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On A-center of a Quasigroup

V.A. Shcherbacov

Abstract

We introduce concept of A-center of a quasigroup and give some its properties.

Keywords: quasigroup, loop, center, abelian group, isotopy.

1 Classical definitions of center

We recall standard Garrison's [6] definition of quasigroup nuclei. Let $(Q, \circ)$ be a quasigroup. Then $N_l = \{a \in Q \mid (a \circ x) \circ y = a \circ (x \circ y)\}$, $N_r = \{a \in Q \mid (x \circ y) \circ a = x \circ (y \circ a)\}$ and $N_m = \{a \in Q \mid (x \circ a) \circ y = x \circ (a \circ y)\}$ are respectively its left, right and middle nuclei [6, 2, 8].

Let $(Q, \cdot)$ be a quasigroup. Nucleus is given by $N = N_l \cap N_r \cap N_m$ [8]. A center of a loop $(Q, \cdot)$ is defined in the following way [1, 4, 8].

Definition 1 *Let $(Q, \cdot)$ be a loop. Then center Z of loop $(Q, \cdot)$ is the following set $Z(Q, \cdot) = N \cap C$, where $C = \{a \in Q \mid a \cdot x = x \cdot a \ \forall x \in Q\}$.*

It is well known that in loop case $Z(Q)$ is normal abelian subgroup of the loop Q [4].

J.D.H. Smith [5, 10, 11, 12] has given definition of center of a quasigroup in the language of universal algebra (Congruence Theory). J.D.H. Smith defined central congruence in a quasigroup. Center of a quasigroup is a coset class of central congruence. Notice that any congruence of quasigroup Q defines a subquasigroup of Q^2 [10]. G.B. Belyavskaya and J.D.H. Smith definitions are close.

Quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms φ and ψ and a fixed element $a \in Q$ such that $x \cdot y = \varphi x + \psi y + a$ for all $x, y \in Q$ [7].

Notice, a quasigroup is central, if it coincides with its center.

Theorem 1 *(Belyavskaya Theorem). A quasigroup is central (in Belyavskaya and Smith sense) if and only if it is a T-quasigroup [3].*

An overview of various definitions of quasigroup center is in [9].

2 A-centers of a loop

Definition 2 *Let Q be a loop. Autotopy of the form $\{(L_a, \varepsilon, L_a) \mid a \in Z(Q)\}$ we shall name the left central autotopy. Group of all left central autotopies we shall denote by Z_l^A.*

Autotopy of the form $\{(\varepsilon, L_a, L_a) \mid a \in Z(Q)\}$ we shall name the right central autotopy. Group of all right central autotopies we shall denote by Z_r^A.

Autotopy of the form $\{(L_a, L_a^{-1}, \varepsilon) \mid a \in Z(Q)\}$ we shall name the middle central autotopy. Group of all middle central autotopies we shall denote by Z_m^A.

Definition 3 *The set of all autotopisms of the form $(\alpha, \varepsilon, \gamma)$ of a quasigroup $(Q, \circ)$, where ε is the identity mapping, is called the left autotopy nucleus (left A-nucleus) of quasigroup $(Q, \circ)$.*

Similarly, the sets of autotopisms of the forms $(\alpha, \beta, \varepsilon)$ and $(\varepsilon, \beta, \gamma)$ form the middle and right A-nuclei of $(Q, \circ)$. We shall denote these three sets of mappings by N_l^A, N_m^A and N_r^A respectively.

Theorem 2 *Let Q be a loop, $K = N_l^A \cdot N_r^A \cdot N_m^A$. Then*

(i) $K/Z_l^A \cong (N_r^A \times N_m^A)/Z_l^A \times N_l^A/Z_l^A$;

(ii) $K/Z_r^A \cong (N_l^A \times N_m^A)/Z_r^A \times N_r^A/Z_r^A$;

(iii) $K/Z_m^A \cong (N_l^A \times N_r^A)/Z_m^A \times N_m^A/Z_m^A$.

3 A-centers of a quasigroup

Definition 4 *Let Q be a quasigroup. We name*

(i) *the group* $(N_r^A \times N_m^A) \cap N_l^A$ *as left A-center of* Q *and denote by* Z_l^A*;*

(ii) *the group* $(N_l^A \times N_m^A) \cap N_r^A$ *as right A-center of* Q *and denote by* Z_r^A*;*

(iii) *the group* $(N_l^A \times N_r^A) \cap N_m^A$ *as middle A-center of* Q *and denote by* Z_m^A*.*

The set of all autotopisms of a quasigroup $(Q, \circ)$ is denoted by $Avt(Q, \circ)$.

Theorem 3 *In any quasigroup* Q *we have* $Z_l^A \trianglelefteq Avt(Q), Z_r^A \trianglelefteq Avt(Q), Z_m^A \trianglelefteq Avt(Q)$*.*

Theorem 4 *In any quasigroup* $(Q, \cdot)$ *the groups* Z_l^A*,* Z_r^A*,* Z_m^A*,* ${}_1Z_l^A$*,* ${}_3Z_l^A$*,* ${}_2Z_r^A$*,* ${}_3Z_r^A$*,* ${}_1Z_m^A$*,* ${}_2Z_m^A$ *are isomorphic to an abelian group.*

Theorem 5 *The group* ${}_1Z_l^A$ *(*${}_3Z_l^A$*,* ${}_2Z_r^A$*,* ${}_3Z_r^A$*,* ${}_1Z_m^A$*,* ${}_2Z_m^A$*) of a quasigroup* $(Q, \cdot)$ *acts on the set* Q *simply transitively if and only if quasigroup* $(Q, \cdot)$ *is abelian group isotope.*

Definition 5 *A-central quasigroup is a quasigroup* Q *with transitive action of at least one from its components of A-centers on* Q*.*

Using this definition we formulate main theorem of the paper.

Theorem 6 *A quasigroup is A-central if and only if it is an isotope of abelian group.*

Corollary 1 *Any central quasigroup in G.B. Belyavskaya and J.D.H. Smith sense is an A-central quasigroup.*

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References

[1] A.A. Albert, *Quasigroups. I*, Trans. Amer. Math. Soc. **54** (1943), 507–519.

[2] V.D. Belousov, *Foundations of the theory of quasigroups and loops*, Nauka, Moscow, 1967, (in Russian).

[3] G.B. Belyavskaya, *Abelian quasigroups are T-quasigroups*, Quasigroups Relat. Syst. **1** (1994), 1–7.

[4] R.H. Bruck, *A survey of binary systems*, third printing, corrected ed., Springer Verlag, New York, 1971.

[5] O. Chein, H.O. Pflugfelder, and J.D.H. Smith, *Quasigroups and loops: Theory and applications*, Heldermann Verlag, 1990.

[6] G. N. Garrison, *Quasi-groups*, Ann. of Math. **41** (1940), no. 2, 474–487.

[7] P. Němec and T. Kepka, *T-quasigroups, I*, Acta Univ. Carolin. Math. Phys. **12** (1971), no. 1, 39–49.

[8] H.O. Pflugfelder, *Quasigroups and loops: Introduction*, Heldermann Verlag, Berlin, 1990.

[9] V.A. Shcherbacov, *Elements of quasigroup theory and some its applications in code theory*, 2003, www.karlin.mff.cuni.cz/ drapal/speccurs.pdf.

[10] J.D.H. Smith, *Mal'cev varieties*, Lecture Notes in Math., vol. 554, Springer Verlag, New York, 1976.

[11] J.D.H. Smith, *Representation theory of infinite groups and finite quasigroups*, Lecture Notes in Mathematics, Universite de Montreal, Montreal, 1986.

[12] J.D.H. Smith, *An introduction to quasigroups and their representation*, Studies in Advanced Mathematics, Chapman and Hall/CRC, London, 2007.

V.A. Shcherbacov

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD−2028, Chişinău
Moldova

Email: scerb@math.md