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Conjugate-orthogonal Quasigroups and Graphs

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Abstract

It is known that the set of conjugates (the conjugate set) of a binary quasigroup can contain 1,2,3 or 6 elements. We describe all six possible conjugate sets with regard to equality ("assembling") of conjugates, study conjugate-orthogonality of quasigroups and the corresponding graphs.

Keywords: quasigroup, T-quasigroup, conjugate, conjugate-orthogonal graph

1 Introduction

A quasigroup is an ordered pair (Q, A) where Q is a set and A is a binary operation defined on Q such that each of the equations $A(a, y) = b$ and $A(x, a) = b$ is uniquely solvable for any pair of elements a, b in Q. It is known that the multiplication table of a finite quasigroup defines a Latin square and six (not necessarily distinct) conjugates (or parastrophes) are associated with each quasigroup (Latin square) [1].

Two quasigroups (Q, A) and (Q, B) are orthogonal if the system of equations $\{A(x, y) = a,\ B(x, y) = b\}$ is uniquely solvable for all $a, b \in Q$. A set $\Sigma = \{A_1, A_2, ..., A_n\}$ of quasigroups, defined on the same set, is orthogonal if any two quasigroups of it are orthogonal.

The notion of orthogonality plays the important role in the theory of Latin squares, also in the quasigroup theory and in distinct applications, in particular in the coding theory and cryptography. Quasigroups which are orthogonal to some of their conjugates or two conjugates of which are orthogonal (known as conjugate-orthogonal or parastrophic-orthogonal quasigroups) have the particular interest.

It is known that the set of conjugates (the conjugate set) of a binary quasigroup can contain 1,2,3 or 6 elements [2]. We describe all six possible conjugate sets with regard to equality (”assembling”) of conjugates. The spectrum of quasigroups six (five) conjugates of which are distinct and pairwise orthogonal is studied. Such quasigroups are called totally conjugate orthogonal quasigroups, shortly, $totCO$-quasigroups (near totally conjugate orthogonal quasigroups). Necessary and sufficient conditions are established under which a T-quasigroup is a $totCO$-quasigroup (a near $totCO$-quasigroup). As a consequence, we obtain some information about the spectrum of Latin squares realizing the complete conjugate-orthogonal Latin square graphs and near complete conjugate-orthogonal Latin square graphs.

2 Conjugate sets of quasigroups and graphs

With any quasigroup (Q, A) the system $\Sigma(A)$ of six (not necessarily distinct) *conjugates (parastrophes)* is connected: $\Sigma(A) = (A,\ {}^{r}\!A,\ {}^{l}\!A,\ {}^{lr}\!A,\ {}^{rl}\!A,\ {}^{s}\!A)$, where ${}^{l}\!A(z, y) = x$, ${}^{r}\!A(x, z) = y$, ${}^{lr}\!A(y, z) = x$, ${}^{rl}\!A(z, x) = y$, ${}^{s}\!A(y, x) = z$.

If $|\Sigma(A)| = 6$, then a quasigroup (Q, A) has six distinct conjugates.

Theorem 1 . *The following conjugate sets of a quasigroup* (Q, A) *are only possible:*

$\overline{\Sigma}_1(A) = \{A\}$;

$\overline{\Sigma}_2(A) = \{A, {}^{s}\!A\} = \{A = {}^{lr}\!A = {}^{rl}\!A,\ {}^{l}\!A = {}^{r}\!A = {}^{s}\!A\}$;

$\overline{\Sigma}_6(A) = \{A,\ {}^{r}\!A,\ {}^{l}\!A,\ {}^{lr}\!A,\ {}^{rl}\!A,\ {}^{s}\!A\}$;

$\overline{\Sigma}_3(A) = \{A, {}^{lr}\!A, {}^{rl}\!A\}$ *and three cases are only possible:*

$\overline{\Sigma}_3^1(A) = \{A = {}^{r}\!A,\ {}^{l}\!A = {}^{lr}\!A,\ {}^{rl}\!A = {}^{s}\!A\}$;

$\overline{\Sigma}_3^2(A) = \{A = {}^{l}\!A,\ {}^{r}\!A = {}^{rl}\!A,\ {}^{lr}\!A = {}^{s}\!A\}$;

$\overline{\Sigma}_3^3(A) = \{A = {}^{s}\!A,\ {}^{r}\!A = {}^{lr}\!A,\ {}^{l}\!A = {}^{rl}\!A\}$.

The described conjugate sets of quasigroups give only six types of the conjugate equality Latin square graphs corresponding to them.

A quasigroup (Q, A) is *a T-quasigroup* if there exist an abelian group $(Q, +)$, its automorphisms φ, ψ and an element $c \in Q$ such that

$A(x, y) = \varphi x + \psi y + c$ for any $x, y \in Q$ [3]. The conjugates of a T-quasigroup are also T-quasigroups.

Theorem 2 [4]. *A T-quasigroup (Q, A): $A(x, y) = \varphi x + \psi y + c$ is a totCO-quasigroup if and only if all maps $\varphi + \varepsilon$, $\varphi - \varepsilon$, $\psi + \varepsilon$, $\psi - \varepsilon$, $\varphi^2 + \psi$, $\psi^2 + \varphi$, $\varphi - \psi$, $\varphi + \psi$, $\psi\varphi - \varepsilon$ are permutations.*

Using this theorem we obtain the following information with respect to the spectrum of *totCO*-quasigroups.

Theorem 3 [4]. *For any number $n \geq 11$ which is relatively prime to 2,3,5 and 7 there exists a totCO-quasigroup of order n.*

In [5] a situation is considered when the orthogonal relationships of the conjugates of a Latin square L are represented by a graph where the conjugates are the vertices and two vertices are joined if and only if the corresponding conjugates are orthogonal. Such a graph is called a conjugate-orthogonal Latin square graph of L. It is still open the problem to determine completely the spectrum of Latin squares realizing the complete conjugate-orthogonal Latin square graph K_6.

It is evident that any finite *totCO*-quasigroup corresponds to the complete conjugate-orthogonal Latin square graph K_6, so from Theorem 3 it follows

Proposition . *For every $n = p_1^{k_1} p_2^{k_2} ... p_s^{k_s}$ where p_i is a prime number, $p_i \neq 2, 3, 5, 7$, $k_i \geq 1$, $i = 1, 2, ..., s$, $s \geq 1$, there exists a Latin square of order n realizing the complete conjugate-orthogonal Latin square graph K_6.*

Denote a quasigroup A by 1 and its conjugate ${}^{\sigma}\!A$ by σ.

Theorem 4 . *If a quasigroup is not a totCO-quasigroup, then the sets $\Sigma_1 = \{l, r, rl, lr, s\}$, $\Sigma_s = \{1, r, l, rl, lr\}$ of its conjugates are orthogonal if and only if all pairs of conjugates, besides of the pair $(1, s)$ are orthogonal;*

the sets $\Sigma_l = \{1, r, rl, lr, s\}$, $\Sigma_{lr} = \{1, r, l, rl, s\}$ of its conjugates are orthogonal if and only if all pairs of conjugates, besides of the pair (l, lr) are orthogonal;

the sets $\Sigma_r = \{1, l, rl, lr, s\}$, $\Sigma_{rl} = \{1, r, l, lr, s\}$ of its conjugates are orthogonal if and only if all pairs of conjugates, besides of the pair (r, rl) are orthogonal.

A near complete graph (a complete graph without one edge) corresponds to any near *totCO*-quasigroup.

Theorem 5 . *A T-quasigroup (Q, A): $A(x, y) = \varphi x + \psi y + c$, which is not a totCO-quasigroup, is a near totCO-quasigroup if and only if from all mappings in Theorem 4 the unique mapping $\varphi - \varepsilon$ ($\psi - \varepsilon$ or $\varphi + \psi$) is not a permutation.*

The following statement gives some information about the spectrum of near *totCO*-quasigroups realizing near complete conjugate orthogonal Latin square graphs.

Theorem 6 . *For any number n which is relatively prime to 2,3 and 5 there exists a near totCO-quasigroup of order n.*

References

[1] J. Déneš, A. D. Keedwell.*Latin squares and their applications.* Acadèmiai Kiado, Budapest and Academic Press, New York, 1974.

[2] C. C. Lindner, D. Steedly. *On the number of conjugates of a quasigroup.* Algebra Univ. 5 (1975), pp. 191-196.

[3] T. Kepka , P. Nemec. *T-quasigroups.I* Acta Universitatis. Carolinae. Math. et Phys., 1971, vol. 12, no. 1, pp. 39-49.

[4] G.B. Belyavskaya, T.V. Popovich. *Totally conjugate orthogonal quasigroups and complete graphs.* (Russian) Fundamental and Applied Mathematics. Moscow, 2010, vol. 16.

[5] C.C. Lindner, E. Mendelsohn, N. S. Mendelsohn, B. Wolk. *Orthogonal Latin square graphs.* J. Graph Theory, 1973, No. 3, pp. 325-328.

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