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Automatons and Topological Algebras

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Abstract

Universal algebras and automatons represent an important field of research in modern mathematics and computer science. In the present article we study the automatons in the category of topological universal algebras.

Keywords: automaton, universal algebra, annular.

1 Introduction

Automatons in distinct categories were examined in [4, 2, 1]. Let $\mathbb{N} = \{0, 1, 2, ...\}$ and E be the discrete sum of topological spaces $\{E_n : n \in N\}$. We say that E_n is the space of symbols of n-ary operations on topological E-algebras. A topological universal algebra of signature E or a topological E-algebra is a non-empty topological space G on which are given the continuous mappings $\{e_{nG} : E_n \times G^n \to G : n \in G\}$. The mappings e_{nG} forms the algebraical structure on G.

Any space is considered to be T_{-1}-space. Let $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$. A class $\mathcal{K}$ of topological E-algebras is called a T_i-quasivariety if: any algebra $G \in \mathcal{K}$ is a T_i-space; if $G \in \mathcal{K}$ and B is a subalgebra of G, then $B \in \mathcal{K}$; the topological product of algebras from $\mathcal{K}$ is a topological algebra from $\mathcal{K}$.

If $\mathcal{K}$ is a T_i-quasivariety of topological E-algebras, then a space X is called $\mathcal{K}$-embeddable if X is homeomorphic to some subspace of any space from $\mathcal{K}$. Denote by If $S(\mathcal{K})$ the class of all non-empty $\mathcal{K}$-embeddable spaces. An E-bigroupoid or a bigroupoid with the operators E is a topological E-algebra G with two binary continuous $\{\circ, *\}$, for which there exists an element $e \in G$ such that $x \circ e = x$ for each $x \in G$. We put $E'_2 = E_2 \cup \{\circ, *\}$ and $E' = E \cup \{\circ, *\}$. Then every E'-algebra is an E-bigroupoid. An E-bigroupoid G is a division E-bigroupoid if for every two elements $a, b \in G$ there exist two elements $c, d \in G$ such that $c \circ a = a \circ d = b$. An E-bigroupoid G is an E-annular if $x * (y \circ z) = (x \circ y) * z$ for all $x, y, z \in G$.

Construction 1 (see [3]). Let $(G_1, +)$ be a topological E-groupoid with a unity e_1 and $(G_2, +)$ be a topological E-groupoid with a unity e_2. Consider that $G = G_1 \times G_2$, $|G_2| = |G_1|$, $e = (e_1, e_2)$ and $\pi(x, y) = x$ for each $(x, y) \in G$. The space G is a topological E-algebra as the topological product of E-algebras $(G_1, +)$ and $(G_2, +)$. Let $g : G_1 \to G$ be a continuous mapping. Now, consider the following two continuous binary operations $(x_1, x_2) \circ (y_1, y_2) = ((x_1 + y_1, x_2 + y_2),\ (x_1, x_2) * (y_1, y_2)) = g(x_1 + y_1)$. By construction:

P1. *$(G, \circ, *, e)$ is an E-bigroupoid.*

P2. *$(G, \circ, *, e)$ is a division bigroupoid if and only if $(G_1, +)$ and $(G_2, +)$ are division groupoids.*

P3. If *$(G_1, +)$ is a semigroup, then $(G, \circ, *, e)$ is an E-annular.*

P4. If *$(G_1, +)$ is a group, then $(G, \circ, *, e)$ is a division annular.*

This construction has a general character and the following fact completes it.

Theorem 2 (M. Choban and L. Chiriac [3]). *Let G be an annular, $x * e = y * e$ if and only if $x = y$ and $g(x) = x * e$ for each $x \in G$. Then:*

1. $x * y = g(x \circ y)$ for all $x, y \in G$.

2. $(G, \circ)$ is a semigroup.

3. If G is a division annular, then $(G, \circ)$ is a group

Corollary 3. *Let G be an annular and $(G, *)$ be a quasigroup. Then $(G, \circ)$ is a semigroup.*

2 Universal Algebras and Automatons

If $n \geq 1$, $1 \leq i \leq n$ and $g \in E_n$, then $g(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n) = a_i$ is an equation on E-bigrupoids. Let φ be a set of equations on E-bigroupoids. By $V(E, \varphi)$ we denote the class of all topological E-

bigroupoids G, on which the equations φ are solutions. By $V(E, u\varphi)$ we denote the class of all E-algebras $G \in V(E, \varphi)$, on which the equations φ are unique solutions.

Theorem 4. *For every Tychonov space X there exists a topological algebra $F(X, E, \varphi) \in V(E, u\varphi)$ with the properties: X is a subspace of $F(X, E, \varphi)$; if $X \subseteq G \subseteq F(X, E, \varphi)$, G is an E-subalgebra and $G \in V(E, u\varphi)$, then $G = F(X, E, \varphi)$; for every continuous mapping $f : X \to G \in V(E, u\varphi)$ there exists a unique continuous homomorphism $f_1 : F(X, E, \varphi) \to G$ such that $f = f_1|X$.*

Theorem 5. (M. Cioban) *Every discrete E-algebra G is a subalgebra of some discrete algebra $\varphi G \in V(E, \varphi)$.*

Let us consider $E = E_0 \cup E_2$, $E_0 = \{e\}$, $E_2 = \{\circ, *\}$ and $V(E)$ be the family of all E-algebras with the identity $x \circ e = x$. Then $V(E)$ is the class of all bigroupoids. Consider the equations $\psi = \{a \circ x = b$, $y \circ a = b\}$ and $\varphi = \{a \circ x = b$, $a * x = b$, $y \circ a = b$, $y * a = b\}$.

Then for every Tichonov space X we have $F(X, E, \varphi) \in V(E, u\varphi)$ and $F(X, E, \psi) \in V(E, u\psi)$. We put $A(E) = \{G \in V(E) : G$ is an annular$\}$, $A(E, \varphi) = A(E) \cap V(E, \varphi)$ and $A(E, \psi) = A(E) \cap V(E, \psi)$.

Corollary 6. *$A(E, \varphi)$ is a φ-quasivariety of E-algebras and $F(X, A(E, \varphi)) \notin V(E, u\varphi)$ for each space X.*

Corollary 7. *$A(E, \psi)$ is a ψ-quasivariety of E-algebras and $F(X, A(E, \psi)) \in V(E, u\varphi)$ for each space X.*

Corollary 8. *$F(X, A(E, \varphi))$ is a subalgebra of the E-algebra $F(X, A(E, \psi))$ for each discrete space X.*

Fix a signature $E = \oplus\{E_n : n \in N\}$ and an E-groupoid I with a unity e. We consider that I is a space of inputs.

An automaton consists of a space A, a space B and two continuous mappings $\circ : A \times I \to A$ and $* : A \times I \to B$, where $(x \circ \alpha) \circ \beta = x \circ (\alpha\beta)$ and $(x \circ \alpha) * \beta = x * (\alpha\beta)$ for all $x \in A$ and $\alpha, \beta \in I$.

An automaton $(A, I, B, \circ, *)$ is an automaton in the category of E-algebras or an E-automaton if A, B are E-algebras and the mapping $x \to x * \alpha$ is a homomorphism of A into B for each $\alpha \in I$.

Theorem 9. *Let φ be a set of equations and $\mathcal{K}$ be a φ-quasivariety. Assume that $A, B \in S(\mathcal{K})$. Then $(F(A, \mathcal{K}), I, F(B, \mathcal{K}), \circ, *)$ is an E-*

automaton, which contains $(A, I, B, \circ, *)$ *as a subautomaton.*

From the above result it follows that the automatons does not exist in some categories of algebras. For instance, an automaton from the category of quasigroups is from the category of groups.

3 Automatons and bigroupoids

Let $(A, I, B, \circ, *)$ be an automat, Q be an E-bigrupoid, I be a subgroupoid of the groupoid $(Q, \circ)$, A and B be subsets of Q. If $x \circ \alpha$ and $x * \alpha$ in automaton coincide with $x \circ \alpha$ and $x * \alpha$ in E'-algebra Q, then we say that $(A, I, B, \circ, *)$ is an automaton in the E-bigroupoid Q. If A and B are E-subalgebras of Q and $(A, I, B, \circ, *)$ is an E-automaton, then we say that $(A, I, B, \circ, *)$ is an E-automaton in the E-bigroupoid Q. Fix some equations φ on the class of E'-algebras. The next result due to (M. Choban and L. Chiriac [3]).

Theorem 10 . *For every automat* $(A, I, B, \circ, *)$ *there exists an* E*-bigroupoid* $Q \in V(E', \varphi)$ *such that* $(A, I, B, \circ, *)$ *is an automat in the* E*-bigroupoid* Q.

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