diagonal.
3. If we find a rectangle whose corners are 1's in the final modified matrix such that the upper right corner is an entry of the main diagonal, then there must be a triangle in the undirected graph input.
4. Otherwise, there is no possible triangle in the undirected graph input.

Theorem 4.1. *After these previous steps there is a triangle in the undirected graph if and only if we find a rectangle whose corners are 1's in the final modified matrix such that the upper right corner is an entry of the main diagonal.*

Proof. Since the $matrix$ is symmetric, then we have $matrix[c][a] = matrix[a][c]$ for every pair of nodes a and c. In addition, since we fill the main diagonal with ones, then $matrix[a][a]$ has the value of 1 for every node a (1 is equivalent to the value of true into the Boolean adjacency $matrix$). Consequently, we have three nodes $b < a < c$ that is a triangle if and only if the entries $matrix[a][b]$, $matrix[a][a]$, $matrix[c][b]$ and $matrix[c][a]$ represent the corners of a rectangle. $\square$

4.1 Runtime

This takes the running time $O(n^2)$ assuming the Binary AND operation could be done in constant time.

4.2 Code

This work is implemented into a GitHub Project programmed in Java [3]. We show the Algorithm 1 in pseudo coding.

4.3 Programming Techniques

We use the Binary AND Operation between two integers for finding the rectangle whose corners are 1's such that the upper right corner is an entry of the main diagonal.

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Algorithm 1 Triangle Finding algorithm

```
/*A two-dimensional Boolean matrix that represents the graph*/
/*In the cells of the matrix the value true represents the number 1 and 0 otherwise*/
procedure ALGORITHM(matrix)
    n ← getRows(matrix)
    numbers ← Array[n]
    power ← Array[n + 1]
    power[0] ← 1
    for i ← 1 to n do
        power[i] ← 2 × power[i − 1]
    end for
    for i ← 0 to n − 1 do
        for j ← 0 to i do
            if i = j ∨ matrix[i][j] then
                numbers[i] ← numbers[i] + power[j]
            end if
        end for
    end for
    for i ← 0 to n − 1 do
        all ← power[i + 1] − 1
        rhs ← power[i]
        for j ← i + 1 to n − 1 do
            /*& is the Binary AND Operation*/
            lhs ← (numbers[j] & all) & numbers[i]
            if lhs > rhs then
                return "yes"
            end if
        end for
    end for
    return "no"
end procedure
```

Section 3

Computer Science

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Distributed System for Real-Time Collective Computing

Victor Ababii, Viorica Sudacevschi, Mariana Oşovschi, Ana Ţurcan, Ana Nistiriuc, Dimitrie Bordian, Silvia Munteanu

Abstract

This paper presents the results of the distributed system of collective computation for the application in solving complex problems development. Distributed data processing is performed based on the Mesh network connected with a large number of Agents. For each Agent, there is predefined a task, which is solved using a strategy based on a mathematical model defined in time. In this paperwork there is elaborated the hierarchical structure of the collective calculation system, the mathematical model, the network topology, the distribution mode of tasks between the Agents, and the model for the quality evaluating of command service.

Keywords: *distributed computing systems, real-time computing, collective computing, target goals, task target, and multi-agent systems.*

1 Introduction

At the moment, distributed computing systems can be considered the most efficient and applied in such fields as: industry; technological systems, robotic and monitoring systems; finance and trade; information societies; creativity and entertainment; medical assistance; education; transport and logistics; science; environment management, etc. The most important characteristics of these systems can be considered: parallelism; competition; lack of a global clock; independent errors; resource sharing and virtualization; heterogeneity; scalability; adaptability, etc. [1].

The real-time concept has been introduced in the computing systems to be able to specify them, from the point of view, of the time variable that has an important role in the correctness of the decision. The characteristics of a real-time system are: response time, multiple processes, determinism, flexibility, adaptability, and irreversibility. Another aspect of real-time systems is the ability to synchronize the processes of calculation and communication between them [2].

Solving complex problems [3] requires the involvement of essential resources of data processing and storage. One method of solving this problem is to apply distributed computing systems, especially collective calculus or collective artificial intelligence [4]. The concept of collective artificial intelligence is also specific for Multi-Agent systems [5], to which is accorded extraordinary attention by researchers from different fields, as a method of solving complex problems by dividing them into smaller individual tasks. Individual tasks are assigned to a set of agents where each agent decides on an appropriate action to solve the task using multiple inputs, for example, the history of actions, interactions with neighboring agents and its purpose.

2 Goal of research

The purpose of the research carried out by the present work, is to design a distributed Multi-Agent system of collective calculation in real-time. The described problem is specific for the areas in which there appears the necessity to combine the advantages of Multi-Agent collective calculation in combination with time restrictions. The specific feature of the Multi-Agent collective calculation is the necessity to perform the exchange of information, which imposes some time restrictions in which the system evaluates. It is obvious that a decision made by a Multi-Agent collective computing system can be restricted in time, because, the decision taken by the system at one point of time can be fatal for the evolution of the system in continuation.

In Figure 1 it is shown the structure of the Multi-Agent collective calculation system. This structure determines the hierarchy between the abstraction levels of the calculation system, where:

Target Goals – target goals collective calculation system;

Strategy – applied strategies to reach the target objectives;

$Task_1, \cdots, Task_N$ – tasks to apply the planned strategies to achieve the target objectives;

$A_1, \cdots, A_N$ – the set of agents that solve appropriate tasks $TT_n, \forall n = \overline{1, N}$;

Communication Environment – the communication environment based on the technologies Wi-Fi, GSM, GPRS, etc.;

Activity Environment – system activity environment.

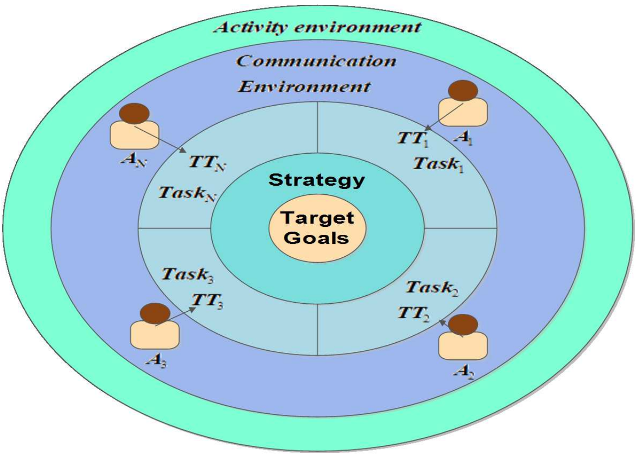

Figure 1. The hierarchical structure of the Multi-Agent collective computing system.

Each level of abstraction can communicate only with the neighbor level. The activity environment provides status information to the set of agents. Each agent solves his task based on the established strategy, to achieve the system's target objectives. The result of the performed calculations by the set of agents is applied in the activity environment to maintain it in the set up limits by the target objectives.

3 The Mathematical Model

Whether the process is defined $P = \bigcup_{n=1}^{N}(p_n)$, where p_n is a set of sub-processes and $p_i \cap p_j \neq \varnothing$, $\forall i = \overline{1,N}, \forall j = \overline{1,N}, i \neq j$; which evaluates in the ***Activity Environment***.

The process P is managed by the mathematical model defined by the system of equations 1 [6]:

$$\mathbf{X} = \mathbf{f}(\mathbf{X}(\mathbf{t}), \mathbf{U}(\mathbf{t}), \mathbf{t}), \mathbf{X}(\mathbf{t_0}) = \mathbf{X_0} \tag{1}$$

where:

$f : \mathbf{R}^N \to \mathbf{R}^N; \mathbf{X}(\mathbf{t}) = \{\mathbf{x_1}(\mathbf{t}), \mathbf{x_2}(\mathbf{t}), \cdots, \mathbf{x_N}(\mathbf{t})\}$ – state of the process vector P in the moment of time $\mathbf{t}$;

$\mathbf{U}(\mathbf{t}) = \{\mathbf{u_1}(\mathbf{t}), \mathbf{u_2}(\mathbf{t}), \cdots, \mathbf{u_N}(\mathbf{t})\}$ – the intervention vector over the process $\mathbf{P}$;

$\mathbf{X} = \{\mathbf{dx_1}(\mathbf{t})/\mathbf{dt}, \mathbf{dx_2}(\mathbf{t})/\mathbf{dt}, \cdots, \mathbf{dx_N}(\mathbf{t})/\mathbf{dt}\}$ – process dynamic $\mathbf{P}$;

$\mathbf{X_0}$ – the initial state of the process $\mathbf{P}$.

The target goals of process control $\mathbf{P}$ are defined by the expression 2:

$$\max_{x(t) \in \mathbf{R}^N} \{\varphi(\mathbf{X}(\mathbf{t}), \mathbf{t})\}, \tag{2}$$

where $\varphi : \mathbf{R}^N \to \mathbf{R}$.

With restrictions:

$$g(\mathbf{X}(\mathbf{t}), \mathbf{U}(\mathbf{t}), \mathbf{t}) = 0 \tag{3}$$

$$h(\mathbf{X}(\mathbf{t}), \mathbf{U}(\mathbf{t}), \mathbf{t}) \geq 0 \tag{4}$$

$$\mathbf{X}(\mathbf{t}) \in \mathbf{X} \subseteq \mathbf{R}^N \tag{5}$$

$$\mathbf{U}(\mathbf{t}) \in \mathbf{U} \subseteq \mathbf{R}^N \tag{6}$$

$$\mathbf{t} \in [\mathbf{t_0}, \mathbf{T}] \tag{7}$$

To achieve the goals provided in 2 it is necessary to find the vector $\mathbf{U(t)}$ respecting the conditions 3-7, which has to ensure the condition 8, it means the solving strategy of the collective calculation problem:

$$\nabla\varphi(x(t))_i = \frac{\partial\varphi}{\partial x(t)_i}(X(t)) > 0, \forall i = \overline{1, N}. \tag{8}$$

4 The Topology of Multi-Agent Network

The collective calculation specific consists in taking decisions based on the particular decisions of the agents. In this context, three types of decisions can be classified:

- Absolute decisions – when all the particular decisions of the agents are taken into consideration;

- Majority decisions – when the particular decisions of most agents are taken into consideration;

- Minority decisions – when the personal decisions of the agents that correspond to the personal agents' interests are taken into account.

The topology of the multi-agent collective computing network is shown in 4

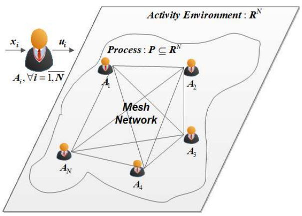

Figure 2. Multi-Agent network for Collective Computing.

In 4 there is presented the set of Agents $A_1, ..., A_N$ that forms a network Mesh. Each Agent acquires the status x_i of the process P, processing the data according to the algorithm defined by mathematical models 1-8 and influences the process with the command value u_i.

5 The distribution of collective computing tasks

In order to optimize the process of calculating and uniform distributing of computational power, the following solution is proposed, where each Agent will perform the set of instructions defined by the mathematical model 9:

$$A_i : \begin{cases} \mathbf{g_i(X(t), u_i(t), t) = 0}, \\ \mathbf{h_i(X(t), u_i(t), t) \geq 0}, \\ \mathbf{f_i(X(t), u_i(t), t) = dx_i(t)/dt}, \\ \max_{x_i(t) \in \mathbf{R}^N} \{\varphi_i(X(t), t)\}, \\ \mathbf{Q_oS_i = q_i(u_i(t)), T_{k-1} < t \leq T_k}, \\ \forall i = \overline{1, N}, \end{cases} \tag{9}$$

where: Q_oS_i – quality criterion of the command service; q_i – function to evaluate the quality criterion; T_{k-1} – the beginning of the time interval and T_k – the end of the time interval for the evaluation quality criterion.

The quality criterion Q_oS is evaluated in the graph shown in 5, where: Q_oS_max – the maximum quality criterion; Q_oS_{opt} – the optimal quality criterion; $Q_oS(t)$ – the evolution of the quality criterion in time; $C(t)$ – the convergence of the command value solution in time; T_{k-1} – the beginning of the evaluation of the quality criterion Q_oS; T_k – the end of the time interval of quality criteria evaluation Q_oS ; $U(T_k)$ – the values of the command vector obtained at the moment of time T_k.

At the beginning of each data processing cycle T_{k-1} there is acquired the state vector $X(t)$, performed by all Agents connected to the network, and the communication between them with this data. At the end of the data processing cycle T_k the action on the process P with vector values $U(T_k)$ takes place.

6 Conclusion

The results of this research are dedicated to solve some specific problems of real-time collective calculation. The collective computa-

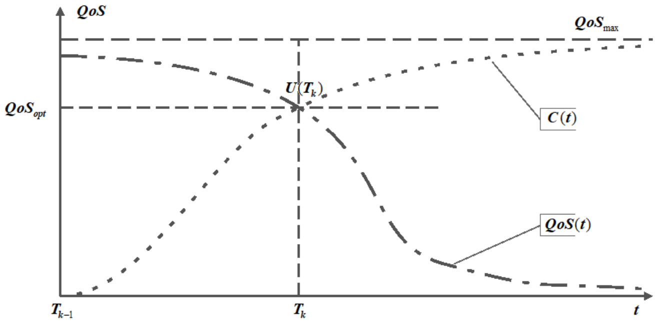

Figure 3. The graph of the quality criterion evolution Q_oS.

tion is presented as a Multi-Agent system which solves a complex problem defined in time (a process). In this paperwork there are presented: the hierarchical structure of the collective calculation system that determines the interaction between different abstract levels of the system; the mathematical model that describes the evolution of the process in time with the definition of the target goals, and the strategic methods applied in solving of the problem; the topology of the Multi-Agent network that presents a Mesh network; and the mathematical model for the distribution of computational tasks among the set of Agents. In order to synchronize the collective calculation process performed by the set of Agents, the quality criterion evaluated in time is defined. For the future, a Multi-Agent system based on the NodeMCU V3 ESP8266 and ESP32 devices development is planned. As a complex problem, the order of a production process is selected, which includes the stages of: logistics, production, marketing and delivery.

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Encapsulating the Concept of Information in Scientific Paradigm of Reality: A Possible Scenario of a Theory of Everything

Veaceslav Albu

Information is information, not matter or energy.
No materialism, which does not admit this, can
survive at the present day.
Norbert Wiener

Abstract

The paper represents an attempt of formulating a theory of everything as a coherent theoretical account how nature works where information gets its fundamental role with same importance that the energy and the matter have in main theories of modern physics. The presented approach, based on central concepts of N. Wiener and B. Mandelbrot is able to solve many existing paradoxes yielded by contemporaneous scientific paradigm.

Keywords: potential information, embedded information, emergent information, universal gravity, observer, homeostasis, Unified Field Theory.

1 Introduction. What is going on here?

To propose a theory of everything today presupposes that one should be prominently skilled in theoretical or quantum physics. Otherwise, this person has maximum probability to fail in his goal to challenge the state of the art theories of modern cosmology. The above-mentioned two fields of physics harmonized together by Einstein's theory of relativity, play the same role as a central black hole does in its own galaxy. Every single object within this galaxy has to obey the power of gravity of this central black hole and surround it within the given paradigm of seemingly free flight. Unfortunately, the time of building up a theory of everything just by power of reason during the romantic contemplation of reality, as Plato and Aristotle did, is gone. Nevertheless, this paper presents such an endeavor.

The aim of the proposed approach is to enrich the existing scientific paradigm with a central concept from Wiener's cybernetics [1] – *the concept of information,* and with the *concept of fractals,* Mandelbrot's [2] view on how the nature topologically behaves. From this point, some minor complications begin. Although, the proposed paradigm remains acting in four-dimensional reality, there is one exception: the *concept of relative change of information* in a given fractal of observed reality relocates the *concept of time.* Another minor novelty required by our reasoning and aimed to reach the harmony of the proposed theory, is that the force of gravity transforms into four-dimensional universal force, where the other free known forces, unified as electromagnetic forces remain three-dimensional. Moreover, the known force of gravity remains in 3D space as the 3D projection of the four-dimensional universal gravity. That novelty does not contradict with their definitions as forces acting within three-dimensional matter. Consequently, the three-dimensional potential energy resulting from Newtonian gravity, in the proposed theory becomes four-dimensional as well.

The very new concepts in this theory are the *concept of potential information,* the *concept of embedded in 3D matter information* and the *concept of emergent information* in given volume of 3D space that we will explain later in this paper.

2 Why information?

We believe there are two reasons why the information concept is fundamental in nature but is missing in contemporaneous scientific paradigm of reality. The first reason suggests that we are living in informational age due to the courage of Norbert Winner to put scientists from non–linked sciences as engineers, biologists, chemists, and psychologists to brainstorm together. He did that with one purpose – to find a new perspective how to build an informational hammer for humans with the same revolutionary results, as the stone hammer had for our ancestors. Wiener's citation from the top of this paper represents the final two sentences that coronate the "Computing Machines and the Nervous System" chapter in his "Cybernetics". Unfortunately, since then no one of the modern cosmology theorists did pay attention to the highest intuition of this idea. By that, they gave a chance to the present attempt to be produced as a remembrance of Wiener's warnings – that there is no way to build a scientific paradigm without use of information as a fundamental block of reality. The second reason, which makes us believe that the concept of information follows from the natural laws of the Universe, represents the fact of absolute similarity of mathematical description of dependence at opposite scales of reality – Newton's formula for the force of gravity and for Coulomb's formula for the force between two charges.

3 The Theory's Road Map. The information

The main concept observed in nature that inspired N. Wiener to create the science of cybernetics was the concept of homeostasis in living organisms as a necessary element for sustaining the evolution of life. He proposed the seminal idea of embedding the same principles into automatic machines and computers as fundamental for cybernetics. He got this insight from his attempt to find an efficient solution for guided missile technology during WWII by implementing the feedback principle. Its automatic missiles system uses the received information about a targeted airplane's current position to a simultaneous change of parameters of the missile's flight towards the target. It is easy to follow the analogy between guided hunting on airplanes and the Schrodinger's cat paradox. Indeed, the one hunting on airplane during the nighttime without feedback of its actions does not know if airplane is still flying or it is dead, by analogy with a cat from the box. The first important conclusion follows from above reasoning. The feedback information in nature and in cybernetic systems' processors plays a role of an **observer**, one of very fundamental concept of quantum mechanics. So, if information is so important, then why it is missing as a fundamental concept in modern scientific paradigm?

The second important conclusion follows from the ability of humans' and contemporaneous Artificial Intelligence (AI) "beings" for planning its future strategies and actions. Any plan of the future, in one's mind or in computer's processor, represents a pure informational picture of future imagined state of affairs. A good example can be a plan of launching a space ship. The proper start of a space ship, in the first day when a person becomes in charge of that space ship's launching, represents just a collection of thoughts and imaginations in the person's mind as pure non-material information. Nevertheless, a push on the START button on the day of launch makes this pure imaginary and informational event real and it really pushes the spacecraft into outer space. On the first day, that person's mind entangles with the future pure informational event and, by controlling his real actions with feedback information of the current state of affairs on the project with the information from the imagined start moment, the launch of the space ship becomes real. One can count terabytes of feedback information about state of affairs during the implementation of the project, but only at the launching moment we will open the "Schrodinger cat's box cover", and we will see if all feedback information was correctly observed and processed. The second conclusion has a twofold structure: First, human mind possesses properties of a quantum device, which can entangle itself with an imagined informational object from future reality. Second, the quantum state of observer can be applied by human or AI "beings" minds to any logically possible imagined targeted event from the future reality, by using feedback information for adjusting its real actions with future's target event to make it become real.

The following question then arises. If we admit that living organisms, non–living structures in nature and robotic systems possess the self–organizing property, what impedes us to admit that the ability of self–planning is also necessary for nature, as a necessary a priori part of self–organization? If we acknowledge all the above said, we should confess that N.Wiener was right and wise, when he spelled out for us his famous idea. If so, one should recognize that information is information, not matter and not energy and it is fundamental for nature, as matter and energy does. For the purpose of present theory, information persists in three well–determined states — **potential information, emergent information** and **information embedded** in matter. The latter one, which is most widely known by humanity, reveals itself in all material objects or phenomena. Digging embedded information from matter by any specific science generates knowledge of that science. Humans perceive the embedded information as a physical object with the same degree of illusion, as they perceive a movie as a whole by looking at a CD with this movie. The emergent information reveals itself on the cloud of processes emerging on the edge of the fractal dimension for each recurrent internal fractal of the Universe's fractal. As an example, one can take emergent information about outputs of different quantum effects occurring in cells of living organisms on atomic and molecular level, as well as in all Universe's stars evolution processes. The emergent information ontologically precedes the embedded information. Finally, the potential information is a state of information that persists as embedded in four–dimensional universal gravity of a "genuine immaterial form" of energy–information singularity of matter by analogy with singularity state of matter in Universe's black holes. The potential information represents all logically possible form of emergent or embedded information. As an example of genuine potential information, one can take the thoughts generated by their mind. The Universe's fractal can be seen as an homeostasis in action in 4D space, where Big-Bang can be taken as burst of corresponding potential information into 3D space as primary emergent information via quantum realm, that formed the all atoms in Universe and proceed to step of creating of embedded information in 3D space, known as matter and energy. All the above conclusions roots in N. Wiener's legacy.

4 The force

A hard problem of the proposed theory is to find the carrier of all informational events described above. As a universal force, which generates all energy appearance in 3D space, we propose the four–dimensional gravity that represents state of energy–informational singularity, bounded by some laws of 4D energy–information conservation by analogy with its 3D analog. Moreover, the proposed four dimensionality of force of gravity do represent also an attempt to present a possible solution for unsolved yet Unified Field Theory, as for 3D space all four forces are governed by the proposed four dimensionality of universal gravity, as fourth dimension of our reality. Saying that, we have no logical impediment for claiming existence of phenomenon of homeostasis within such singularity that can locally activate respective amount of potential energy from where the fractal of our Universe roots. Following, from the definition of homeostasis we can logically deduct the existence of a feedback organized as cause-and-effect chain of generating emergent and embedded in 3D space information, under the conditions controlled by given set of our Universe's constants. Such feedback forms a circuit or a loop, with Bing Bang singularity as an output and routed back through black holes' singularities as an input, or vice versa. Important is that such system feeds back into itself developing as our Universe's fractal. In addition, we shall admit logically possible existence by that of many others feedback loops as other universe's fractals possibly governed by the same or another set of universal constants. For that reason, we shall admit the existence of four-dimensional force carrier, which reveals itself in 3D due to Heisenberg uncertainty principle, depending on released emergent information in the given point of space. As in the proposed paradigm the reality is four-dimensional, the commonsensical reasoning in modern physics of the dimensionless point in 3D space should be relocated to +1 dimension of information point, or four-dimensional point, in 4D space.

5 The fractals

Finally, to complete the performance of the great spectacle of our Universe's evolution conducted by the proposed theory, we need a "stage", materiality of which will be easy to defend from any attempts to claim the idealistic nature of the proposed theory, as N. Wiener suggested. Legacy of another modern great thinker, B. Mandelbrot [2] claims that fractals are the ideal carriers of all infinitely complicated objects in the nature. They have some specific properties that are very appropriate for fulfilling the needs of our "stage". Indeed, fractals possess an emergent hierarchical self–similar structure. Internal fractals repeat the structure of external ones and vice versa. We will take density of emergent information in matter directed by laws of quantum physics in given volume of 3D space as fractal dimension. That fractal dimension is the same for entire hierarchical structure of our Universe's fractal. As fractal in Universe acts for 3D matter, together with fractal dimension, mentioned above the dimension of any 3D fractal has to have a fractional fractal dimension between 3 and 4 integer. As B. Mandelbrot did not describe such entity as the Universal fractal, for the sake of presented theory we will take the following fractal's definition: "Fractals are the emergent properties of iterative feedback systems that exhibit both unpredictable and deterministic behaviors, forming patterns that manifest as complex coherent structures, with the property of scale invariance and self-similarity, displaying very specific boundary conditions, with complex morphologies that have a fractal dimension that uniquely quantifies the level of complexity of the emergent patterns within the system" [3]. Important are the specific particularities of Universe's fractal, see [4]. Each fractal manifests itself between its own pair of singularities and only both emergent and embedded information transcend such singularities. The homeostasis and feedback loop, inclusively between level of emergent information and embedded information level, remains the main principle under what the self–organization of matter occurs in given fractal. As an example of such fractals, we can mention from the single living cell up to a living body, any plant or coral, any star system, etc.

6 Conclusion

The paper presents the possible scenario for theory of everything, which encapsulates the concept of information in modern scientific paradigm of reality with a clear aim, to prove the N. Wiener's wise vision of specific and fundamental role of information for performance of reality. The presented approach keeps the existing four dimensions of reality, but as the fourth dimension, the theory regards the four-dimensional gravity, as universal carrier of all changes of real world.

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Rule Forms in Rewriting Systems

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Abstract

We discuss which sets of patterns for rules are sufficient for reaching computational completeness in various rewriting systems, for example, sequential generating grammars and accepting machines as Turing machines and circular Post machines; special attention is paid to the parallel model of membrane systems.

Keywords: theoretical computer science, rewriting systems, membrane computing, normal form.

1 Introduction

Following various normal forms for type-0 grammars and register machines, we aim at a systematic investigation of the forms of rules sufficient to reach computational completeness, besides sequential models also focusing on the parallel model of membrane systems. In this context we see computational completeness as the capability of a formal computing model to generate or accept all recursively enumerable sets of strings (in the case of string rewriting systems) and all recursively enumerable sets of vectors of non-negative integers (in the case of multiset rewriting systems). In this context we allow for ignoring a bounded amount of superfluous symbols (for example, the state symbol or catalysts) and/or a bounded amount of strings (e.g., the empty string/multiset).

The formal computing models of our interest are limited to string or multiset rewriting systems, working in the sequential or parallel derivation mode. Questions about descriptional complexity parameters, such as the size of the alphabet or sub-alphabets or the number of membrane labels, catalysts etc., are out of the scope of this paper, yet we mainly focus on the maximal number of rule forms and the maximal length of the strings describing these rule forms.

2 Definitions

For the standard definitions and the main results in formal language theory we refer the reader to handbooks as, for example, [10].

The simplest variant of a rewriting system can be defined as a tuple (V, R), where V is a finite alphabet and $R \subset V^* \times V^*$ is a finite set of string or multiset rewriting rules; $(u, v) \in R$ is typically written as $u \to v$. Of course, this basic variant does not yet specify

1. the *initial* string or multiset w,
2. *sub-alphabets*, for example the terminal alphabet for extended grammars or the states for machines,
3. special sub-alphabets (for additional restrictions/normal forms, e.g., catalysts),
4. the *derivation relation* $\Rightarrow$, for example, $\{xuy \to xvy \mid x, y \in V^*,\ (u, v) \in R\}$ for sequential grammars,
5. the *halting condition*, for example, the absence of non-terminal symbols for grammars, the appearance of the final state for machines, or the inapplicability of any further rule in the case of P systems.

We should also mention that rules may be more complicated than $u \to v$ in case of certain control mechanisms or distributed systems, and the notation sometimes needs to include special symbols to represent additional ingredients (e.g., permitting/forbidden context, membrane labels/polarizations), yet in principal we shall restrict ourselves to the basic rule forms in this paper:

A *rule form* is a pattern of the form $u \to v$, where u, v are strings of variables (and, possibly, special symbols). For some formal computing model M, we say that a system G of type M is constructed according to a finite set F of rule forms if each rule of G is obtained from some rule form in F by replacing each variable by an element from an alphabet specified for this variable. In the most common case, we distinguish between the total alphabet V, the set of terminal symbols T ($\subseteq V$) and the set of non-terminal symbols $N := V \setminus T$.

For example, $A \to BC$ ($A, B, C \in V$) is a rule form, but $A \to u$ ($A \in V$, $u \in V^*$) is not a rule form, as it can only be written as an infinite set of rule forms, one for each value of $|u| \in \mathbb{N}$.

We say that a set F of rule forms is sufficient for obtaining computational completeness for some formal computing model M if the family of sets of strings or vectors of non-negative integers generated or accepted by systems of M using only rules constructed according to F equals RE (the family of recursively enumerable string languages) or $PsRE$ (the family of recursively enumerable sets of vectors of non-negative integers), thereby possibly ignoring specific symbols or a bounded number of strings or multisets.

3 Sequential rewriting systems

We proceed with presenting rule forms for computationally complete sequential string rewriting models. This is certainly not a complete list, but we cover some known interesting cases.

3.1 String grammars

In case of sequential string grammars, two rule forms are sufficient: $A \to BC$ and $AB \to C$. Indeed, all that is needed for computational completeness is cooperation and the ability to lengthen and shorten the sentential form. As a technical detail, we mention that we neglect the empty string to be generated, which would require additional rules of the form $A \to \lambda$. We will omit such a comment throughout the rest of the paper as long it is obvious from the context.

If we consider accepting, no matter by halting or by final state, then a single rule form $AB \to CDE$ is enough, because we can use copies of an additional symbol E for lengthening especially the right-hand side of rules, i.e., we take $AB \to CEE$ instead of $AB \to C$. Information can propagate over these "garbage" symbols E and these symbols do not influence accepting.

These results are classic and straightforward, so the proofs can be left to the reader.

Other remarkable examples of rule forms for the type-0 grammars can easily be derived from some well-known normal forms:

3.1.1 Penttonen normal form

In [7] it is shown that every recursively enumerable language can be generated by a type-0 grammar with rules of the forms $A \to BC$, $AB \to AD$, $A \to a$, and $E \to \lambda$, where A, B, C, D, E are non-terminal symbols in N and a is a terminal symbol in T.

Based on these rule forms, we immediately infer that we only need rule forms $A \to BC$, $AB \to AD$, and $E \to \lambda$, where A, C, D, E are non-terminal symbols in N and $B \in N \cup T$, as every rule $A \to a$ can be replaced by the rules $A \to aE$ and $E \to \lambda$. In fact, only one such rule $E \to \lambda$ is needed.

3.1.2 Geffert normal form

Among other variants, in [4] it is shown that every recursively enumerable language can be generated by a type-0 grammar with rules of the forms $S \to uSv$ and $S \to u$, where S is a special non-terminal symbol and $u, v \in (\{A, B, C, D\} \cup T)^*$, where A, B, C, D are non-terminal symbols and T is the set of terminal symbols, as well as the two non-context-free λ-rules $AB \to \lambda$ and $CD \to \lambda$, which constitute the rule form $XY \to \lambda$. Introducing intermediate non-terminal symbols, the rules of the forms $S \to uSv$ and $S \to u$ can be replaced by rules only using the single rule form $X \to YZ$. In total, only the two rule forms $X \to YZ$ and $XY \to \lambda$ are needed for obtaining computational completeness, which strengthens the result given at the beginning of this section.

3.2 Insertion-deletion systems

Insertion-deletion systems are another grammar-like computing model, with rules of the form $\lambda \to v$ and $u \to \lambda$, $u, v \in V^*$.

Two rule forms are already sufficient (for example see [5] and [6]), e.g., we only need $\lambda \to AB$ and $ABC \to \lambda$; alternatively, one can use $\lambda \to ABC$ and $AB \to \lambda$.

3.3 Multiset grammars

If the underlying data structure is a multiset instead of a string, then without additional control the power of sequential systems is equivalent to the power of partially blind register machines, i.e., they are not computationally complete.

Of course, using additional mechanisms, computational completeness can be reached again. For example, we can use inhibitors, thus obtaining rule forms $AB \to C$ and $A \to BC|_{\neg F}$, where $|_{\neg F}$ in $A \to BC|_{\neg F}$ indicates that the rule $A \to BC$ can only be applied if no F appears in the underlying sentential form. Not surprisingly, a single rule form $AB \to CDE|_{\neg F}$ is enough in the accepting mode. In Section 5 we avoid inhibitors by switching from the sequential derivation mode to the maximally parallel derivation mode.

4 Turing machines and circular Post machines

For Turing machines, in general the transition rules are of the form $(p, A) \to (q, B, d)$, where $p, q \in Q$ are *states*, $A, B \in V$ are tape symbols, and $d \in \{L, R\}$ is the direction of the head move. As rewriting rules, these transition rules can be written/simulated as follows:

- $(p, A) \to (q, B, R)$ is simulated by $pA \to Bq$, and
- $(p, A) \to (q, B, L)$ is simulated by $CpA \to qCB$, for any $C \in V$.

However, at the expense of intermediate states, it is possible to split rules into two categories: rules with rewriting, but without moving and rules without writing but moving, which, in Turing machine notation, yields transition rules of the form $(p, A) \to (r, B, S)$, where S indicates that the tape head stays on its position, as well as $(r, B) \to (q, B, R)$ and $(r, B) \to (q, B, L)$, respectively. These transition rules correspond with rewriting rules in the following way:

- the rules $(p, A) \to (r, B, S)$ and $(r, B) \to (q, B, R)$ are simulated by $pA \to rB$ and $rB \to Bq$, and
- the rules $(p, A) \to (r, B, S)$ and $(r, B) \to (q, B, L)$ are simulated by $pA \to rB$ and $Cr \to rC$, for any $C \in V$.

The relevant rule forms then are $PA \to RB$, $RB \to BR$, and $BR \to RB$, with $P, R \in Q$ and $A, B \in V$.

An interesting variant of Turing machines is called circular Post machines: the tape is circular, moving is always to the right, but it is possible to delete and insert tape cells. Circular Post machines can be interpreted as string rewriting systems (Post systems of a very specific type) where the symbols are consumed on the left and written on the right.

We now present rule forms derived from the normal form called CPM5 (for example, see [2]):

$pA \to q$ and $p \to Aq$, with $p, q \in Q$ and $A \in V$.

These rules are to be interpreted as working on circular strings, changing the state from p to q, together with deleting symbol A on the left or inserting it on the right, respectively.

5 P systems

We now turn our attention from sequential to parallel rewriting systems. As a special model of that kind, we consider membrane systems, i.e., P systems, with non-extendable multisets of rules acting on multisets in parallel, and the computation halts if no rule can be applied any more. Whereas the general model of P systems allows for the evolution of objects in different membrane regions as well as the passing of the objects through the membranes to inner membrane regions or to the outer membrane region, we here restrict ourselves to the trivial case of one-membrane systems. We refer to [8] and [9] for the main definitions and results in the area of membrane systems as well as to the P systems website [11] for actual news in this area.

Even in one-region P systems, the rule forms $A \to BC$ and $AB \to C$ are sufficient for obtaining computational completeness in the generating case, and the single rule form $AB \to CDE$ is enough for the accepting case. These rule forms are similar to the ones being sufficient in the case of sequential string grammars, whereas we remind the reader that in the multiset case the sequential application of multiset rules of these forms is not enough.

Co-operativity can be restricted in various ways. For example, using anti-matter objects and anti-matter rules, we get a Geffert-like set of rule forms $A \to BC$ and $AB \to \lambda$ (e.g., see [1]).

5.1 Purely catalytic P systems

Using catalytic rules only, we get rule forms $\{cA \to cB_1 \cdots B_k \mid k \in F\}$, where $c \in C$ stands for one of the symbols in the set of catalysts C, i.e., symbols which are never changed, and the B_i represent other symbols $\notin C$. For a given finite set F the lengths of the right sides are bounded, therefore this is a finite set of rule forms. As already shown in [3], $F = \{0, 1, 2, 3\}$ is sufficient in the generating case. Using an additional dummy symbol e, we immediately get that $F = \{0, 3\}$ is sufficient, too. As $F = \{0, 1\}$ would not allow the underlying multiset to grow, the challenging question for future research is whether $F = \{0, 2\}$ is already sufficient, too.

6 Further variants

In the case of sequential string grammars, further variants of rule forms necessary and sufficient for obtaining computational completeness may be investigated, especially based on different variants of the Geffert normal form.

In the case of parallel rewriting systems, especially many further variants of P systems remain to be investigated carefully, for example, P systems with membrane generation and membrane dissolution as well as P systems with active membranes.

7 Conclusions

We have presented a number of generating and accepting devices and corresponding sets of rule forms sufficient for obtaining computational completeness. We first discussed sequential grammars as well as Turing machines and circular Post machines as specific models for computations on strings. For sequential string grammars, several sets of rule forms sufficient for obtaining computational completeness were described, especially using specific well-known normal forms as the Penttonen normal form and the Geffert normal form.

Moreover, we then considered the model of membrane system (P systems), where multisets are evolving by applying non-extendable multisets of rules in parallel. We have especially focused on one-membrane P systems and investigated rule forms necessary for several variants of P systems. For purely catalytic P systems we have raised the challenging open question if the number of non-catalysts on the right-hand side of the catalytic rules can be restricted to two, whereas we know that three of them are sufficient in the generating case.

Finally, we shortly discussed further variants to be investigated in the future.

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Can Computers Catch the Authors Style?

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Abstract

The paper presents the experiments on authorship attribution for scientific articles written in Russian and Ukrainian. The main aim of this research is to explore the topic influence on author classification.

Keywords: Text classification, authorship attribution, topic classification, machine learning methods, character based learning.

1 Introduction

Authorship Identification is a hot topic in many areas and especially in science and education as a part of plagiarism detection effort. The goal of this study is to verify how strong the topic of the text influences its authorship attribution. We used machine learning methods to classify the scientific articles from different domains by their authors. While the domain specific words could affect the classification, we selected sets of texts from the same domain and performed authorship attribution within the domain. However, even within one domain the papers describe different topics that still affect the classification. We are organizing the experiments in such a way that training and test sets contain different papers of the same author in order to avoid topic influence.

2 Related work

The methods used in the domain of authorship attribution vary from purely manual meticulous analysis of the different text elements to the absolutely automate statistical methods such as machine learning. While machine learning is mostly a classification task, classification by the text author is always influenced by the text topic. This phenomena was explored in [1] and the results showed that most of the stylometric variables are actually discriminating topics rather than authors.

Our work is the continuation of the author classification experiments presented in [2] on the scientific papers written in Russian and Ukrainian language. Although there has been made an attempt to avoid the topic influence by experimenting with the papers form one domain (Economics), there was still a significant probability that topics of the papers influenced the classification and helped to obtain such high results (f-measure from 0.85 to 0.95 on different datasets). In this paper we reorganize the experiments in order to avoid topic classification as much as possible.

3 Experiments Description

The Dataset. We used the same dataset described in [2]. There were two data sets: (1) the whole one that included 271 Ukrainian and 77 Russian articles written by 32 and 8 authors respectively (we experimented with them apart) and (2) the part consisting of 175 Economics and 65 IT articles written by 18 and 10 authors respectively (experimented apart as before).

The Method. We worked with classification method on the base of PPM compression algorithm. It demonstrated its ability to classify the short forum posts by the author with impressive accuracy of almost 90% [3]. The method uses as the features all sequences of characters of length 5, 4, 3, 2 and 1 character from texts. We tested two variations of this method: (1) absolutely all characters from texts including upper and lower case letters, numbers, spaces, all kind of punctuation marks and other specific characters which appear in scientific publications; (2) only words composed of alphabet letters converted to lowercase and spaces between them.

Text Organization. The main difference from the previous experiments consist in the text organisation. The main problem with the text we worked with was the differences in their sizes: it differs from 150 words to 50 pages and such differences may influence the classification results. This was the cause for the decision to split all texts in fragments of approximately 150 words and classify these fragments by author. However, we suppose that the fragments of the same article were classified not by their author but by their common topic. Thus, in the current set of experiments we does not divide the texts in fragments and work with them as they are dividing them in training and test sets in such way as every author is presented in both sets. We used 5-fold cross-validation and there were around 2 texts from each author in each test set.

Table 1. Results of the author classification experiments using PPM and two variations of the features: all characters (PPM char) and only lowercase letters (PPM letters) on four sets of texts

	Ukrainian	Russian	Economics	IT
PPM(char)	0.601	0.950	0.604	0.800
PPM(letters)	0.585	0.913	0.580	0.782

Table 1 presents the results of the experiments. In comparison with the previous work we can say that high results remained only for Russian texts. The previous were 0.972 and 0.979; slightly higher. For the rest of the sets the results are much worse. For example, previous results for Ukrainian were 0.854-0.865 and for IT 0.943-0.958. In case of Russian texts, the high results are explained by the differences in topics the 8 authors in question wrote about.

4 Conclusion

In this paper we tested the hypothesis that author classification is highly influenced by the text topics. The results of our experiments confirmed this. There is still possibility that the differences of text sizes influenced the results; we plan to explore the possibilities to even out the text sizes. The other plan is to investigate the influence of the number of the authors on the classification accuracy.

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Risk Detection Algorithm Design: Officer Fall Case Study*

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Abstract

This paper is an overview of the latest research results achieved within the ongoing NATO SPS program project – WITNESS. In particular, some details are presented regarding the algorithm for the state detection of a security officer deployed on the ground. We will focus on the sudden fall or drop attack case studies.

Keywords: surveillance, drone, sensor, data analysis, data fusion.

1 Introduction

The massive evolution of technologies has opened new ways to research and solve problems from a wide range of fields: social-human; security; health; transport; monitoring etc. In the context of ensuring security, one of the problems is the detection of (crisis) risk situations for security agents. As an information provider in this case it can be a small digital device (not to burden the agent), with a sufficient operating battery life during a mission (up to 8-10 hours). Such a device can be the agent's mobile phone device. Modern smartphones are equipped with a vast variety of sensors: gyroscope, accelerometer, magnetometer, Global Positioning System (GPS) proximity sensor, ambient light sensor, microphone, fingerprint sensor, pedometer, barometer, heart

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rate sensor, thermometer, air humidity sensor, etc. The accelerometer detects acceleration, vibration and tilt to determine the exact movement and orientation along the three axes: x, y and z. This sensor can be used to determine the position of the phone (vertically or horizontally oriented), the position of the screen (facing up or down), the speed of movement of the phone in any linear direction.

In the following section we will describe how we can use the accelerometer to detect the risk situation for the security agent – the fall.

2 Fall detection

The parameter provided by the fall analysis function (F) is a very important one, capable of improving the accuracy of detecting the critical situation. The fall of the police officer could signal a risk of attack on him (injury, hit, etc.). The agent will carry the phone close to the waist (in a pocket, fastened by a belt clip, etc.).

Function F will have four modules:

1. *the interoperability module* will ensure the transition between the phone's accelerometer and the system that will retrieve the data;

2. *the motion data transformation module* (acceleration values x, y and z), provided by the accelerometer, will be passed through a low-order filter 1st order IIR (Infinite Impulse Response) [1] and will be annotated and mapped into a comma-separated value file format (comma-separated value file format); each filtered acceleration value will be stored locally for use in the next module (data analyzer);

3. *the data analyzer* will be responsible for processing and analyzing the acceleration values (x, y and z) to detect if these values represent a drop. One can use a classification model based on decision trees from knowledge of past events [2]. This model is to be developed a priori and will contain similar patterns of fall behavior.

It will be kept in the cloud and will be filled with new patterns every time the data analyzer finds a drop. Such a model, based on decision trees, can be built from the SisFall dataset [3]. The decision tree is widely used for data classification and is one of the best methods of classification and regression among many other Machine Learning algorithms because it uses a non-parametric structural design, which makes it efficient for managing data sets with large dimensions.

4. *the alert manager* will send notifications about the fall, including to the cloud service (to update the mentioned model).

We mention that the methodology and the process of data collection are of particular importance. This is why our decision was to collect data collection in batches of different sizes (by default 10):

– vertical displacement of the phone, in particular if the phone falls 25, 50 cm or more;

– horizontal displacement of the phone, the phone makes a sudden move of 25, 50 cm or more on the x-axis;

– a derivative of previous observation – the agent falls (that is, it suddenly reaches a horizontal position) with the phone in the pocket of the shirt;

– a derivative of previous observation – the agent falls (that is, it suddenly reaches a horizontal position) with the phone near its belt.

Decisions on the first parameter regarding the risk situation can be made based on the coefficient k $= \dfrac{A}{g}$, where A is the movement acceleration of the phone, and g – the gravitational acceleration.

The acceleration A is given by the formula $A = \sqrt{a_x^2 + a_y^2 + a_z^2}$, with a_x, a_y, a_z being the accelerations of motion along the x, y and z-axes. The value of k will be close to 0 when the phone is idle (i.e. no sudden displacements). The value of k will exceed 1 as long as the phone is in motion. The value of k is close to 1 when the phone is in free fall, that is, when the phone falls, or when it has an equivalent to that displacement in other dimensions. Besides this approach, the model additionally relies on standard probabilistic methods to detect outliers using $z - score$ or standard deviation from the mean of the data collected prior to model deployment.

3 Conclusion

The parameter provided by the fall analysis function is a very important one, capable of improving the accuracy of detecting the critical situation. The methodology and the process of data collection is particularly important.

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Enhancement of Historical Documents Scans by P system based Image Processing Techniques

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Abstract

In the presented work, the developed by author segmentation technique based on P system is applied to pre-OCR enhancing of document images. Developed for solving the noisy medical imaging problems, the proposed technique has the advantage features, which ensure successful application to the historical document images enhancement problem. The first advanced feature is the high sensitivity of the P system based algorithm, which recognizes the tiny details. Another advantage of the P system based algorithm is its natural parallelism that allows parallel processing of regions of a huge image. The use case of the proposed work is old books printed in Romanian Cyrillics.

Keywords: document image enhancing, P system, image processing

1 Introduction

Current work presents the P system based technique for pre-OCR enhancing of document images. This techniques is developed as the component of the technology of free access to cultural heritage in the form of historical texts. Today, most research in this domain focuses on converting vast archives of scanned historical documents into accessible text form. The implementation of such converting has a lot of challenges. The presented work refers to the answer to the challenge that is considered as the main one [1]. This challenge springs from high dependency of OCR techniques on scans quality. All kinds of document images have the usual problems: noise, stains, shadings. In addition, the historical documents images have specific problems, pages can be damaged by poor preservation and can have different textures: parchment, papyrus, palm leaf. To solve these problems, the main stages of pre-processing are the enhancement and binarization of document images [2]. Document image enhancement here is the improving of the perceptual quality for maximum restoring of document initial look, the binarization is the separation of texts and background. Modern researchers implement the stages of enhancement and binarization of document images, applying the full range of existing image processing techniques. In recent works, deep learning methods are mainly applied, but modern versions of classical algorithms are also widely used. The presented work applies the P system based segmentation algorithm to the document image enhancing. P system computing is a particular branch of unconventional computing. The P system model is based on the cell-like hierarchical arrangement of membranes, which are the main computing "devices". Membranes define containers filled with objects named multisets. The P system computing engine works by executing rules on multisets in parallel. P systems demonstrate challenging results in applications to images processing. The P system based segmentation algorithm was developed to process the noisy grayscale medical ultrasound images [3]. These specific features provoke the idea of applying this processing to historical document images.

2 Enhancement

Applying of the P system based segmentation algorithm to historical document image makes enhancement and some elements of binarization. As any segmentation, presented algorithms allow separating color of letters and color of noise. But the historical document images processing has to implement the specific feature that is very suitable for the P system based algorithm. This feature is the need to recognize the tiny details: specific marks between the lines, flourish and nonstandard letters, and so on. P system based algorithm is very sensitive and can find even tiny segments. Another advantage of the P system based algorithm is its natural parallelism. Pre-OCR processing can be done in parallel, having just a slight interlacing to avoid information losses.

Figure 1. Results of presented use case simulation

The use case of the proposed work is old books printed in Romanian Cyrillics. Actually, this use case springs the presented research because of the hard requirement of the mentioned above specific feature. The Romanian Cyrillics books have: the letters with non-standard typeface; placed outside the line specific signs that reflect the regional specificity of reading. All these details are of particular interest to linguists and historians, who are the main users of historical test archives. Thus, when processing this use case, the enhancement algorithm should preserve the indeterminate character rather than remove it.

Like most branches of unconventional computing, P systems one has no real computing devices and runs on the simulator. The used simulator is based on P lingua [4] framework. After several circles of simulations/rules adjusting, during which the gray levels of a particular book are mapped in the rules, P system based algorithm shows the acceptable results (Fig.1).

3 Conclusion

The presented research was done as the answer to the challenge of converting vast archives of scanned historical documents into accessible text form. Following the tendency of applying existing image processing methods to historical documents images, the segmentation algorithm based on P system was tested on this problem. The presented algorithm was initially developed for solving the noisy medical imaging problems. The advantage features, required to process this type of image, ensure the successful application of the P system-based segmentation algorithm to enhancement of historical documents scans.

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A new approach of designing the Intelligent Tutoring System "GeoMe"

Olesea Caftanatov

Abstract

This paper provides a study on intelligent tutoring system's architecture. Additionally, it is proposed a new architecture approach for our ITS "GeoMe", based on seven-model, which emphasizes the learners affected state, affords and identifies students' learning style without their direct involvement.

Keywords: Intelligent tutoring system, affective model, geometry tutoring system.

1 Introduction

Intelligent Tutoring System (ITS) is a part of a new breed of instructional computer programs with the aim to provide immediate support one-on-one and personalized feedback to learners. Canfield [1] defines ITS as a system that is able to diagnose and adapt to student's knowledge and skills. Along the years, there are many definitions regarding what intelligent tutoring systems are, however, the common point is that, it uses Artificial Intelligence (AI) techniques in order to track learner's needs and respond with an appropriate feedback.

2 Intelligent Tutoring System "GeoMe"

GeoMe stands for "geometry for me". It is a ITS designed to help pupils in learning geometry by personalizing their learning paths. Geometry proving theorem is known to be very challenging for students to learn. Thus, almost all ITSs proposed for geometry are dedicated to learning proof-writing with constructions. We intend to deliver learning material for elementary geometry for elementary schoolchildren.

Therefore the knowledge model emphasizes the identification of basic shapes, properties of shapes, the shape's comparison etc.

3 ITS GeoMe Architecture

Despite the fact that all ITSs share the same goal: to provide tutorial services that support learning, but almost all of them show a vast variety of approaches to conceptualize, design and develop these services. There is a range of ITS architectures developed for specific application domains with a philosophy behind, from traditional trinity model (domain, student, tutoring) to classical standard model architecture with four components (domain, student, tutoring, interface).

In the same vein, we suggest a seven-model architecture for ITS "GeoMe", see Fig.1. The seven-model architecture retained the four major components of classical model and added ***"affective model"***, ***"system control"***, ***"learning style"*** components.

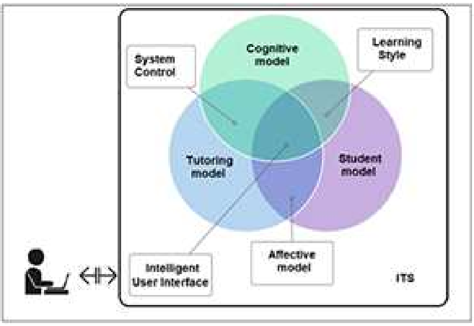

Figure 1. ITS Architecture proposed for GeoMe tutoring system.

- ***Domain model*** – also called "expert knowledge" or "cognitive model" – contains concepts, facts, rules and problem-solving strategies of the domain to be learned. In other words, emphasis is on *what is needed to be learned.*
- ***Student model*** – refers to the dynamic representation of the emerging knowledge and skill of the student. Nwana [2] claims that "no intelligent tutoring can take place without an understanding of the student", thus, developing student model becomes

a main theme for many researchers, starting with Hartley & Sleeman [3], Woolf [4], Self [5], and many others. In the best possible way, this model should include data regarding learner's behaviour, skills, performance, knowledge level, preference in learning style, affective state etc.

- ***Tutoring model*** – also called "instructional model" or "pedagogic module" – contains information about effective tutorial practices and monitors the status of the student model. According to Marouf et. al. [6] one of the function of tutoring model is to *"track the learner's progress, to build a profile of strengths and weaknesses relative to the production rules (termed as "knowledge-tracing")"*.
- ***Intelligent User Interface*** – also called "Inference module" or "communication module" – Along the years, the researches on user interface made this component to evolve from it primitive form into intelligent, adaptive or even invisible interface. The major requirement to interfaces is to be easy to use and to be attractive, thus, students will learn quickly how to use it, and then they can focus all their attention to learning materials.
- ***Affective model*** – this component delivers "emotional elements" to learners, its main function is to engage students in learning process, by using multiple forms of learning environment, including elements from gamification, simulations, sharing achievements, hypermedia, avatars etc.
- ***System control*** – this component detects errors and provides helpful feedback on student's input. Respond if it cannot diagnose an error or intervene to remediate a misconception or a missing concept.
- ***Learning style*** – this component identifies student's learning style by using bayesian approach. Students with different learning styles have different needs in the learning process. There are many researches in this field but most of them need learners' direct involving by collecting data through questionnaire. However, our approach consists in collecting needed information with

indirect learner's involving, in other words, we track their behaviour during 9 sessions, afterwords, we detect the patterns of specific type of material and then deliver information on user's preference.

4 Conclusion

The novelty of an ITS and its interactive components is quite engaging, and we consider that developing an ITS for mobile devices has a lot of educational potential. Analyzing the existed tutoring system purposed for geometry, we observed that almost all of them are dedicated to proof solving subject. Thus, we pursue this topic, by emphasizing on elementary geometry content. Additionally, we consider that classical ITS with four-model architecture must have the affective, learning style and system controls components, because the aggregation of several models allows the organizing learning process more flexible.

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Generation of the Romanian Cyrillic lexicon for the period 1967 – 1989

Constantin Ciubotaru, Valentina Demidova, Tudor Bumbu

Abstract

This paper is dedicated to the issue of generation of the Romanian Cyrillic lexicon used between 1967 and 1989. The rules for transliteration of words in the modern Romanian lexicon in their equivalents written in Cyrillic and vice versa are specified and argued. The respective algorithms have been developed and implemented, which is an expert tool in the lexicon generation process. The activity of the expert is reduced to the verification of the transliterated variants and the modification of the transliteration rules.

Keywords: lexicon, transliteration, Moldovan Cyrillic, cyrillization, romanization of Cyrillic, morpho-syntactic tagsets.

1 Introduction

The problem of digitizing and preserving the linguistic historical heritage is a priority domain of the digital agenda for Europe. The digitization process requires solving a series of problems related to the recognition, editing, translation, and interpretation of printed texts. The solution of these problems has to do with specific difficulties: a large number of periods in the evolution of the language, a small volume of resources widely distributed, a large diversity of alphabets used in their printing.

The paper addresses the issues related to the digitization and transliteration of the historical linguistic heritage printed with Cyrillic alphabet in the period 1967–1989 on the territory of the Moldovan Soviet Socialist Republic following the linguistic norms of the modern Romanian language. The alphabet used in this period (the Moldovan Cyrillic alphabet, AlfCYR) is, in fact, the alphabet of the Russian language from which the letters "ё", "щ" and "ъ" were excluded and extended in 1967 by the introduction of the letter "ӂ". Specific difficulties: lack of resources in electronic format, fragmentary grammatical descriptions that allow ambiguous interpretations.

According to the definition of dexonline [1] *transliteration* is "the transcription of a text from one alphabet to another, rendering the letters by their equivalents, without regard to the phonetic value of the signs".

The process of transliterating the Romanian words in their written equivalents with characters of the AlfCYR alphabet we will call *cyrillization*. For instance, *puiului*⇒*пуюлуй* (chicken), *fiului*⇒*фиулуй* (son), *cenușiu*⇒*ченушиу* (greyish), *viermi*⇒*вермь* (worms), *vierii*⇒*виерий* (boars).

The inverse procedure for cyrillization will be called *romanization of Cyrillic*, e.g. *пуюлуй* ⇒ *puiului* (chicken), *бьет* ⇒ *biet* (poor), *боер*⇒*boier* (boyar), *пепт*⇒*piept* (chest).

Taking into account the lack of Romanian Cyrillic resources for this period, the following lexicon generation algorithm is proposed. Sets of rules are defined after which the algorithms for cyrillization and romanization of Cyrillic are constructed. The cyrillization algorithm is applied to the Romanian lexicon (we used the lexicon developed at the "Al. I. Cuza", Iasi [2]), noted by LexROM. There will be obtained a variant of the Cyrillic lexicon which can be further processed by the algorithm for the romanization of Cyrillic, thus obtaining a new variant for the Romanian lexicon. The ideal situation would be for these two lexicons to coincide. If mismatches occur, the expert(s) intervenes, who can modify the rules of cyrillization \romanization of Cyrillic, repeat the whole process or intervene with corrections on the constructed

Cyrillic lexicon. The diagram of this process is inserted in Figure 1. The accuracy of the obtained Cyrillic lexicon largely depends on the qualification of the expert(s).

2 Cyrillization

If the problem of digitizing and recognizing the printed text is solved relatively simply, then the problem of cyrillization is more difficult. The rules of transliteration according to cyrillization can be divided into two categories: general rules and context-sensitive rules. The general rules establish the situations when the transliteration directly substitutes letter with a letter(s) ignoring the contextual dependencies. Thus, the letter *b* will be always transliterated into *б*, we will note this through *b* ⇒ *б*. The same action will be performed for the following pairs: *a* ⇒ *а*, *ă* ⇒ *э*, *â* ⇒ *ы*, *c* ⇒ *к*, *d* ⇒ *д*, *e* ⇒ *е*, *f* ⇒ *ф*, *g* ⇒ *г*, *h* ⇒ *х*, *i* ⇒ *и*, *î* ⇒ *ы*, *j* ⇒ *ж*, *k* ⇒ *к*, *l* ⇒ *л*, *m* ⇒ *м*, *n* ⇒ *н*, *o* ⇒ *о*, *p* ⇒ *п*, *r* ⇒ *р*, *s* ⇒ *с*, *ș* ⇒ *ш*, *t* ⇒ *т*, *ț* ⇒ *ц*, *u* ⇒ *у*, *v* ⇒ *в*, *x* ⇒ *кс*, *z* ⇒ *з*.

For instance, *chirilizarea* ⇒ *кхирилизареа* (cyrillization). Of course, that *chirilizarea* ⇒ *кирилизаря* would be correct. In this case, the contextual dependencies intervene. Because such dependencies are many and quite complicated, we will only expose a few. More information on this topic can be found in [4]. From the example above we notice that the transliterations *chi* ⇒ *ки* and *ea* ⇒ *я* are correct. A few more contextual rules: *gi* ⇒ *жи*, *ghi* ⇒ *ги*, *ci* ⇒ *чи*, *ci* ⇒ *ч*, *iu* ⇒ *ю*, *iu* ⇒ *иу* (the examples in paragraph 1).

Many difficulties arise when transliterating the letter *i* at the end of the word. All the possible transliterations: *i* ⇒ *и* (*citi* ⇒ *чити* (read)), *i* ⇒ *й* (*pui* ⇒ *пуй* (chicken)), *i* ⇒ *ь* (*arici* ⇒ *аричь* (hedgehog), plural), *i* ⇒ (*arici* ⇒ *арич*, singular). To make the correct decisions, contextual rules can sometimes be supplemented with morpho-syntactic information. We use MSD (morpho-syntactic-description) tags present in the LexROM lexicon [2]. For example, for infinitive verbs the letter *i* at the end of the word will be translated into *и*, the masculine nouns in the plural nominative-accusative case the articulated form will end in *ий*, and the transliteration $i \Rightarrow$ *ь* at the end of the word is characteristic for nouns in the plural dative-genitive case, and also for verbs, the second person present, past, more than perfect.

The order of the two categories of transliteration rules is very important. Contextual dependencies always take precedence over general rules.

CYRILLIZATION ALGORITHM

0. ***Start***

1. The lexicon of the modern Romanian language is given [2] (we will note it LexROM_1) and the rules of cyrillization (general and context sensitive).
* We will build the Romanian Cyrillic lexicon for the period 1967–1989 (we will note it by LexCYR) *\

2. Initially LexCYR = ∅

3. ***For all*** words *wrom* from LexROM_1:

- **3.1.** Apply on *wrom* the context-sensitive rules for cyrillization. Note the result by $wcyr_1$.
- **3.2.** Apply on $wcyr_1$ general rules for cyrillization. Note the result by *wcyr*.
- **3.3.** Include *wcyr* in LexCYR.

4. ***Stop***

3 Romanization of Cyrillic

Romanization of Cyrillic is facing the same problems as cyrillization. The general and contextual rules for this procedure are also defined. The general rules are relatively simple, for example, *а* ⇒ *a*, *р* ⇒ *r*, *ю* ⇒ *iu*, *ь* ⇒ *i*. If only the general rules apply to transliteration, we obtain, for instance, *пуюлуй* ⇒ *puiului*, *бьет* ⇒ *biet*, *боер* ⇒ *boer*, *пепт* ⇒ *pept*. The last two transliterations are incorrect and their correct variants are: *боер* ⇒ *boier*, *пепт* ⇒ *piept*. In this case contextual rules are also needed. E.g, $г\omega \Rightarrow gh\omega$, if $\omega \in\{е,и,я,ю,ь\}$ and $г\omega \Rightarrow g\omega$, if $\omega \notin\{е,и,я,ю,ь\}$ (*георгинэ*⇒*gheorghинэ* (dahlia), *гогоашэ*⇒ *gogoашэ* (donut)). Rules for letter *я*: *я* ⇒ *ia* (usually at the beginning of the word), *ия* ⇒ *ia* (usually at the end of the word), *я* ⇒ *ea*. It is very difficult to make the right decision for the presence of the letter *я* inside the word. In such situations the algorithm will use the rule *я* ⇒ [*ia*][*ea*], leaving the correct decision to the expert.

Another difficult problem to mention is the transliteration of the letter *ы*, which can be substituted either by î or by â. Our algorithm follows the recommendations of the Romanian Academy regarding this spelling.

ROMANIZATION OF CYRILLIC ALGORITHM

0. ***Start***

1. The Romanian Cyrillic Lexicon for the period 1967–1989 (LexCYR) and romanization of Cyrillic rules (general and context sensitive) are given
* We will build the modern Romanian lexicon (we will note it by LexROM_2) by applying the transliteration method *\

2. Initially $\text{LexROM}_2 = \varnothing$

3. ***For all*** words *wcyr* from LexCYR:

- **3.1.** Apply on *wcyr* the context-sensitive rules for romanization of Cyrillic. Note the result by $wrom_1$.
- **3.2.** Apply on $wrom_1$ general rules for romanization of Cyrillic. Note the result by $wrom_2$.
- **3.3.** Apply on $wrom_2$ rules of transliteration for letter *ы*. Note the result by *wrom*.
- **3.4.** Include *wrom* in LexROM_2.

4. ***Stop***

4 Lexicon generation technology

As mentioned above, the electronic resources for the period 1967–1989 are completely absent, complete exposure of the grammar used is missing, many of the interpretations of the transliterated words are ambiguous. Therefore, the expert plays a major role in the lexicon generation process. The proposed technology aims to automate this process. With the available cyrillization and romanization of Cyrillic algorithms, but also access to the formalized rules, the lexicon generation process can take a few iterations. At each iteration, the expert(s) intervene(s) to modify the set of rules and, possibly, directly, the Cyrillic lexicon. This scheme is described in detail in Figure 1.

5 Conclusion

The paper proposes a technology for the generation of the Romanian Cyrillic lexicon for the period 1967–1989 applying the transliteration method. Starting from the lexicon of the modern Romanian language elaborated at the University "Al. I. Cuza", Iasi [2] (1.096.674 words) the cyrillization and romanization of Cyrillic algorithms are applied consecutively. The intermediate results are available to the experts, who can modify or extend the set of rules applied to transliteration, but also directly correct the obtained Cyrillic lexicon. The final lexicon will be a result of performing several such iterations. The main problems that need to be solved by the expert(s) are the ambiguities that arise as a result of cyrillization\romanization of Cyrillic. For example, *iu* ⇒ [*ю*] [*иу*], *eie* ⇒ *е* [*е*] [*йе*] [*ие*] – at cyrillization, *я* ⇒ [*ia*] [*ea*] – at romanization of Cyrillic.

For all the words in LexROM starting with the letter "c", in total 171846 words, 6381 ambiguities were found at the first iteration, which represents 3.7 %. Two iterations were needed to overcome these ambiguities. Of course, the accuracy depends considerably on the qualification of the expert. Also here, at the first iteration comparing the lexicon LexROM_1 with LexROM_2 we obtained 8960 words that do not

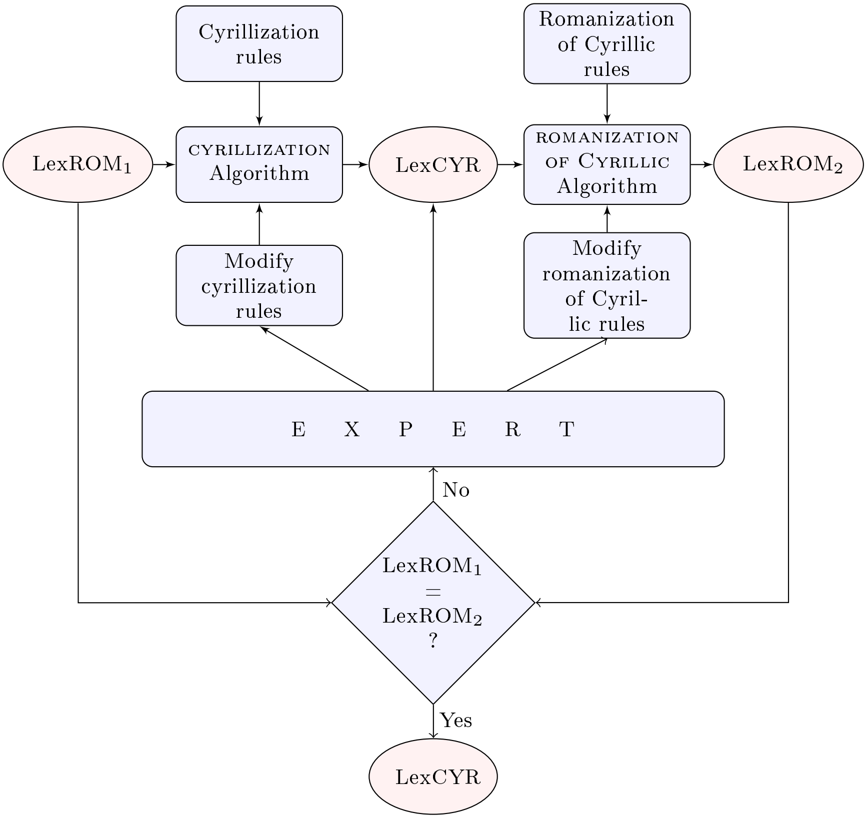

Figure. 1. The scheme for generating the Romanian Cyrillic lexicon

coincide, which represents about 5.21%. Most of these mismatches are incorrect general rules for the romanization of the Cyrillic. The proposed technology allows returning to the previous intermediate variants, thus revising the lexicon. To better understand the role of the expert and contextual dependencies, we applied on LexROM lexicon only the general rules (paragraph 2) of cyrillization. As a result, we got only

42.2% correct words.

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Information system support for Cerebrovascular Accident prophylaxis

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Abstract

This paper briefly presents the results obtained in the process of information system StrokeMD elaboration and predictive models creation based on the data got from the database StrokeMD.

Keywords: stroke, risk factors, stroke prevention, information systems, database StrokeMD, data on patients, predictive models.

1 Introduction

According to data presented by World Health Organization [1] cardiovascular diseases are the number 1 cause of death globally. The 85% of these deaths are caused by heart attack and stroke. In the World Stroke Organization Annual Report (2018) it is specified that 13.7 millions of stroke are recorded each year, and 5.5 millions of people died due to stroke per year [2]. We will mention that starting with 2018 in International Classification of Diseases, Revision 11 (ICD 11) [3] Stroke is no more placed under diseases of the circulatory system being reclassified along with all cerebrovascular diseases to form a single block under diseases of the nervous system.

It is important to emphasize that most cardiovascular diseases can be prevented by reduction of the behavioral risk factors and by keeping under control the factors related to medical conditions, especially metabolic risks. It is established that the behavioral factors are responsible for 72.1% of stroke, and the metabolic ones are registered in

66.3% of them [3]. Based on these data one can deduce the importance of stroke prevention activities.

A support for prophylaxis measures can be provided by information systems, which contain records with data on people who have undergone a routine medical control (primary prophylaxis) or have suffered a stroke (secondary prophylaxis). Such systems are developed at international, national or local level, see for example [4–6].

In order to reduce the number of people who have suffered or are in the risk group of stroke, it was decided to create an information system StrokeMD for the Republic of Moldova. The StrokeMD system is intended for the collection, processing and visualization of data on patients with primary or secondary stroke risk. The main purpose of the application is to facilitate the efficient introduction and processing of patient data and the subsequent extraction of knowledge, based on which predictive models for stroke can be developed.

2 StrokeMD database: the specificity, structure, functionality

The system StrokeMD has the following functions: it serves as an electronic patient register, offers the possibility of clustering the population into risk groups to start stroke prevention campaigns. Also it serves as a research platform in the field of stroke prevention, treatment, etc.

The database of the system contains 9 modules, connected to the examination protocol for primary and secondary prophylaxis, developed by the team of academician S. Groppa.

StrokeMD provides elaboration of statistical reports, graphical visualization of data, highlighting deviations from the normal indicators with offering suggestions. For convenience in use, the system interface is implemented in two languages – Romanian and English. The working version is also elaborated on mobile devices (tablet, smartphone).

3 Mathematical modeling of stroke risk

The information registered in the StrokeMD system gives the possibility of developing mathematical models, which could be applied to the personalized risk assessment of stroke. As a tool for the elaboration of such models, the Weka system was chosen – a collection of machine learning algorithms used for data mining.

We operated with information about 339 patients included into DB. Each record contains 123 parameters, some of them having no value or being marked as unknown. Using different methods supplied by Weka, we have selected 21 most significant parameters for all the applied methods, and then we have left only 10 of them, besides the target parameter “stroke on the date of clinical examination”. In our second approach the targeted selection of parameters for model creation – different sets of 10 params were done basing on various considerations: either the features considered as common risk factors, or the features used in published prognostic models. According to these parameters in both cases we applied the methods based on splitting data in training and validation sets, as well as the cross validation method, and found that the last approach gives better results.

It is significant that some parameters from our list of the selected ones are related to the Doppler examinations. This examination is difficult of access and the respective values can be missing at a number of patients, so it is a usual practice that when creating different prognostic models they are intentionally excluded from the list of examined parameters, or the comparative analysis is carried out: models with Doppler indicators as against models without them. We used this approach as well, and our next step was excluding parameters with Doppler examinations from our dataset. All the procedure with parameters ranking and models creation was reiterated. Surely, the selected ranked parameters differed from the previous set with Doppler parameters. But, the results with models creation were much the same when considered the effectiveness of methods for model construction.

4 Conclusion

The objectives for creation of the system StrokeMD and the above described mathematical models are:

- collecting information on stroke cases during time period and identifying risk groups;

- improving the quality of care offered to stroke patients;
- prevention of stroke in various population groups, including the young;
- statistical processing of data and research of evolution over time.

The information registered in the StrokeMD system gives the possibility of developing mathematical models, which could be applied to the personalized risk assessment of stroke. Our goal for today is to find significant parameters for our specific dataset. We tried various ways to find them (including ranking and targeted selection). The work is to be continued.

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On Digitization of Documents with Script Presentable Content

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Abstract

The paper is dedicated to details of the digitization of printed documents that include formalized script presentable content, in connection with the revitalization of the cultural heritage. We discuss the process and the necessary software by an example of music, as the recognition of scores is a solved task.

Keywords: information technologies, digitization of heterogeneous content, script presentable content, optical music recognition.

1 Introduction

We met a need for optical recognition of heterogeneous content at the revitalization of the printed cultural heritage.

The whole bunch of connected problems and challenges is presented in [1].

In the paper we'll discuss: widespread script presentations of music; the process of score digitization illustrated by an example; results of digitization and their possible use; organization and necessary support of work with heterogenious documents.

Review of OMR (Optical Music Recognition) software and details of its work can be found in a comprehensive survey [2].

2 Script presentation of music

Historically, content had been presented mainly as plain text with drawings. From other heterogeneous content elements, scores had been the earliest. The oldest interpreted record of a religious song (*Hurrian Hymn to Nikkal*) is dated back to 1400 B.C.[1], and it is in fact a script. Later, the graphical presentation of music was developed.

Modern script presentations of musical content emerged with computer music typesetting.

Exchange standards between music software are MIDI[2] and MusicXML[3].

- MIDI is a standard industrial communication protocol for electronic musical instruments. Information is transmitted in the form of binary messages.
- MusicXML is an XML file with specific tags and attributes for music.

3 Process of score digitization

We will use the score of *The wedding waltz* by Eugen Doga. Stages of the process are: scan; image refinement; OMR; manual correction of the result in a music editor.

The scan should be performed in black-and-white, 300 DPI. We need BMP or TIFF without compression image format. This is dictated by SharpEye[4] that is our recognizer. Other OMR programs may permit other formats. We applied afterscan image refinement tool Scan Tailor[5]. Fragment of such scanned image is in Tab. 1(A).

SharpEye can recognize several images at once in its batch mode. The result of recognition is opened for viewing as in Tab. 1(B). It

[1] http://urkesh.org/attach/duchesne-guillermin%201984%20the%20 discovery%20of%20mesopotamian%20music.pdf

[2] Musical Instrument Digital Interface; https://www.midi.org/

[3] https://www.musicxml.com/

[4] http://www.visiv.co.uk/

[5] https://scantailor.org/

may be immediately edited. However, dedicated music editors provide much more possibilities. To pass the recognition result to external music editor, we exported it into MusicXML.

There are several restrictions for SharpEye, for example, it recognizes only the Latin script. To correct possible recognition errors we opened the resulting XML file with Musescore[6]. We see in Tab. 1(C) that some details disappeared, for example, glissando and a badly printed note at the end of measure 9. Some mistakes are clearly audible at playing. Then the file was thoroughly checked visually and corrected. The score was also complemented with title and author name. See the fragment of final result in Tab. 1(D).

4 Result and its use

The obtained script presentation of the score has a lot of advantages and ways to use: compactness; generation of visual presentation with different decor and design; easiness of transposition and orchestration; editing (adding, deleting, and replacing notes); replacement of textual components into the score; search by diverse criteria; export in different graphical and script formats; playing; re-publishing; insertion of musical fragments, and other types of content, into the text of a heterogeneous document[7]; montage of music from fragments of different pieces, etc.

5 Conclusion

At present we haven't any possibility to combine formatted text and score in a uniform script. The heterogeneous document is assembled in the text processor by converting any other type of content into images that are inserted on document pages. This motivates the need in an integrated platform to work with heterogeneous texts [1].

[6]https://musescore.org/en

[7]An example mix of text, music, and math formulas:
https://math.ru/lib/files/pdf/Taneev_S_Podvizhnoy_kontrapunkt_strogo_pisma.pdf

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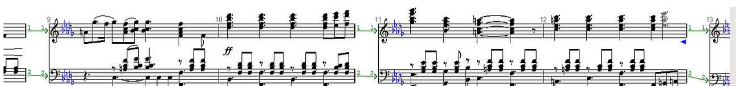
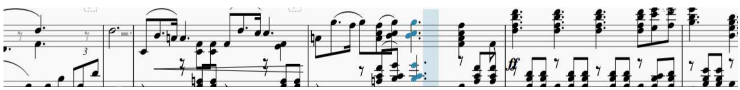
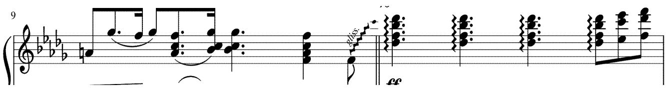
Table 1. Score digitization in progress

A: Scanned	
B: Recognized	
C: Played in editor	
D: Corrected	

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On a Notion of a Calculus Efficiency

Alexandre Lyaletsky

Abstract

An approach to solving the problem "to be an effective logical calculus" is proposed and discussed from a semantic point of view.
Keywords: calculus, category, efficiency, partial algebra.

In spite of the wide use of semantic approaches to solving different problems of modern mathematics, a great attention is focused on the consideration of logical calculi, which are "convenient" from some point of view. Often, such "convenience" is motivated by a personal preference; as a result, the problem of constructing a relation "to be more convenient" (in a certain sense) under a set of "suitable"" calculi is still open for a significant their part.

The described research is devoted to the learning of a "convenience" of logical calculi, when this "convenience" is understood as an "efficiency". At that, the possibility to construct procedures transforming calculi under consideration to more effective ones was investigated.

The research has been made in two directions.

In the first direction, some investigations have been performed for such preorder relations over sets of calculi that can be considered as ones that simulate relations of efficiency in the sense of improving efficiency of logical inference search. For this, in particular, the notion of a "local given preorder relation" was introduced in the category language. As a result, there appeared the possibility to solve the problem of the reduction of such relations to others. (The notion of "density" for partially ordered sets could be useful for this reduction.) But there was made the decision to restrict the consideration by only one special class of local given relations, namely, by so-called relations of an

E_Σ-efficiency, where Σ denotes a category serving as a basis for the construction of E_Σ. The reason for this was the fact that these relations are suitable for simulating the notion of efficiency in a "reasonable" sense. It was shown that *the investigation of "classical" local given relations "ordering" calculi under consideration according to the length of inference search can be reduced to the investigation of E_Σ-relations.*

In the second direction, the main attention was paid to the investigation of constructive partial operations that transform each "constructive" calculus into a more E_Σ-effective one and that does not have fixed points different from the E_Σ-greatest calculi. *Necessary and sufficient conditions of such "effectiveness-giving" operations for so-called logical α-categories have been found*, where α-categories correspond in a certain sense to logics under consideration. At that, a leading role plays an operation $\uplus$ satisfying the following properties:

1. Let Σ be a logical α-category with objects $\mathbb{A}$ and $\mathbb{B}$ being partial algebras in the same signature. Then $\mathbb{A}E_\Sigma(\mathbb{A}\uplus\mathbb{B})$ and $\mathbb{B}E_\Sigma(\mathbb{A}\uplus\mathbb{B})$.

2. Let Σ be such an α-category with objects $\mathbb{A}$ and $\mathbb{B}$ being partial algebras in the same signature that $\mathbb{A}E_\Sigma\mathbb{B}$. Then there exist such objects $\mathbb{A}'$ and $\mathbb{B}'$ in Σ, that $\mathbb{A} \sim_\Sigma \mathbb{A}'$ and $\mathbb{A}'\uplus\mathbb{B}' = \mathbb{B}$, where $\mathbb{A} \sim_\Sigma \mathbb{A}'$ denotes that $\mathbb{A}$ and $\mathbb{A}'$ are E_Σ-equivalent.

The obtained results imply in particular that constructive calculi corresponding to classical and intuitionistic propositional logics have "effectiveness-giving" constructive operations. This fact leads to the possibility to design and realize tools for creating "more and more" effective classical and intuitionistic propositional calculi. The author expresses hope that similar results will take place for first-order logics.

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Features of building analytical enterprise-systems in R

Vladimir Popukaylo

Abstract

Paper describes the main components necessary for Business intelligence systems building using a technology stack based on the R language. The R language is very popular in the academic environment, but it is almost never used for commercial development in our region. The review aims to show what basic tools should be studied for developing full-fledged enterprise applications.

Keywords: R language, buisness intelligence systems, data analysis.

1 Introduction

Currently, the programming language R is undoubtedly one of the most commonly used tools for solving academic problems associated with data analysis. So, according to research [1], the language manual is the most cited non-academic publication in academic papers, and the h-index of the language developers "R Core team" in the Google Scholar system exceeds 50 [2]. However, currently, this language is widely used outside the academic community, which confirms its presence in the upper parts of the ratings of programming languages (TIOBE, Pypl, IEEE Spectrum). Despite the fact that the trend shows a gradual decline in the popularity of this language, it is currently a convenient tool for solving a large number of business problems, some examples of which can be found, for example, in the proceedings of the EARL conference (Enterprise Applications of the R language) [3].

2 Problem statement

Taking into account the fact that each task has its own optimal tools, we will describe the main business case when the use of R is appropriated. This is the development of systems in which it is necessary to aggregate heterogeneous data stored in a structured and unstructured form in various sources, validate them, bring them into a form convenient for understanding, and on their basis obtain information about problem areas that require special, close attention of management, as well as the issuance of information sections, submissions and recommendations to make the best decision. Under such definition to some extent it is possible to refer to a large number of tasks including: audit systems, activity-based costing, log-based analytics, business-process monitoring, texts analytics, capacity management, salary analytics and other.

3 Resolution process

The majority of the listed above tasks comes down to data manipulation, which is the strength of the R language. At the same time, R uses mainly a paradigm meta - and functional programming that allows to develop quickly building block applications that is especially important in a fast changing business environment.

So, we will consider the main packages necessary for the development of a full-fledged BI system based on the ecosystem of the R language.

Considering the fact that the main objective of such system is data processing in business processes of the company, and the main users are managers of different levels, we will begin the review with tools which allow constructing the convenient web interface. First of all, this is the "Shiny" package: its use allows you to build a branched interface that can include almost all structural html elements that will be reactively updated when interacting with the user. Also, the use of the "shinyJS" and "shinyBS" packages for working with JavaScript and Bootstrap, respectively, can provide additional opportunities. To build interactive reports and graphs, you should pay attention to the packages ”flexdashboard”, ”plotly”, ”DT”. Despite the simplicity and convenience of the web-based interface, it is often necessary to solve the problem of saving the displayed results in the form convenient for presentation in presentations and reports. Packages such as “knitr” solve this problem, which provides a single interface for using Markdown markup and receiving .html, .doc, .docx, .pdf, .tex and other dynamically generated reports. If you need to receive formatted reports for MS Office, you should pay attention to such packages as “openxlsx” and “officer”. To obtain data from external sources, there is a large number of specialized packages (”googleshhets”, ”RSQLite”, ”RFacebook” and others), as well as more general ones (”RODBC”, ”nodbi”, ”httr” and others). Thus, easy integration is achieved, both with relational and non-relational databases, as well as with cloud storage and Internet services. Passing to processing and manipulation with data, irreplaceable packages will be:

”ggplot2” – provides extensive visualization capabilities;

”dplyr” – a package that solves most of the tasks of data manipulation;

“tidyr” – a package for “cleaning” data and bringing it into a form convenient for processing;

and also a large number of the packages which are also widely used in the academic environment for working with text data, dates and times, categorical variables and more.

Using the above tools allows you to quickly build prototypes of enterprise solutions aimed at optimizing business processes, which should give a concrete result in the absence of a clear technical task for the project. Using the developed logging system (“futile.logger”), working with data routing (“magnittr”), evaluating system performance (“benchmarkme”), and the possibility of data validation (“checkmate”, “validate”, “ruler”) allows you to write complex highly loaded systems working 24/7 and comprehensively solving complex analytical problems.

To start and maintain such a system, the necessary components are Linux Server, Shiny Server, Rstudio Server, cron-manager. A useful addition will be working with monitoring systems (Zabbix) and containerization (docker).

It is important to note that all of the solutions listed are open-source projects and are distributed with an open license and similar code, which allows them to be used in both academic and business tasks. It is worth noting that existing solutions also have proprietary versions that allow you to get round-the-clock support and with improved performance and functionality.

4 Conclusion

This paper presents the basic R packages needed to develop full-fledged web-oriented enterprise applications for analytics of various business processes.

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Petri nets approach to solve logical problems

Inga Titchiev

Abstract

The aim of this paper is to perform theoretical research with practical applicability that will contribute to solving some logical problems by applying Petri nets formalism. Their facilities offer a new approach that allows the solution to be determined by a method different from the existing ones, but which is quite suggestive and easy to understand.

Keywords: Petri nets, logical problem, modeling, reachability graph

1 Introduction

Development of the information technologies, of the formal methods and of the means of communication have contributed to the development in a very fast time of the tools that can be applied to complex processing and which lead to appearance of new approaches in solving problems.

In this research, the formalism of Petri nets is proposed for solving some logical problems. Their solution is possible by applying the Petri nets verification methods [3] like reachability graph [2]. For example the solution of well-known logical problem with wolf, goat and cabbage (Figure 1) was done.

2 Petri nets

Petri nets (PNs) [1, 2] provide a good framework for the design, specification, validation, and verification of the modeling systems. Modeling concerns the abstracting and representing the systems under consideration using PNs, and analysis deals with effective ways to study the behaviors and properties of the resulting PNs models. Simulation by the tools [1] is effective and efficient in property verification.

Petri Nets is a structure $PN = (P; T; F; W; M_0)$:

- P is a finite set of elements called Places, T is a finite set of elements called Transitions $P(P \bigcap T = \emptyset;)$.
- $F = (\mathrm{Px}T) \bigcup (T\mathrm{x}P)$ is the flow relation (arcs).
- $W : F \rightarrow N \backslash 0$ is the arc weight.
- M_0 is the initial marking.

A transition of a net is enabled at a marking if all its input places (the places from which some edge leads to it) contain at least as many tokens as is the weight of arcs connecting them. An enabled transition can fire: it removes tokens from each of the input places, and adds tokens to each of its output places and exactly as much tokens as the weight of arcs connecting them. This is called the *firing rule*.

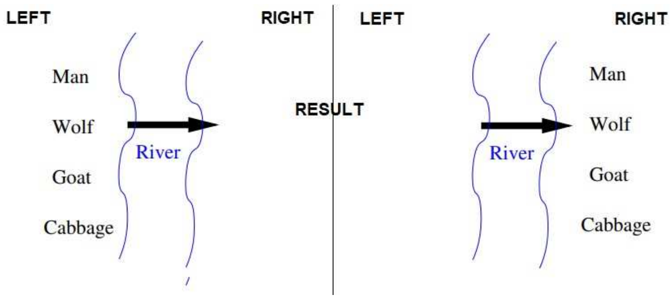

Figure 1. Logical problem

3 Modeling of the logical problem

In order to solve the logical problem, four steps will be done in modeling it through Petri nets.

- *First step.* There will be modeled the states when all the involved actors are in the left or right side of the river.
- *Step two.* There will be modeled the actions when all involved actors crossing the river from left to right and vice-versa.
- *Step three.* There will be modeled the critical actions: wolf eats goat, goat eats cabbage, when men is on other side of river.
- *Step four.* The previous steps will be integrated in order to obtain the final Petri net (Figure 2).

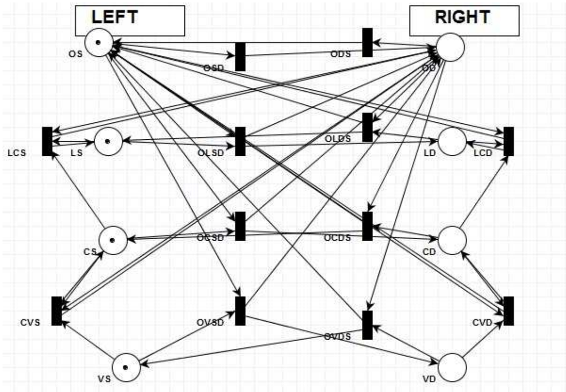

Figure 2. Logical problem modeled by Petri net

3.0.1 Reachability graph

The verification method of Petri nets like reachability graph offers facilities in determining the solution of the problem.

The *reachability graph* is a rooted, directed graph $G = (V, E, v_0)$, where

- $V = reach(N)$ is the set of vertices, i.e. each reachable marking is a vertex;.
- $F = (P\text{x}T)\bigcup(T\text{x}P)$ is the flow relation (arcs).
- $v_0 = M_0$, i.e. the initial marking is the root node.
- $E = \{(M, t, M)M \in V$ and $Mt \to M\}$ is the set of edges, i.e. there is an edge from each marking (resp. vertex) M to each of its successor markings, and the edge is labelled with the firing transition.

For problem with wolf, goat and cabbage, after simulation the reachability graph from Figure 3 was obtained. From this one we can extract the information about the existence of a way from marking S_0 to marking S_{29}, which contains the actions that must be executed to solve the problem successfully.

4 Conclusion

In this paper the new aproach for solving some logical problems by applying the Petri nets was done. This one is easy to understand and apply.

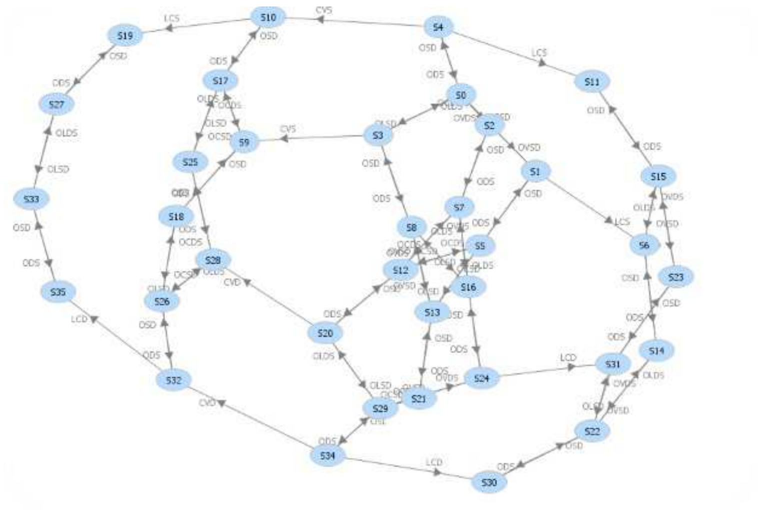

Figure 3. Reachability graph

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Steepest Descent Method in the Wolfram Language and Mathematica System

Valeriu Ungureanu

Abstract

The gradient method, known also as the steepest descent method, includes related algorithms with the same computing scheme based on a gradient concept. The illustrious French mathematician Augustin-Louis Cauchy proposed the method in 1847, much more before the appearing of modern computers. The method is very simple and may be successfully used for computing local minima, generally in combination with other methods as Newton's method. The Wolfram Language and Mathematica system are ideal tools for expositions and experiments.

Keywords: optimization methods, method of steepest descent, the Wolfram language, Mathematica system.

1 Introduction

We consider the problem:

$$f(x) \to \min, x \in R^n,$$

that requires to find an approximation x^k to some local minimum x^* with an accuracy $\varepsilon > 0$. The accuracy can refer to:

- argument: $||x^k - x^*|| < \varepsilon_1$,
- function value: $|f(x^k) - f(x^*)| < \varepsilon_2$,

- argument and function value: $max\{||x^k - x^*||, |f(x^k) - f(x^*)|\} < \varepsilon_3$,
- gradient: $||f'(x^k)|| < \varepsilon_4$,
- etc.

Computations are performed according to a very known iterative formula:

$$x^{k+1} = x^k - \alpha_k f'(x^k), k = 0, 1, 2, ...,$$

where

- x^0 is the starting point,
- $f'(x^k)$ is the objective function gradient,
- α_k is the step length on the anti-gradient direction.

First, we expose briefly advantages of the Wolfram language, in general.

Second, we present application of the Wolfram Language and Mathematica System in applying and experimenting steepest descent method, in particular.

Finally, we present investigations of Pareto-Nash-Stackelberg Games and Control Processes [1] in the Wolfram Language and Mathematica System.

2 Applications of built-in functions

We have to observe that α_k may be computed differently in the above formula. For example, α_k may be determined as:

- $\alpha_k = \arg\min f(x^k - \alpha_k f'(x^k)), \alpha_k > 0$ – method of optimal step,
- $\alpha_k = \alpha\lambda^s$, where: $\alpha, \lambda > 0$ are fixed numbers, $s \geq 0$ is the least power which ensures $f(x^{k+1}) < f(x^k)$ – method of a step fragmentation,

- $\alpha_k = \alpha > 0$ – method of constant steps.

Now, we can construct the Wolfram code of the steepest descent method. Surely, we need to use ending conditions, too.

$\varepsilon = 10^{-10}; X = \{x, y\};$
$grad[\{x\_, y\_\}] = D[f_2[X], X]; X = \left\{-\frac{12}{10.}, 1\right\};$
$While[||grad[X]|| \geq \varepsilon,$
$X = X - \alpha grad[X]/.Minimize[\{f_2[X - \alpha grad[X]], \alpha \leq 1\}, \alpha][[2]]];$

We present a list of benchmark examples for finding local and global minima. We provide an express analysis of the results.

3 Monitoring the Computations

To understand better the algorithm and code, we monitor the computations. There are some known test functions used to verify optimization programs. One of them is, e.g., the Rosenbrock function:

$$f_3[\{x\_, y\_\}] := 100(-y - x^2)^2 + (1 - x)^2 + 1.$$

For the Rosenbrock function we construct a graphical illustration of the computation process. Analogical graphical monitoring we provide for other benchmark functions.

In the process of monitoring we need to observe application of the standard Wolfram Mathematica function FindMinimum[], that finds local minimum of a function.

We apply the above code, too. Unfortunately, it needs much-much more time.

So, we identify difficulties arising in the application of gradient descent method both with built-in functions and our code.

First, we need to know properties of the objective function that guarantee the convergence of computing process. Second, we need to know requirements for the Wolfram code which will guarantee a high convergence speed.

4 Test Problems and Analysis

Wolfram Mathematica has a special package for unconstrained optimization problems.

The package is loaded simply:

$$<< Optimization`UnconstrainedProblems`$$

The package includes 35 test functions. More the more, it includes some useful plotting functions that may be used directly.

We use some special Pareto-Nash-Stackelberg game and control processes as the test problems.

We prove the convergence theorem of the steepest descent method by applying Wolfram Mathematica as a special tool.

5 Conclusion

The method of steepest descent is very simple and convenient to use especially in the Wolfram Language and Mathematica System. Unfortunately, the convergence theorem guarantees the convergence to stationary points that are not obligatory local minima. Nevertheless, the method is successfully used in practice, especially in combination with Newton's method, for different classes of functions when there are reasons to argue optimality of the found approximate solution.

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