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m-polar fuzzy Lie ideals of Lie algebras

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Abstract. We introduce the notion of an m-polar fuzzy Lie ideal of a Lie algebra and investigate some properties of nilpotency of m-polar fuzzy Lie ideals. We introduce the concept of m-polar fuzzy adjoint representation of Lie algebras and discuss the relationship between this representation and nilpotent m-polar fuzzy Lie ideals. We also define Killing form in the m-polar fuzzy case and study some of its properties.

1. Introduction

The concept of Lie groups was first introduced by Sophus Lie in nineteenth century through his studies in geometry and integration methods for differential equations. The importance of Lie algebras in mathematics and physics have become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. It is noted that Lie theory has applications not only in mathematics and physics but also in diverse fields such as continuum mechanics, cosmology and life sciences. A Lie algebra has nowadays even been applied by electrical engineers in solving problems in mobile robot control [8].

In 1965, Zadeh [15] introduced the concept of fuzzy subset of a set. A fuzzy set on a given set X is a mapping $A: X \to [0,1]$. In 1994, Zhang [16] extended the idea of a fuzzy set and defined the notion of bipolar fuzzy set on a given set X as a mapping $A: X \to [-1,1]$, where the membership degree 0 of an element x means that the element x is irrelevant to the corresponding property, the membership degree in $(0,1]$ of an element x indicates that the element satisfies the property, and the membership degree in $[-1,0)$ of an element x indicates that the element somewhat satisfies the implicit counter-property. In 2014, Chen *et al.* [7] introduced the notion of m-polar fuzzy sets as a generalization of bipolar fuzzy set and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions and that we can obtain concisely one from the corresponding one in [7]. The idea behind this is that "multipolar information" (not just bipolar information which correspond to two-valued logic) exists because data for a real world problems are sometimes from n agents ($n \geqslant 2$). For example, the exact degree of telecommunication safety of mankind is a point in $[0,1]^n (n \approx 7 \times 10^9)$ because different person has been monitored different times. There are many

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examples such as truth degrees of a logic formula which are based on n logic implication operators ($n \geqslant 2$), similarity degrees of two logic formula which are based on n logic implication operators ($n \geqslant 2$), ordering results of a magazine, ordering results of a university and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.

The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were considered first in [13] by Yehia. Since then, the concepts and results of Lie algebras have been broadened to the fuzzy setting frames [1-6, 10, 13]. In this paper, we introduce the notion of an m-polar fuzzy Lie ideal of a Lie algebra and investigate some properties of nilpotency of m-polar fuzzy Lie ideals. We introduce the concept of m-polar fuzzy adjoint representation of Lie algebras and discuss the relationship between this representation and nilpotent m-polar fuzzy Lie ideals. We also define Killing form in the m-polar fuzzy case and study some of its properties. The definitions and terminologies that we used in this paper are standard. For other notations, terminologies and applications, the readers are refereed to [8-12, 17].

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper. A *Lie algebra* is a vector space $\mathscr{L}$ over a field $\mathbb{F}$ (equal to $\mathbb{R}$ or $\mathbb{C}$) on which $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in \mathscr{L}$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathscr{L}$ (Jacobi identity).

Throughout this paper, $\mathscr{L}$ is a Lie algebra and $\mathbb{F}$ is a field. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is *anticommutative*, i.e., $[x, y] = -[y, x]$. A subspace H of $\mathscr{L}$ closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra*.

A *fuzzy set* $\mu : \mathscr{L} \to [0, 1]$ is called a *fuzzy Lie ideal* [1] of $\mathscr{L}$ if

(a) $\mu(x + y) \geqslant \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geqslant \mu(x)$,

(c) $\mu([x, y]) \geqslant \mu(x)$

hold for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$. The addition and the commutator [,] of $\mathscr{L}$ are extended by Zadeh's extension principle [15], to two operations on I^L in the following way:

$$(\mu \oplus \lambda)(x) = \sup\{\min\{\mu(y), \lambda(z)\} \,|\, y, z \in \mathscr{L},\, y + z = x\},$$

$$\ll \mu, \lambda \gg (x) = \sup\{\min\{\mu(y), \lambda(z)\} \mid y, z \in \mathscr{L},\ [y, z] = x\},$$

where μ, λ are fuzzy sets on $I^{\mathscr{L}}$ and $x \in \mathscr{L}$. The scalar multiplication αx for $\alpha \in \mathbb{F}$ and $x \in \mathscr{L}$ is extended to an action of the field $\mathbb{F}$ on $I^{\mathscr{L}}$ denoted by $\odot$ as follows for all $\mu \in I^L$, $\alpha \in \mathbb{F}$ and $x \in \mathscr{L}$:

$$(\alpha \odot \mu)(x) = \begin{cases} \mu(\alpha^{-1}x) & \text{if } \alpha \neq 0, \\ 1 & \text{if } \alpha = 0,\ x = 0, \\ 0 & \text{if } \alpha = 0,\ x \neq 0. \end{cases}$$

The two operations of the field $\mathbb{F}$ can be extended to two operations on $I^{\mathbb{F}}$ in the same way. The operations are denoted by $\oplus$ and $\circ$ as well [15]. The zeros of $\mathscr{L}$ and $\mathbb{F}$ are denoted by the same symbol 0. Obviously $0 \odot \mu = 1_0$ for every $\mu \in I^{\mathscr{L}}$ and every $\mu \in I^{\mathbb{F}}$, where 1_x is the fuzzy subset taking 1 at x and 0 elsewhere.

Definition 2.1. [7] An *m-polar fuzzy set* (or a $[0,1]^m$-set) on X is a mapping $A : \mathscr{L} \to [0,1]^m$. The set of all m-polar fuzzy sets on $\mathscr{L}$ is denoted by $m(\mathscr{L})$.

Note that $[0,1]^m$(m-power of [0, 1]) is considered a poset with the point-wise order $\leqslant$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n \mid n < m\}$ when $m > 0$), $\leqslant$ is defined by $x \leqslant y \Leftrightarrow p_i(x) \leqslant p_i(y)$ for each $i \in m$ ($x, y \in [0,1]^m$), and $p_i : [0,1]^m \to [0,1]$ is the i-th projection mapping ($i \in m$). $\mathbf{0} = (0, 0, \ldots, 0)$ is the smallest element in $[0,1]^m$ and $\mathbf{1} = (1, 1, \ldots, 1)$ is the largest element in $[0,1]^m$.

Definition 2.2. Let C be an m-polar fuzzy set on a set $\mathscr{L}$. An *m-polar fuzzy relation* on C is an m-polar fuzzy set D of $\mathscr{L} \times \mathscr{L}$ such that for all $x, y \in X$ and $i = 1, 2, 3, \ldots, m$ we have $p_i \circ D(xy) \leqslant \inf(p_i \circ C(x), p_i \circ C(y))$.

3. m-polar fuzzy Lie ideals

Definition 3.1. An m-polar fuzzy set C on $\mathscr{L}$ is called an *m-polar fuzzy Lie ideal* if the following conditions are satisfied:

(1) $C(x + y) \geqslant \inf(C(x), C(y))$,

(2) $C(\alpha x) \geqslant C(x)$,

(3) $C([x, y]) \geqslant C(x)$ for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$.

That is,

(1) $p_i \circ C(x + y) \geqslant \inf(p_i \circ C(x), p_i \circ C(y))$,

(2) $p_i \circ C(\alpha x) \geqslant p_i \circ C(x)$,

(3) $p_i \circ C([x, y]) \geqslant p_i \circ C(x)$

for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$, $i = 1, 2, 3, \dots, m$.

Example 3.2. Let $\mathbb{R}^3 = \{(x, y, z) \,|\, x, y, z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors. Then $\mathbb{R}^3$ with the bracket $[\cdot, \cdot]$ defined as the usual cross product, i.e., $[x, y] = x \times y$, forms a real Lie algebra. We also define an m-polar fuzzy set $C : \mathbb{R}^3 \to [0, 1]^m$ by

$$C(x, y, z) = \begin{cases} (0.8, 0.8, \dots, 0.8) & \text{if } x = y = z = 0, \\ (0.1, 0.1, \dots, 0.1) & \text{otherwise.} \end{cases}$$

By routine computations, we can verify that the above m-polar fuzzy set C is an m-polar fuzzy Lie ideal of the Lie algebra $\mathbb{R}^3$.

Example 3.3. A subalgbera $sl_2(\mathbb{C})$ of all 2×2 matrices with trace 0 is an ideal of $gl_2(\mathbb{C})$. The basis of $sl_2(\mathbb{C})$ are: $h = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The commutators are $[e, f] = h$, $[h, f] = -2f$ and $[h, e] = 2e$.

We define an m-polar fuzzy set $C : gl_2(\mathbb{C}) \to [0, 1]^m$ by

$$C(g) = \begin{cases} (1, 1, \dots, 1), & g \in sl_n(\mathbb{C}) \\ (0, 0, \dots, 0), & \text{otherwise.} \end{cases}$$

By routine computations, we see that C is an m-polar fuzzy ideal.

We sate the following theorem without its proof.

Theorem 3.4. Let C be an m-polar fuzzy Lie ideal in a Lie algebra $\mathscr{L}$. Then C is an m-polar fuzzy Lie ideal of $\mathscr{L}$ if and only if the non-empty upper s-level cut $C_{[s]} = \{x \in \mathscr{L} | C(x) \geqslant s\}$ is Lie ideal of $\mathscr{L}$, for all $s = (s_1, s-2, \dots, s_m) \in [0, 1]^m$.

Example 3.5. Consider the group algebra $\mathbb{C}[S_3]$, where S_3 is the Symmetric group. Then $\mathbb{C}[S_3]$ assumes the structure of a Lie algebra via the bracket (commutator) operation.

Clearly, the linear span of the elements $\hat{g} = g - g^{-1}$ for $g \in S_3$ is the subalgebra of $\mathbb{C}[S_3]$, which is also known as Plesken Lie algebra and denoted by $\mathscr{L}(S_3)_{\mathbb{C}}$. It is easy to see that $\mathscr{L}(S_3)_{\mathbb{C}} = Span_{\mathbb{C}}\{\widehat{(1, 2, 3)}\}$ and $\widehat{(1, 2, 3)} = (1, 2, 3) - (1, 3, 2)$.

We define an m-polar fuzzy set $C : \mathscr{L}(S_3)_{\mathbb{C}} \to [0, 1]^m$ by

$$C(g) = \begin{cases} (t_1, t_2, \dots, t_m), & g = \gamma(1, 2, 3) - \gamma(1, 3, 2), \text{ where } \gamma \in \mathbb{C}, g \in \mathbb{C}[S_3] \\ (s_1, s_2, \dots, s_m), & \text{otherwise, where } s_i < t_i \end{cases}$$

By routine calculations, we have $\{g \in \mathbb{C}[S_3] : C(g) > (s_1, s_2, \dots, s_m)\} = \mathscr{L}(S_3)_{\mathbb{C}}$. Then we see that $\mathscr{L}(S_3)_{\mathbb{C}}$ can be realized $C_{[s]}$ as an upper s_i-level cut and C is an m-polar fuzzy Lie ideal of $\mathscr{L}(S_3)_{\mathbb{C}}$.

Definition 3.6. Let $C \in I^{\mathscr{L}}$, an m-polar fuzzy subspace of $\mathscr{L}$ generated by C will be denoted by $[C]$. It is the intersection of all m-polar fuzzy subspaces of $\mathscr{L}$ containing C. For all $x \in \mathscr{L}$, we define:

$$[C](x) = \sup\{\inf C(x_i) \,|\, x = \sum \alpha_i x_i, \alpha_i \mathbb{F}, x_i \in \mathscr{L}\}.$$

Definition 3.7. Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a homomorphism of Lie algebras which has an extension $f : I^{\mathscr{L}_1} \to I^{\mathscr{L}_2}$ defined by:

$$f(C)(y) = \sup\{C(x), x \in f^{-1}(y)\}.$$

for all $C \in I^{\mathscr{L}_1}$, $y \in \mathscr{L}_2$. Then $f(C)$ is called the *homomorphic image* of C.

Proposition 3.8. *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a homomorphism of Lie algebras and let C be an m-polar fuzzy Lie ideal of $\mathscr{L}_1$. Then*
(i) *$f(C)$ is an m-polar fuzzy Lie ideal of $\mathscr{L}_2$,*
(ii) *$f([C]) \supseteq [f(C)]$.*

Proposition 3.9. *If C and D are m-polar fuzzy Lie ideals in $\mathscr{L}$, then $[C, D]$ is an m-polar fuzzy Lie ideal of $\mathscr{L}$.*

Theorem 3.10. *Let C_1, C_2, D_1, D_2 be m-polar fuzzy Lie ideals in $\mathscr{L}$ such that $C_1 \subseteq C_2$ and $D_1 \subseteq D_2$, then $[C_1, D_1] \subseteq [C_2, D_2]$.*

Proof. Indeed,

$$\begin{aligned} \ll C_1, D_1 \gg (x) &= \sup\{\inf(C_1(a), D_1(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x\} \\ &\geqslant \sup\{\inf(C_2(a), D_2(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x\} \\ &= \ll C_2, D_2 \gg (x). \end{aligned}$$

Hence $[C_1, D_1] \subseteq [C_2, D_2]$. □

Let C be an m-polar fuzzy Lie ideal in $\mathscr{L}$. Putting

$$C^0 = C,\ C^1 = [C, C_0],\ C^2 = [C, C_1],\ \dots,\ C^n = [C, C^{n-1}]$$

we obtain a descending series of an m-polar fuzzy Lie ideals

$$C^0 \supseteq C^1 \supseteq C^2 \supseteq \ldots \supseteq C^n \supseteq \ldots$$

and a series of m-polar fuzzy sets $D^n = \sup\{C^n(x) \,|\, 0 \neq x \in \mathscr{L}\}$.

Definition 3.11. An m-polar fuzzy Lie ideal C is called *nilpotent* if there exists a positive integer n such that $D^n = \mathbf{0}$.

Theorem 3.12. *A homomorphic image of a nilpotent m-polar fuzzy Lie ideal is a nilpotent m-polar fuzzy Lie ideal.*

Proof. Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a homomorphism of Lie algebras and let C be a nilpotent m-polar fuzzy Lie ideal in $\mathscr{L}_1$. Assume that $f(C) = D$. We prove by induction that $f(C^n) \supseteq D^n$ for every natural n. First we claim that $f([C,C]) \supseteq [f(C), f(C)] = [D,D]$. Let $y \in \mathscr{L}_2$, then

$$\begin{aligned} f(\ll C, C \gg)(y) &= \sup\{\ll C, C \gg)(x) \mid f(x) = y\} \\ &= \sup\{\sup\{\inf(C(a), C(b)) \mid a, b \in \mathscr{L}_1, [a,b] = x, f(x) = y\}\} \\ &= \sup\{\inf(C(a), C(b)) \mid a, b \in \mathscr{L}_1, [a,b] = x, f(x) = y\} \\ &= \sup\{\inf(C(a), C(b)) \mid a, b \in \mathscr{L}_1, [f(a), f(b)] = y\} \\ &= \sup\{\inf(C(a), C(b)) \mid a, b \in \mathscr{L}_1, f(a) = u, f(b)] = v, [u,v] = y\} \\ &\geqslant \sup\{\inf(\sup_{a \in f^{-1}(u)} C(a), \sup_{b \in f^{-1}(v)} C(b)) \mid [u,v] = y\} \\ &= \sup\{\inf(f(C)(u), f(C)(v)) \mid [u,v] = y\} = \ll f(C), f(C) \gg (y), \end{aligned}$$

Thus

$$f([C,C]) \supseteq f(\ll C, C \gg) \supseteq \ll f(C), f(C) \gg = [f(C), f(C)].$$

For $n > 1$, we get

$$f(C^n) = f([C, C^{n-1}]) \supseteq [f(C), f(C^{n-1})] \supseteq [D, D^{n-1}] = D^n.$$

Let m be a positive integer such that $C^m = \mathbf{0}$. Then for $0 \neq y \in L_2$ we have

$$D^m(y) \leqslant f(\mu^P_{C^n})(y) = f(0)(y) = \sup\{0(a) \,|\, f(a) = y\} = \mathbf{0}.$$

This completes the proof. □

Let C be an m-polar fuzzy Lie ideal in $\mathscr{L}$. Putting

$$C^{(0)} = C,\ C^{(1)} = [C^{(0)}, C^{(0)}],\ C^{(2)} = [C^{(1)}, C^{(1)}], \ldots, C^{(n)} = [C^{(n-1)}, C^{(n-1)}]$$

we obtain series

$$C^{(0)} \subseteq C^{(1)} \subseteq C^{(2)} \subseteq \ldots \subseteq C^{(n)} \subseteq \ldots$$

of m-polar fuzzy Lie ideals and a series of m-polar fuzzy sets $D^{(n)}$ such that

$$D^n = \sup\{C^n(x) \,|\, 0 \neq x \in \mathscr{L}\}.$$

Definition 3.13. An m-polar fuzzy Lie ideal C is called *solvable* if there exists a positive integer n such that $D^{(n)} = \mathbf{0}$.

Theorem 3.14. *A nilpotent m-polar fuzzy Lie ideal is solvable.*

Proof. It is enough to prove that $C^{(n)} \subseteq C^n$ for all positive integers n. We prove it by induction on n and by the use of Theorem 3.10:

$$C^{(1)} = [C,C] = C^1, \qquad C^{(2)} = [C^{(1)}, C^{(1)}] \subseteq [C, C^{(1)}] = C^2.$$

$$C^{(n)} = [C^{(n-1)}, C^{(n-1)}] \subseteq [C, C^{(n-1)}] \subseteq [C, C^{(n-1)}] = C^n.$$

This completes the proof. □

Definition 3.15. Let C and D be two m-polar fuzzy Lie ideals of a Lie algebra $\mathscr{L}$. The sum $C \oplus D$ is called a *direct sum* if $C \cap D = \mathbf{0}$.

Theorem 3.16. *The direct sum of two nilpotent m-polar fuzzy Lie ideals is also a nilpotent m-polar fuzzy Lie ideal.*

Proof. Suppose that C and D are two m-polar fuzzy Lie ideals such that $C \cap D = \mathbf{0}$. We claim that $[C, D] = \mathbf{0}$. Let $x(\neq 0) \in \mathscr{L}$, then

$$\ll C, D \gg (x) = \sup\{\inf(C(a), D(b)) \mid [a, b] = x\} \leqslant \inf(C(x), D(x)) = \mathbf{0}.$$

This proves our claim. Thus we obtain $[C^m, D^n] = \mathbf{0}$ for all positive integers m, n. Now we again claim that $(C \oplus D)^n \subseteq C^n \oplus D^n$ for positive integer n. We prove this claim by induction on n. For $n = 1$,

$$(C \oplus D)^1 = [C \oplus D, C \oplus D] \subseteq [C, C] \oplus [C, D] \oplus [D, C] \oplus [D, D] = C^1 \oplus D^1.$$

Now for $n > 1$,

$$\begin{aligned}(C \oplus D)^n &= [C \oplus D, (C \oplus D)^{n-1}] \subseteq [C \oplus D, C^{n-1} \oplus D^{n-1}] \\ &\subseteq [C, C^{n-1}] \oplus [C, D^{n-1}] \oplus [D, C^{n-1}] \oplus [D, D^{n-1}] = C^n \oplus D^n.\end{aligned}$$

Since there are two positive integers p and q such that $C^p = D^q = \mathbf{0}$, we have $(C \oplus D)^{p+q} \subseteq C^{p+q} \oplus D^{p+q} = \mathbf{0}$. □

In a similar way we can prove the following theorem.

Theorem 3.17. *The direct sum of two solvable m-polar fuzzy Lie ideals is a solvable m-polar fuzzy Lie ideal.*

Definition 3.18. For any $x \in \mathscr{L}$ we define the function $adx : \mathscr{L} \to \mathscr{L}$ putting $adx(y) = [x, y]$. It is clear that this function is a linear homomorphism with respect to y. The set $H(\mathscr{L})$ of all linear homomorphisms from $\mathscr{L}$ into itself is made into a Lie algebra by defining a commutator on it by $[f, g] = f \circ g - g \circ f$. The function $ad : \mathscr{L} \to H(\mathscr{L})$ defined by $ad(x) = adx$ is a Lie homomorphism which is called the *adjoint representation* of $\mathscr{L}$.

The adjoint representation $adx : \mathscr{L} \to \mathscr{L}$ is extended to $\overline{ad}x : I^{\mathscr{L}} \to I^{\mathscr{L}}$ by putting

$$\overline{ad}x(\gamma)(y) = \sup\{\gamma(a) : [x, a] = y\}$$

for all $\gamma \in I^{\mathscr{L}}$ and $y \in \mathscr{L}$.

Theorem 3.19. *Let C be an m-polar fuzzy Lie ideal in a Lie algebra $\mathscr{L}$. Then $C^n \subseteq [C_n]$ for any $n > 0$, where an m-polar fuzzy subset $[C_n]$ is defined by*

$$[C_n](x) = \sup\{C(a) \mid [x_1, [x_2, [\ldots, [x_n, a] \ldots]]] = x, \quad x_1, \ldots, x_n \in \mathscr{L}\}.$$

Proof. It is enough to prove that $\ll C, C^{n-1} \gg \subseteq [C_n]$. We prove it by induction on n. For n=1 and $x \in \mathscr{L}$, we have

$$\begin{aligned}\ll C, C \gg (x) &= \sup\{\inf(C(a), C(b)) \mid [a,b] = x\}\\ &\geqslant \sup\{C(b) \mid [a,b] = x, a \in \mathscr{L}\} = [C_1](x).\end{aligned}$$

For $n > 1$,

$$\begin{aligned}
&\ll C, C^{(n-1)} \gg (x) = \sup\{\inf(C(a), C^{(n-1)}(b)) \mid [a,b] = x\}\\
&= \sup\{\inf(C(a), [C(b), C^{(n-2)}(b)]) \mid [a,b] = x\}\\
&\geqslant \sup\{\inf(C(a), \sup\{\ll C, C^{(n-2)} \gg (b_i) \mid b = \sum \alpha_i b_i\}) \mid [a,b] = x\}\\
&\geqslant \sup\{\inf(C(a), \sup\{[C_{n-1}](b_i) \mid b = \sum \alpha_i b_i\}) \mid [a,b] = x\}\\
&\geqslant \sup\{\inf(C(a), [C_{n-1}](b_i)) \mid \sum \alpha_i [a, b_i] = x\}\\
&\geqslant \sup\{\inf(C(a), \sup\{C_{n-1}(c_i) \mid b_i = \sum \beta_i c_i\}) \mid \sum \alpha_i [a, b_i] = x\}\\
&\geqslant \sup\{\inf(C(a), C_{n-1}(c_i)) \mid \sum \gamma_i [a, c_i] = x\}\\
&\geqslant \sup\{\inf(C(a), \sup\{C(d_i)) \mid [x_1, [x_2, [\ldots, [x_{n-1}, d_i] \ldots]]] = c_i\} \mid \sum \gamma_i [a, c_i] = x\}\\
&\geqslant \sup\{\inf(C(a), C(d_i)) \mid \sum \gamma_i [a, [x_1, [x_2, [\ldots, [x_{n-1}, d_i] \ldots]]]] = x\}\\
&\geqslant \sup\{C_n(d_i) \mid \sum \gamma_i [a, [x_1, [x_2, [\ldots, [x_{n-1}, d_i] \ldots]]]] = x\} \geqslant [C_n](x).
\end{aligned}$$

This complete the proof. □

Theorem 3.20. *If for an m-polar fuzzy Lie ideal C there exists a positive integer n such that*

$$(a\overline{d}x_1 \circ a\overline{d}x_2 \circ \ldots \circ a\overline{d}x_n)(C) = \mathbf{0}.$$

for all $x_1, \ldots, x_n \in \mathscr{L}$, then C is nilpotent.

Proof. For $x_1, \ldots, x_n \in \mathscr{L}$ and $x(\neq 0) \in \mathscr{L}$, we have

$$(a\overline{d}x_1 \circ \cdots \circ a\overline{d}x_n)(C)(x) = \sup\{C(a) \mid [x_1, [x_2, [\ldots, [x_n, a] \ldots]]] = x\} = \mathbf{0}.$$

Thus $[C_n] = \mathbf{0}$. From Theorem 3.19, it follows that $C^n = \mathbf{0}$. Hence C is a nilpotent m-polar fuzzy Lie ideal. □

The mapping $K : \mathscr{L} \times \mathscr{L} \to \mathbb{F}$ defined by $K(x, y) = Tr(adx \circ ady)$, where Tr is the *trace* of a linear homomorphism, is a symmetric bilinear form which is called the *Killing form*. It is not difficult to see that this form satisfies the identity $K([x, y], z) = K(x, [y, z])$. The form K can be naturally extended to $\overline{K} : I^{\mathscr{L} \times \mathscr{L}} \to I^{\mathbb{F}}$ defined by putting

$$\overline{K}(C)(\beta) = \sup\{C(x, y) \mid Tr((adx \circ ady)) = \beta\}.$$

The Cartesian product of two m-polar fuzzy sets C and D is defined as

$$(C \times D)(x, y) = \inf(C(x), D(y)).$$

Similarly we define

$$\overline{K}(C\times D)(\beta)=\sup\{\inf(C(x),D(y))\,|\,Tr((adx\circ ady))=\beta\}.$$

Theorem 3.21. *Let C be an m-polar fuzzy Lie ideal of Lie algebra $\mathscr{L}$. Then $\overline{K}(C\times 1_{(\alpha x)})=\alpha\odot\overline{K}(C\times 1_x)$ for all $x\in\mathscr{L}$, $\alpha\mathbb{F}$.*

Proof. If $\alpha=0$, then for $\beta=0$ we have

$$\begin{aligned}\overline{K}(C\times 1_0)(0)&=\sup\{\inf(C(x),1_0(y))\,|\,Tr(adx\circ ady)=0\}\\&\geqslant\inf(C(0),1_0(0))=\mathbf{0}.\end{aligned}$$

For $\beta\neq 0$ $Tr((adx\circ ady)=\beta$ means that $x\neq 0$ and $y\neq 0$. So,

$$\overline{K}(C\times 1_0)(\beta)=\sup\{\inf(C(x),1_0(y))\,|\,Tr((adx\circ ady)=\beta\}=\mathbf{0}.$$

If $\alpha\neq 0$, then for arbitrary β we obtain

$$\begin{aligned}\overline{K}(C\times 1_{\alpha x})(\beta)&=\sup\{\inf(C(y),1_{\alpha x}(z))\,|\,Tr((ady\circ adz)=\beta\}\\&=\sup\{\inf(C(y),\alpha\odot 1_x(z))\,|\,Tr((ady\circ adz)=\beta\}\\&=\sup\{\inf(C(y),1_x(\alpha^{-1}z))\,|\,\alpha Tr((ady\circ ad(\alpha^{-1}z))=\beta\}\\&=\sup\{\inf(C(y),1_x(\alpha^{-1}z))\,|\,Tr((ady\circ ad(\alpha^{-1}z))=\alpha^{-1}\beta\}\\&=\overline{K}(C\times 1_x)(\alpha^{-1}\beta)=\alpha\odot\overline{K}(C\times 1_x)(\beta).\end{aligned}$$

This completes the proof. □

Theorem 3.22. *Let C be an m-polar fuzzy Lie ideal of a Lie algebra $\mathscr{L}$. Then $\overline{K}(C\times 1_{(x+y)})=\overline{K}(C\times 1_x)\oplus\overline{K}(C\times 1_y)$ and $\overline{K}(C\times 0_{(x+y)})=\overline{K}(C\times 0_x)\oplus\overline{K}(C\times 0_y)$ for all $x,y\in\mathscr{L}$.*

Proof. Indeed,

$$\begin{aligned}\overline{K}(C\times 1_{(x+y)})(\beta)&=\sup\{\inf(C(z),1_{x+y}(u))\,|\,Tr((adz\circ adu)=\beta\}\\&=\sup\{C(z)\,|\,Tr(adz\circ ad(x+y))=\beta\}\\&=\sup\{C(z)\,|\,Tr(adz\circ adx)+Tr(adz\circ ady)=\beta\}\end{aligned}$$

$$\begin{aligned}&=\sup\{\inf(C(z),\inf(1_x(v),1_y(w)))\,|\,Tr(adz\circ adv)+Tr(adz\circ adw)=\beta\}\\&=\sup\{\inf(\sup\{\inf(C(z),1_x(v))\,|\,Tr(adz\circ adv)=\beta_1\},\\&\qquad\sup\{\inf(C(z),1_y(w))\,|\,Tr(adz\circ adw)=\beta_2\}\,|\,\beta_1+\beta_2=\beta)\}\\&=\sup\{\inf(\overline{K}(C\times 1_x)(\beta_1),\ \overline{K}(C\times 1_y)(\beta_2))\,|\,\beta_1+\beta_2=\beta\}\\&=\overline{K}(C\times 1_x)\oplus\overline{K}(C\times 1_y)(\beta).\end{aligned}$$

This completes the proof. □

We conclude that:

Corollary 3.23. *For each m-polar fuzzy Lie ideal C and all $x,y\in\mathscr{L}$, $\alpha,\beta\in\mathbb{F}$ we have*

$$\overline{K}(C\times 1_{(\alpha x+\beta y)})=\alpha\odot\overline{K}(C\times 1_x)\oplus\beta\odot\overline{K}(C\times 1_y).$$

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