On entropicity in \(n\)-ary semigroups

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\textbf{Abstract.} We investigate entropicity and the generalized entropic property in \(n\)-ary semigroups derived from binary semigroups satisfying for some fixed \(k \geq 2\) the identity \(x^k = x\).

1. We say that an \(n\)-ary semigroup \((S, f)\), where \(n > 2\), has the entropic property or is entropic \textit{(medial} in other terminology), if it satisfies the identity

\[
\begin{align*}
&f\left(f(x_1, \ldots, x_n), f(x_{11}, \ldots, x_{2n}), \ldots, f(x_{n1}, \ldots, x_{nn})\right) = \\
&f\left(f(x_2, \ldots, x_n), f(x_{12}, \ldots, x_{2n}), \ldots, f(x_{n2}, \ldots, x_{nn})\right) = \\
&\ldots = \\
&f(f(x_1, \ldots, x_n), f(x_{21}, \ldots, x_{2n}), \ldots, f(x_{n1}, \ldots, x_{nn})).
\end{align*}
\]

If in \((S, f)\) there exist \(n\)-ary terms \(t_1, t_2, \ldots, t_n\) such that \((S, f)\) satisfies the identity

\[
\begin{align*}
&f\left(f(x_1, \ldots, x_n), f(x_{11}, \ldots, x_{2n}), \ldots, f(x_{n1}, \ldots, x_{nn})\right) = \\
&f\left(f(x_1, \ldots, x_n), f(x_{12}, \ldots, x_{2n}), \ldots, f(x_{n1}, \ldots, x_{nn})\right) = \\
&\ldots = \\
&f\left(f(x_1, \ldots, x_n), f(x_{21}, \ldots, x_{2n}), \ldots, f(x_{n1}, \ldots, x_{nn})\right),
\end{align*}
\]

then we say that \((S, f)\) has the \textit{generalized entropic property}. These two properties, studied by many authors with various names, are not equivalent in general. The entropicity of \(n\)-ary semigroups is a generalization of \textit{mediality}:

\[
xy \cdot zu = xz \cdot yu,
\]

and \textit{semimediality}:

\[
xy \cdot zx = xz \cdot yx
\]

of binary algebras (cf. for example [17] or [22]).

The entropicity and the generalized entropicity in idempotent \(n\)-ary semigroups were studied in [15]. Below we give very simple (almost trivial) proofs of results given in this paper. We also present some generalizations of these results.

2. We start with some comments on entropic \(n\)-ary groups \((n > 2)\).

In [11] it is proved that an \(n\)-ary group \((S, f)\) is entropic if and only if it is \textit{semiabelian}, i.e., if

\[
f(x_1, x_2, \ldots, x_{n-1}, x_n) = f(x_n, x_2, \ldots, x_{n-1}, x_1)
\]

for all \(x_1, x_2, \ldots, x_n \in S\).

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\end{itemize}
From [5] (Corollary 15) it follows that an n-ary group \((S, f)\) is entropic if and only if for some \(a \in S\) and all \(x, y \in S\) we have

\[
f(x, a, \ldots, a, y) = f(y, a, \ldots, a, x).
\]

Thus, by Gluskin-Hosszú theorem (cf. [12, 13]) any entropic n-ary group \((S, f)\) can be presented in the form

\[
f(x_1, x_2, \ldots, x_n) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \cdots \circ \varphi^{n-2}(x_{n-1}) \circ x_n \circ b,
\]

where \((S, \circ)\) is an abelian group, \(\varphi\) its automorphism such that \(\varphi^{n-1} = \text{id}\) and \(\varphi(b) = b\) for some fixed element \(b \in S\) (cf. [7]). Moreover, as it is proved in [24] (see also [10]), \((A, \circ), \varphi\) and \(b\) are uniquely determined.

3. Mal’cev n-semigroups, i.e., n-ary semigroups \((S, f)\) satisfying the identities

\[
f(x, y, \ldots, y) = x \quad \text{and} \quad f(y, \ldots, y, x) = x, \tag{1}
\]

studied in [15], are in fact n-ary groups. This follows from Proposition 3.1 in [8]. It also can be deduced from results proved in [25]. Hence, Mal’cev n-semigroups (as n-ary groups) are cancellative, i.e.,

\[
f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) \Rightarrow a = b \tag{2}
\]

for all \(i = 1, \ldots, n\) and \(a, b, x_1, \ldots, x_n \in S\).

On the other hand, an n-ary semigroup is cancellative if and only if it satisfies (2) for some \(i = 2, 3, \ldots, n-1\) or, equivalently, for \(i = 1\) and \(i = n\) (Lemma 2 in [6]). Hence, in an idempotent \(i\)-cancellative n-ary semigroup \((S, f)\) we have

\[
f(x, y, \ldots, y) = f(x, y, \ldots, y, f(y, \ldots, y)) = f(f(x, y, \ldots, y), y, \ldots, y),
\]

which implies the first identity of (1). Analogously we obtain the second identity. (It is Lemma 3.2 in [15]). Thus an idempotent n-ary semigroup \(i\)-cancellative for some \(i = 2, \ldots, n-1\) or for \(i = 1\) and \(i = n\) is an n-ary group satisfying (1). Hence, Proposition 3.3 in [15] is trivial.

As a simple consequence, we obtain Theorem 3.5 from [15]: a Mal’cev n-semigroup is entropic if and only if it is semiabelian.

4. By Gluskin-Hosszú theorem, for any ternary Mal’cev semigroup \((S, f)\), as for a ternary group, there exists a group \((S, \cdot)\), its automorphism \(\varphi\) and an element \(b \in S\) such that \(\varphi(b) = b\) and

\[
f(x, y, z) = x \cdot \varphi(y) \cdot \varphi^2(z) \cdot b.
\]

Since a ternary Mal’cev semigroup is idempotent, \(\varphi(x) \cdot \varphi^2(x) \cdot b = e\). Hence, \(\varphi(x) = x^{-1} \cdot b^{-1}\) and \(b^{-1} = \varphi(b^{-1}) = e\). Therefore, \(b = e\) and \(\varphi(x) = x^{-1}\). Thus,
for any ternary Mal’cev semigroup \((S, f)\) there is an abelian group \((A, +)\) such that
\[
f(x, y, z) = x - y + z.
\]

From this, as a simple consequence, we obtain all results proved in Section 4 in [15]. Theorem 4.5 in [15] is a special case of Artamonov’s Proposition 6 (cf. [2]). It also can be deduced from the description of free \(n\)-ary groups presented by Shchuchkin [14, 18, 19] and Sioson [23].

5. We say that an \(n\)-ary semigroup \((S, f)\) is derived from a semigroup \((S, \cdot)\) if
\[
f(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n \quad \text{for all } x_1, \ldots, x_n \in S.
\]
Obviously, an \(n\)-ary semigroup derived from an entropic semigroup is entropic, too. For a surjective semigroup, i.e., a semigroup \((S, \cdot)\) with the property \(S^2 = S\), we have a stronger result.

**Proposition 1.** An \(n\)-ary semigroup \((S, f)\) derived from a surjective semigroup \((S, \cdot)\) is entropic if and only if \((S, \cdot)\) is entropic.

**Proof.** An entropic \(n\)-ary semigroup \((S, f)\) derived from a surjective semigroup \((S, \cdot)\) is semiabelian and each its element can be presented as a multiplication of \(n - 1\) elements of \(S\). Thus, for any \(x, y, a, b \in S\), we have
\[
xa \cdot by = x(a_2a_3\cdots a_n) \cdot (b_2b_3\cdots b_n)y = x(a_2a_3\cdots a_nb_2)b_3\cdots b_ny
\]
\[
= x(b_2a_3\cdots a_2a_3)b_3\cdots b_ny = xb_2(a_3a_4\cdots a_na_2b_3)b_4\cdots b_ny
\]
\[
= xb_2(b_3a_4\cdots a_n a_2a_3)b_4\cdots b_ny = xb_2b_3(a_4\cdots a_n a_2a_3b_4)b_5\cdots b_ny
\]
\[
= \ldots = x(b_2b_3\cdots b_n) \cdot (a_2a_3\cdots a_n)y = xb \cdot ay,
\]
which completes the proof. \(\square\)

We say that a semigroup \((S, \cdot)\) is \(k\)-idempotent \((k > 1)\), if for all \(x \in S\) we have \(x^k = x\). An \(n\)-ary semigroup derived from an \(n\)-idempotent semigroup is obviously idempotent, but a \((k+1)\)-ary semigroup derived from a \(k\)-idempotent semigroup is not idempotent. So, results proved for \(n\)-ary semigroups derived from \(k\)-idempotent semigroups are a significant generalization of results proved in [15].

**Lemma 2.** Any \(k\)-idempotent semigroup has at least one idempotent.

**Proof.** In a \(2\)-idempotent semigroup each element is idempotent. It is clear. For \(k > 2\), we have \(a^{k-1} = a^ka^{k-2} = a^{k-1}a^{k-1}\), which means that in a \(k\)-idempotent semigroup each element \(a^{k-1}\) is idempotent. \(\square\)

**Proposition 3.** A \(k\)-idempotent semigroup is entropic if and only if it is semimedial.

**Proof.** An entropic semigroup is obviously semimedial. To prove the converse statement observe that in a semimedial semigroup
\[(xyzu)^2 = (xyzu)xzu = (xzu)(yzu) = x(yz(xu))(zu)u = (xzu)(yzu)\]
and
\[(xyzu)(xyzu) = xzy(yzu)xzu = xz((yu)zx(yu)) = (xzyu)(xyzu) = (xzyu)^2.

Thus, in a \(k\)-idempotent semigroup
\[xyzu = (xyzu)^k = (xyzu)^2(xyzu)^{k-2} = (xzyu)^2(xyzu)^{k-2} = \ldots = (xyzu)^k = xzyu, \quad \text{if } k \text{ is even.}

For \(k = 2t + 1\) we have
\[xyzu = (xyzu)^{2t+1} = (xyzu)^2(xyzu)^{2t-1} = \ldots = (xzyu)^{2t}(xyzu) = (xzyu)^{2t-1}(xzyu)^2 = (xzyu)^{2t+1} = xzyu,
which completes our proof. \(\square\)

As a consequence, we obtain

**Corollary 4.** [15, Lemma 6.3] An \(n\)-idempotent semigroup is entropic if and only if it is semimedial.

**Proposition 5.** An \(n\)-ary semigroup derived from a \(k\)-idempotent semigroup is semiabelian if and only if it is commutative.

**Proof.** Let \((S, f)\) be a semiabelian \(n\)-ary semigroup derived from a \(k\)-idempotent semigroup \((S, \cdot)\). Then for all \(x, y \in S\) we have
\[xy = xx^{k-1}y = xx^{k-1}x^{k-1}y = \ldots = xx^{k-1}x^{k-1}x^{k-1}y = f(x, x^{k-1}, \ldots, x^{k-1}y) = f(y, x^{k-1}, \ldots, x^{k-1}x) = yx^{k-1}x^{k-1}x = yx,
which means that \((S, \cdot)\) is a commutative semigroup. Consequently, \((S, f)\) is commutative, too.

The converse statement is obvious. \(\square\)

**Corollary 6.** [15, Corollary 6.6] For an \(n\)-ary semigroup \((S, f)\) derived from an \(n\)-idempotent semigroup \((S, \cdot)\) the following statements are equivalent:

(a) \((S, f)\) is semiabelian,
(b) \((S, \cdot)\) is commutative,
(c) \((S, f)\) is commutative.

6. Below we present simple proofs of some other results presented in [15]. For this we will use the concept of the covering semigroup.

As is well known (cf. for example [3] or [4]) each \(n\)-ary semigroup \((S, f)\) can be isomorphically embedded into some semigroup \((S^*, \cdot)\), called the covering or enveloping semigroup, in this way that \(f(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n\) for all
The construction of such semigroup is very similar to the construction of the covering group for an \(n\)-ary group (cf. \[16\]). Unfortunately, as it was observed in \([9]\), two non-isomorphic \(n\)-ary semigroups (groups) may have the same covering semigroup (group).

**Proposition 7.** \([15, \text{Proposition 5.1}]\) An associative and idempotent \(n\)-ary operation \(f\) satisfying the identities

\[
f(x_1, \ldots, x_n) = f(y, x_2, \ldots, x_n) = f(x_1, \ldots, x_{n-1}, y)
\]

or

\[
f(x_1, \ldots, x_n) = f(y, x_1, \ldots, x_{n-1}, y)
\]

is commutative.

**Proof.** Let \((S, f)\) be an idempotent \(n\)-ary semigroup satisfying the above identities. Then in its covering semigroup \((S^*, \cdot)\), for every \(x, y \in S\), we have \(x^n = x\) and \(x^{n-1}y = yx^{n-1} = xyy^{n-2}\). So,

\[
xy = x^n y = x \cdot x^{n-1}y = x \cdot yx^{n-1} = xyx^{n-2} \cdot x = yx^{n-1}, \quad x = yx^n = yx
\]

for every \(x, y \in S\). Hence \(f\) is a commutative operation.

An \(n\)-ary semigroup \((S, f)\) is called a left zero \(n\)-semigroup if it satisfies the identity \(f(x_1, \ldots, x_n) = x_1\). If it satisfies the identity \(f(x_1, \ldots, x_n) = x_n\), then it is called a right zero \(n\)-semigroup.

**Proposition 8.** \([15, \text{Proposition 5.3}]\) Let \((S, f)\) be an \(n\)-ary semigroup. If \((S, f)\) satisfies the identity \(f(x_1, \ldots, x_n) = x_1\), then it is a left zero semigroup. If \((S, f)\) satisfies the identity \(f(y, x_1, \ldots, x_{n-1}) = x_3\), then it is a right zero semigroup.

**Proof.** In the covering semigroup of \((S, f)\) for all \(x, y \in S\) we have \(x^{n-1}y = x\) and \(x^n = x\). Thus \(xy = x^n y = x \cdot x^{n-1}y = xx\), and consequently,

\[
f(x_1, x_2, \ldots, x_n) = (x_1x_2)x_3x_4 \cdots x_n = (x_1x_1)x_3x_4 \cdots x_n = x_1(x_1x_3)x_4 \cdots x_n
\]

\[
= x_1x_1x_1x_4 \cdots x_n = \ldots = x_1 \cdots x_1(x_1x_n) = x_1
\]

for all \(x_1, \ldots, x_n \in S\). Hence \((S, f)\) is a left zero semigroup.

The second sentence can be proved analogously.

7. In the case of an \(n\)-ary semigroup \((S, f)\) derived from a binary semigroup \((S, \cdot)\), the generalized entropic property has the form

\[
\begin{align*}
(x_{11} \cdots x_{1n}) \cdot (x_{21} \cdots x_{2n}) \cdots (x_{n1} \cdots x_{nn}) &= \quad (3) \\
t_1(x_{11}, \ldots, x_{n1}) \cdot t_2(x_{12}, \ldots, x_{n2}) \cdots t_n(x_{1n}, \ldots, x_{nn}),
\end{align*}
\]

where \(t_1, t_2, \ldots, t_n\) are some \(n\)-ary terms of \((S, f)\).
We start with the following two lemmas which are a generalization of results proved in [15] for idempotent $n$-ary semigroups derived from binary semigroup containing an idempotent element. We do not assume that considered semigroups are idempotent.

**Lemma 9.** If an $n$-ary semigroup $(S, f)$ derived from a binary semigroup $(S, \cdot)$ with an idempotent $e$ satisfies the generalized entropic property (3), then for every $a \in S$ we have
\[
t_1(e, a, e, \ldots, e) \cdot e = eae = e \cdot t_n(a, e, \cdot, e).
\]

*Proof.* The proof is the same as the proof of Lemma 6.4 in [15].

**Lemma 10.** If an $n$-ary semigroup $(S, f)$ derived from a binary semigroup $(S, \cdot)$ with an idempotent $e$ satisfies the generalized entropic property (3), then
\[
eabe = ebeae
\]
for all $a, b \in S$.

*Proof.* The proof is the same as the proof of Lemma 6.5 in [15].

**Theorem 11.** If an $n$-ary semigroup $(S, f)$ derived from a $k$-idempotent semigroup $(S, \cdot)$ has the generalized entropic property, then $(S, \cdot)$ is entropic.

*Proof.* For any $a \in S$ consider the set
\[
S_a = \{a^{k-1}sa^{k-1} : s \in S\}.
\]

It is not difficult to see that $(S_a, \cdot)$ is a semigroup and $e = a^{k-1}$ is its neutral element. So, $(S_a, f)$ is an $n$-ary subsemigroup of $(S, f)$. Moreover, from the generalized entropic property (3), for all $x_1, \ldots, x_n \in S_a$ we have
\[
x_1 \cdot x_2 \cdot \cdots \cdot x_n = (x_1 \cdot e \cdot \cdots \cdot e) \cdot (x_2 \cdot e \cdot \cdots \cdot e) \cdots (x_n \cdot e \cdot \cdots \cdot e)
\]
\[
= t_1(x_1, x_2, \ldots, x_n) \cdot t_2(e, e, \ldots, e) \cdots t_n(e, e, \ldots, e)
\]
\[
= t_1(x_1, x_2, \ldots, x_n).
\]

Thus $t_1(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_n$. Analogously we obtain $t_k(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_n$ for other terms $t_2, \ldots, t_n$.

So, in this case, (3) has the form
\[
(x_{11} \cdot \cdots \cdot x_{1n}) \cdot (x_{21} \cdot \cdots \cdot x_{2n}) \cdots (x_{n1} \cdot \cdots \cdot x_{nn}) =
(x_{11} \cdot \cdots \cdot x_{n1}) \cdot (x_{12} \cdot \cdots \cdot x_{n2}) \cdots (x_{1n} \cdot \cdots \cdot x_{nn}),
\]
which, for $x_{1n} \neq e$, $x_{n1} \neq e$ and $x_{ij} = e$ in other cases, gives the commutativity of $(H_a, \cdot)$. Therefore, for all $a, b, c \in S$ we have
\[
a^{k-1}b \cdot ca^{k-1} = a^{k-1}ba^{k-1} \cdot a^{k-1}ca^{k-1} = a^{k-1}ca^{k-1} \cdot a^{k-1}ba^{k-1} = a^{k-1}c \cdot ba^{k-1}.
\]

So, $a^{k-1}b \cdot ca^{k-1} = a^{k-1}c \cdot ba^{k-1}$. This implies $ab \cdot ca = ac \cdot ba$. Thus $(S, \cdot)$ is semimedial. Proposition 3 completes the proof. 

\[\square\]
Corollary 12. [15, Theorem 6.7] Let \((S, f)\) be an idempotent \(n\)-ary semigroup derived from an \(n\)-idempotent semigroup \((S, \cdot)\) and assume that \((S, f)\) has the generalized entropic property. Then \((S, \cdot)\) is entropic.

Theorem 13. A \(k\)-idempotent semigroup has the generalized entropic property if and only if it is entropic.

Proof. Let \((S, \cdot)\) be a \(k\)-idempotent semigroup satisfying the generalized entropic property. Then there are terms \(t_1\) and \(t_2\) such that
\[
xy \cdot zu = t_1(x, z)t_2(y, u)
\]
for all \(x, y, z, u \in S\). By Lemma 2, the set \(E_S\) of all idempotents of a semigroup \((S, \cdot)\) is non-empty. Since \(ef \cdot ef = t_1(e, e)t_2(f, f) = ef\) for all \(e, f \in E_S\), \((E_S, \cdot)\) is a subsemigroup of \((S, \cdot)\). By [1, Proposition 3.11], \((E_S, \cdot)\) is entropic. This completes the proof for \(k = 2\) because in this case \(S = E_S\).

For \(k > 2\), for every \(a \in S\) we have \(a = axa\) and \(ax = xa\), where \(x = a^{k-2}\). So, \((S, \cdot)\) is a completely regular semigroup whose idempotents forms a subsemigroup, i.e., a normal band. Hence, in this case, the proof is identical with the second part of the proof of Theorem 6.8 in [15]. Namely, by [20, Theorem 4.1 and Corollary 4.4], a semigroup \((S, \cdot)\) is a normal band of groups. Thus, by [21, Theorem 3.2], it is a subdirect product of a band \(B\) and a semilattice \(L\) of groups. Since, by the definition of subdirect product, \(B\) and \(L\) are homomorphic images of a semigroup \((S, \cdot)\), they satisfy all identities satisfied by \((S, \cdot)\). Hence, they satisfy the generalized entropic property. This means that a band \(B\) is entropic (see the case \(k = 2\)). Every group with the generalized entropic property is commutative [1, Proposition 4.7] and a semilattice of commutative groups is commutative. Thus a semilattice \(L\) is commutative and hence entropic. Consequently, \((S, \cdot)\), as a subdirect product of entropic \(B\) and \(L\), is entropic. \(\square\)

Corollary 14. [15, Theorem 6.8] An \(n\)-idempotent semigroup \((S, \cdot)\) has the generalized entropic property if and only if it is entropic.

References


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