Quasigroups and Related Systems **24** (2016), 75 − 80

On the principal (m,n)-ideals in the direct product of two semigroups

Panuwat Luangchaisri and Thawhat Changphas

Abstract. We characterize properties of the equivalence relation determined by two (m,n)-ideals of a semigroup S and describe properties of this relation in the direct product of two semigroups.

1. Preliminaries

Let m, n be non-negative integers. A subsemigroup A of a semigroup S is called an (m,n)-*ideal* of S if

$$A^m S A^n \subseteq A$$

(Here, $A^0 S = S A^0 = S$). This notion was first introduced by S. Lajos [3] in 1961. The principal (m,n)-ideal of S generated by $a \in S$ will be denoted by $[a]_{(m,n)}$, and it is of the form

$$[a]_{(m,n)} = \bigcup_{i=1}^{m+n} \{a^i\} \bigcup a^m S a^n$$

(see [2]).

Now, let T be a semigroup, and thus the direct product $S \times T$ is a semigroup under the coordinate wise multiplications. In this paper we introduce the equivalence relation $\mathcal{J}_{(m,n)}$ on S by, for any $a, b \in S$,

$$a \mathcal{J}_{(m,n)} b \;\longleftrightarrow\; [a]_{(m,n)} = [b]_{(m,n)}.$$

2. Main results

Throughout this section, let m, n be non-negative integers and S be a semigroup. We begin this section with the following lemmas:

Lemma 2.1. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$. Then*

$$(s,t)^m (S \times T)(s,t)^n = s^m S s^n \times t^m T t^n.$$

2010 Mathematics Subject Classification: 20M20.
Keywords: semigroup, principal (m,n)-ideal, maximal $\mathcal{J}_{(m,n)}$-class.
The first author is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

Hence,

$$[(s,t)]_{(m,n)} = \bigcup_{i=1}^{m+n}\{(s,t)^i\} \cup s^mSs^n \times t^mTt^n.$$

Proof. This follows by

$$(s,t)^m(S\times T)(s,t)^n = (s^m,t^n)(S\times T)(s^n,t^n) = s^mSs^n \times t^mTt^n. \qquad \square$$

Lemma 2.2. *Let s be an element of a semigroup S. Then*

$$[s]_{(m,n)} = s^mSs^n \;\longleftrightarrow\; s \in s^mSs^n.$$

Proof. It is clear that $[s]_{(m,n)} = s^mSs^n$ implies $s \in s^mSs^n$.
Conversely, if $s \in s^mSs^n$, then

$$[s]_{(m,n)} = \bigcup_{i=1}^{m+n}\{s^i\} \cup s^mSs^n \subseteq s^mSs^n.$$

By $s^mSs^n \subseteq [s]_{(m,n)}$, $[s]_{(m,n)} = s^mSs^n$. $\square$

Lemma 2.3. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$. Then*

$$[(s,t)]_{(m,n)} \subseteq [s]_{(m,n)} \times [t]_{(m,n)}.$$

Proof. This follows by Lemma 2.1. $\square$

We now prove the first main purpose of this paper.

Theorem 2.4. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$. Then $[(s,t)]_{(m,n)} = [s]_{(m,n)}\times[t]_{(m,n)}$ if and only if at least one of the following conditions holds:*

(1) $s^mSs^n = \{s\}$,

(2) $t^mTt^n = \{t\}$,

(3) $s \in s^mSs^n$ *and* $t \in t^mTt^n$.

Proof. Assume first that $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$. Suppose that $s \notin s^mSs^n$ or $t \notin t^mTt^n$. If $s \notin s^mSs^n$, then $s \neq s^k$ for all $k \in \{2,3,\ldots\}$; hence

$$\{s\} \times t^mTt^n = \{(s,t)\}.$$

This implies that $t^mTt^n = \{t\}$. Similarly, if $t \notin t^mTt^n$, then $s^mSs^n = \{s\}$.

Conversely, we assume that (1), (2) or (3) holds. If $s^mSs^n = \{s\}$, then, by Lemma 2.2, it follows that

$$\begin{aligned}[s]_{(m,n)} \times [t]_{(m,n)} &= \{s\} \times [t]_{(m,n)} = \bigcup_{i=1}^{m+n}\{(s,t^i)\} \bigcup \{s\} \times t^mTt^n \\ &= \bigcup_{i=1}^{m+n}\{(s^i,t^i)\} \bigcup s^mSs^n \times t^mTt^n = [(s,t)]_{(m,n)}.\end{aligned}$$

Thus $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$.

Similarly, if $t^mTt^n = \{t\}$, then $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$.

Finally, we assume that $s \in s^m S s^n$ and $t \in t^m T t^n$. By Lemmas 2.2 − 2.3, we have

$$[(s,t)]_{(m,n)} \subseteq [s]_{(m,n)} \times [t]_{(m,n)} = s^m S s^n \times t^m T t^n \subseteq [(s,t)]_{(m,n)}.$$

Therefore $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$, as required. □

Now, we consider the second aim of this paper.

Lemma 2.5. *For any $s \in S$, if $J_{(m,n),s} \cap s^m S s^n \neq \emptyset$, then $J_{(m,n),s} \subseteq s^m S s^n$.*

Proof. For $J_{(m,n),s} \cap s^m S s^n \neq \emptyset$ there exists $u \in J_{(m,n),s} \cap s^m S s^n$. Thus

$$s \in [s]_{(m,n)} = [u]_{(m,n)}.$$

We have $s \in s^m S s^n$. Indeed, if $s = u^i$ for some $i \in \{1, 2, \ldots, m+n\}$, then $s \in s^m S s^n$. And, if $s \in u^m S u^n$, then $s \in u^m S u^n \subseteq (s^m S s^n) S (s^m S s^n) \subseteq s^m S s^n$, and so $s \in s^m S s^n$.

Now, if $v \in J_{(m,n),s}$, then $[v]_{(m,n)} = [s]_{(m,n)}$; hence $v \in [s]_{(m,n)}$. This implies $v \in [s]_{(m,n)} = s^m S s^n$ by Lemma 2.2. Therefore $J_{(m,n),s} \subseteq s^m S s^n$. □

Lemma 2.6. *If for $s \in S$ the cardinality $|J_{(m,n),s}| > 1$, then $J_{(m,n),s} \subseteq s^m S s^n$.*

Proof. For $|J_{(m,n),s}| > 1$ there exists $u \in J_{(m,n),s}$ such that $u \neq s$. We have

$$u \in [u]_{(m,n)} = [s]_{(m,n)}.$$

If $u \in s^m S s^n$, then $J_{(m,n),s} \cap s^m S s^n \neq \emptyset$. So, $J_{(m,n),s} \subseteq s^m S s^n$, by Lemma 2.5. Let $u = s^i$ for some $i \in \{2, 3, \ldots, m+n\}$. If $s = u^j$ for some $j \in \{2, 3, \ldots, m+n\}$, then $s \in s^m S s^n$. Therefore, by Lemma 2.5, it follows that $J_{(m,n),s} \subseteq s^m S s^n$. If $s \in u^m S u^n$, then $J_{(m,n),s} = J_{(m,n),u} \subseteq u^m S u^n = s^{mi} S s^{ni} \subseteq s^m S s^n$. This completes the proof. □

Let S and T be any two semigroups. Define $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$, respectively, by:

$$(s,t)\pi_S = s \text{ for all } s \in S \quad \text{and} \quad (s,t)\pi_T = t \text{ for all } t \in T.$$

Then π_S (resp. π_T) is a projection from $S \times T$ onto S (resp. T). Moreover, for any $(s,t) \in S \times T$ we have $[(s,t)]_{(m,n)}\pi_S = [s]_{(m,n)}$ and $[(s,t)]_{(m,n)}\pi_T = [t]_{(m,n)}$.

Theorem 2.7. *Let S and T be any two semigroups, and let $(s,t) \in S \times T$. Then*

(1) $J_{(m,n),(s,t)} \subseteq J_{(m,n),s} \times J_{(m,n),t}$, *and*

(2) *if $J_{(m,n),(s,t)}$ is a proper subset of $J_{(m,n),s} \times J_{(m,n),t}$, then $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes in $S \times T$.*

Proof. To prove (1), let $(u,v) \in J_{(m,n),(s,t)}$. Then $[(s,t)]_{(m,n)} = [(u,v)]_{(m,n)}$,

$$[s]_{(m,n)} = [(s,t)]_{(m,n)}\pi_S = [(u,v)]_{(m,n)}\pi_S = [u]_{(m,n)}$$

and

$$[t]_{(m,n)} = [(s,t)]_{(m,n)}\pi_T = [(u,v)]_{(m,n)}\pi_T = [v]_{(m,n)}.$$

Thus $(u,v) \in J_{(m,n),s} \times J_{(m,n),t}$.

(2). Let $(u,v) \in J_{(m,n),s} \times J_{(m,n),t} \setminus J_{(m,n),(s,t)}$. Then $[u]_{(m,n)} = [s]_{(m,n)}$ and $[v]_{(m,n)} = [t]_{(m,n)}$. Thus

$$J_{(m,n),(u,v)} \subseteq J_{(m,n),u} \times J_{(m,n),v} = J_{(m,n),s} \times J_{(m,n),t}. \qquad \square$$

Corollary 2.8. *Let S and T be any two semigroups, and let $(s,t) \in S \times T$. If $J_{(m,n),s} = \{s\}$ and $J_{(m,n),t} = \{t\}$, then*

$$J_{(m,n),(s,t)} = J_{(m,n),s} \times J_{(m,n),t} = \{(s,t)\}.$$

Theorem 2.9. *Let S and T be any two semigroups, and let $s \in S, t \in T$. Then $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$ if and only if at least one of the following conditions holds:*

(1) $J_{(m,n),s} = \{s\}$ *and* $J_{(m,n),t} = \{t\}$,

(2) $s \in s^mSs^n$ *and* $t \in t^mTt^n$.

Proof. Assume that $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$. If $|J_{(m,n),(s,t)}| = 1$, then

$$J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)} = \{(s,t)\}.$$

That is, $J_{(m,n),s} = \{s\}$ and $J_{(m,n),t} = \{t\}$.

If $|J_{(m,n),(s,t)}| > 1$, then, $(s,t) \in J_{(m,n),(s,t)} \subseteq s^mSs^n \times t^mTt^n$, by Lemma 2.2.

Conversely, if (1) holds, then $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$, by Corollary 2.8. By (2) and Theorem 2.4, we get $[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)}$. By Theorem 2.7, $J_{(m,n),(s,t)} \subseteq J_{(m,n),s} \times J_{(m,n),t}$.

To prove the reverse inclusion let $(u,v) \in J_{(m,n),s} \times J_{(m,n),t}$.

Case 1: $(u,v) = (s,t)$. Then $(u,v) \in J_{(m,n),(s,t)}$, and so $J_{(m,n),s} \times J_{(m,n),t} \subseteq J_{(m,n),(s,t)}$.

Case 2: $u \neq s$. By Lemma 2.6, we have $u \in J_{(m,n),u} \subseteq u^mSu^n$, because $s, u \in J_{(m,n),s} = J_{(m,n),u}$.

Case 2.1: $v = t$. We have $v \in v^mTv^n$. By Theorem 2.4,

$$[(u,v)]_{(m,n)} = [u]_{(m,n)} \times [v]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)} = [(s,t)]_{(m,n)}.$$

Thus $(u,v) \in J_{(m,n),(s,t)}$. Therefore $J_{(m,n),s} \times J_{(m,n),t} \subseteq J_{(m,n),(s,t)}$.

Case 2.2: $v \neq t$. We have $v \in J_{(m,n),v} \subseteq v^mTv^n$. As Case 2.1 we have $(u,v) \in J_{(m,n),(s,t)}$, and thus

$$J_{(m,n),s} \times J_{(m,n),t} \subseteq J_{(m,n),(s,t)}.$$

Case 3: $u = t, v \neq t$. Analogously as Case 2.1.

This completes the proof. $\square$

Using the thereom above we have the following.

Corollary 2.10. *Let S and T be any two semigroups, and let $s \in S, t \in T$. If $|J_{(m,n),s}| > 1$ and $|J_{(m,n),t}| > 1$, then $J_{(m,n),(s,t)} = J_{(m,n),s} \times J_{(m,n),t}$.*

Corollary 2.11. *Let S and T be any two semigroups, and let $s \in S, t \in T$. If $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes, then necessarily either $|J_{(m,n),s}| > 1, J_{(m,n),t} = \{t\}$ or $|J_{(m,n),t}| > 1, J_{(m,n),s} = \{s\}$.*

Theorem 2.12. *Let S and T be any two semigroups, and let $s \in S, t \in T$. Then $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes if and only if either*

$$|J_{(m,n),s}| > 1, \quad J_{(m,n),t} = \{t\}, \quad t \notin t^m T t^n$$

or

$$|J_{(m,n),t}| > 1, \quad J_{(m,n),s} = \{s\}, \quad s \notin s^m S s^n.$$

Proof. Assume that $J_{(m,n),s} \times J_{(m,n),t}$ is the union of at least two $\mathcal{J}_{(m,n)}$-classes. By Corollary 2.11,

$$|J_{(m,n),s}| > 1, \ \ J_{(m,n),t} = \{t\}$$

or

$$|J_{(m,n),t}| > 1, \ \ J_{(m,n),s} = \{s\}.$$

CASE 1: $|J_{(m,n),s}| > 1$, $J_{(m,n),t} = \{t\}$. Then $t \notin t^m T t^n$ because otherwise, $s \in J_{(m,n),s} \subseteq s^m S s^n$ and $t \in t^m T t^n$ imply that $J_{(m,n),s} \times J_{(m,n),t} = J_{(m,n),(s,t)}$.

CASE 2: $|J_{(m,n),t}| > 1$, $J_{(m,n),s} = \{s\}$. This can be proceed analogously, and hence $s \notin s^m S s^n$.

For the opposite direction, it suffices to consider the case $|J_{(m,n),s}| > 1$, $J_{(m,n),t} = \{t\}$, $t \notin t^m T t^n$. Let $u \in J_{(m,n),s}$ such that $u \neq s$. Then $(u,t) \in J_{(m,n),s} \times J_{(m,n),t}$. Since $t \notin t^m T t^n$, we have $(s,t) \notin s^m S s^n \times t^m T t^n$. Thus, by Lemma 2.6, we have $(u,t) \notin \{(s,t)\} = J_{(m,n),(s,t)}$. □

The rest of this paper, relationships between maximal $\mathcal{J}_{(m,n)}$-classes in $S \times T$ and maximal $\mathcal{J}_{(m,n)}$-classes in S and in T will be investigated.

Theorem 2.13. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$ be such that $(s,t) \in s^m S s^n \times t^m T t^n$. Then, for any $u \in S, v \in T$, $[(s,t)]_{(m,n)} \subseteq [(u,v)]_{(m,n)}$ if and only if $[s]_{(m,n)} \subseteq [u]_{(m,n)}$ and $[t]_{(m,n)} \subseteq [v]_{(m,n)}$.*

Proof. Assume that $[(s,t)]_{(m,n)} \subseteq [(u,v)]_{(m,n)}$. Then

$$[s]_{(m,n)} = [(s,t)]_{(m,n)}\pi_S \subseteq [(u,v)]_{(m,n)}\pi_S = [u]_{(m,n)},$$

$$[t]_{(m,n)} = [(s,t)]_{(m,n)}\pi_T \subseteq [(u,v)]_{(m,n)}\pi_T = [v]_{(m,n)}.$$

Hence $[s]_{(m,n)} \subseteq [u]_{(m,n)}$ and $[t]_{(m,n)} \subseteq [v]_{(m,n)}$.

Assume that $[s]_{(m,n)} \subseteq [u]_{(m,n)}$ and $[t]_{(m,n)} \subseteq [v]_{(m,n)}$. Since $s \in s^m S s^n$ and $t \in t^m T t^n$, it follows by Theorem 2.4 and Lemma 2.2 that

$$[(s,t)]_{(m,n)} = [s]_{(m,n)} \times [t]_{(m,n)} = s^m S s^n \times t^m T t^n.$$

If $(x,y) \in [(s,t)]_{(m,n)}$, then

$$(x,y) \in s^mSs^n \times t^mTt^n \subseteq u^mSu^n \times v^mTv^n \subseteq [(u,v)]_{(m,n)}.$$

Thus $[(s,t)]_{(m,n)} \subseteq [(u,v)]_{(m,n)}$. □

Theorem 2.14. *Let S and T be any two semigroups, and let $s \in S$, $t \in T$ be such that $(s,t) \in s^mSs^n \times t^mTt^n$. Then $J_{(m,n),(s,t)}$ is a maximal $\mathcal{J}_{(m,n)}$-class in $S \times T$ if and only if $J_{(m,n),s}$ and $J_{(m,n),t}$ are maximal $\mathcal{J}_{(m,n)}$-classes in S and in T, respectively.*

Proof. Assume first that $J_{(m,n),(s,t)}$ is a maximal $\mathcal{J}_{(m,n)}$-class in $S \times T$. Suppose that $J_{(m,n),s}$ is not a maximal $\mathcal{J}_{(m,n)}$-class in S. Then there exists $u \in S$ such that $[s]_{(m,n)} \subset [u]_{(m,n)}$. By Theorem 2.13, $[(s,t)]_{(m,n)} \subseteq [(u,t)]_{(m,n)}$. We have

$$(u,t) \notin [s]_{(m,n)} \times [t]_{(m,n)} = [(s,t)]_{(m,n)}.$$

Thus $[(s,t)]_{(m,n)} \subset [(u,t)]_{(m,n)}$. This contradicts to the maximality of $J_{(m,n),(s,t)}$. In the same manner, if $J_{(m,n),t}$ is not a maximal $\mathcal{J}_{(m,n)}$-class in T, then we get a contradiction.

Conversely, we assume that $J_{(m,n),s}$ and $J_{(m,n),t}$ are maximal $\mathcal{J}_{(m,n)}$-classes in S and in T, respectively. Suppose that $J_{(m,n),(s,t)}$ is not maximal. Then there exists $(u,v) \in S \times T$ such that $[(s,t)]_{(m,n)} \subset [(u,v)]_{(m,n)}$. Thus

$$[s]_{(m,n)} \times [t]_{(m,n)} = [(s,t)]_{(m,n)} \subset [(u,v)]_{(m,n)} \subseteq [u]_{(m,n)} \times [v]_{(m,n)}.$$

This is a contradiction. □

References

[1] **I. Fabrici**, *One-sided principal ideals in the direct product of two semigroups*, Math. Bohemica **118** (1993), 337 – 342.

[2] **D.N. Krgović**, *On (m,n)-regular semigroups*, Publ. Inst. Math. (Beograd) **18(32)** (1975), 107 – 110.

[3] **S. Lajos**, *Generalized ideals in semigroups*, Acta Sci. Math. Szeged **22** (1961), 217 – 222.

Received January 19, 2016

P. Luangchaisri
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
E-mail: desparadoskku@hotmail.com

T. Changphas
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
Centre of Excellence in Mathematics, CHE, Si Ayuttaya Rd., Bangkok 10400, Thailand
E-mail: thacha@kku.ac.th