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On 2-absorbing ideals in commutative semirings

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Abstract. We study 2-absorbing ideals in a commutative semiring S with $1 \neq 0$ and prove some important results analogous to ring theory. More general form of the Prime Avoidance Theorem is also given. We also prove that if $I = \langle a_1, a_2, \ldots, a_r \rangle$ is a finitely generated ideal of a semiring S and $P_1, P_2, \ldots, P_n$ are subtractive prime ideals of S such that $I \nsubseteq P_i$ for each $1 \leqslant i \leqslant n$, then there exist $b_2, \ldots, b_r \in S$ such that $c = a_1 + b_2a_2 + \ldots + b_ra_r \notin \bigcup\limits_{i=1}^{n} P_i$.

1. Introduction

The semiring is an important algebraic structure which plays a prominent role in various branches of mathematics like combinatorics, functional analysis, topology, graph theory, optimization theory, cryptography etc. as well as in diverse areas of applied science such as theoretical physics, computer science, control engineering, information science, coding theory etc. The concept of semiring was first introduced by H. S. Vandiver [14] in 1934. After that several authors have apllied this concept in various disciplines in many ways.

A *commutative semiring* is a commutative semigroup $(S, \cdot)$ and a commutative monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$, both are connected by ring like distributivity. A subset I of a semiring S is called an *ideal* of S if $a, b \in I$ and $r \in S$, $a + b \in I$ and $ra, ar \in I$. An ideal I of a semiring S is called *subtractive* if $a, a+b \in I$, $b \in S$ then $b \in I$. A proper ideal P of a semiring S is said to be *prime* (resp. *weakly prime*) if for some $a, b \in S$ such that $ab \in P$ (resp. $0 \neq ab \in P$), then either $a \in P$ or $b \in P$.

Throughout this paper, semiring S will be considered as commutative with identity $1 \neq 0$.

2. Prime ideals

The concept of prime ideal plays an important role in ring and semiring theory. we refer ([8], [10], [13]), for more understanding about prime ideals. In this section, we give the more general form of The Prime Avoidance Theorem for semirings. We start this section with the statement of the following lemma.

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Lemma 2.1 ([15], Lemma 2.5). *Let P_1, P_2 be subtractive ideals of a commutative semiring S and I be an ideal of S such that $I \subseteq P_1 \cup P_2$. Then $I \subseteq P_1$ or $I \subseteq P_2$.*

Theorem 2.2 ([15], Theorem 2.6). (The Prime Avoidance Theorem)
Let S be a semiring and $P_1, \ldots, P_n$ ($n \geqslant 2$) be subtractive ideals of S such that almost two of $P_1, \ldots, P_n$ are not prime. Let I be an ideal of S such that $I \subseteq \bigcup\limits_{i=1}^{n} P_i$. Then $I \subseteq P_j$ for some $1 \leqslant j \leqslant n$.

The next theorem is the more general form of the Prime Avoidance Theorem of semirings.

Theorem 2.3. (Extented version of the Prime Avoidance Theorem)
Let S be a semiring and $P_1, \ldots, P_n$ be subtractive prime ideals of S. Let I be an ideal of S and $a \in S$ such that $aS + I \nsubseteq \bigcup\limits_{i=1}^{n} P_i$. Then there exists $c \in I$ such that $a + c \notin \bigcup\limits_{i=1}^{n} P_i$.

Proof. Assume that $P_i \nsubseteq P_j$ and $P_j \nsubseteq P_i$ for all $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. Suppose that a lies in all of $P_1, P_2, \ldots, P_k$ but none of $P_{k+1}, \ldots, P_n$. If $k = 0$, then $a = a + 0 \notin \bigcup\limits_{i=1}^{n} P_i$, which is required. So, let $k \geqslant 1$. Now, $I \nsubseteq \bigcup\limits_{i=1}^{k} P_i$, for otherwise, by the Prime Avoidance Theorem, we would get $I \subseteq P_j$ for some $1 \leqslant j \leqslant k$, which gives $aS + I \subseteq P_j \subseteq \bigcup\limits_{i=1}^{n} P_i$, which contradicts to the hypothesis. Thus, there exists $d \in I \setminus \bigcup\limits_{i=1}^{k} P_i$. Also, $P_{k+1} \cap \ldots \cap P_n \nsubseteq P_1 \cup \ldots \cup P_k$. Otherwise, if $P_{k+1} \cap \ldots \cap P_n \subseteq P_1 \cup \ldots \cup P_k$, by the Prime Avoidance Theorem, we would get a contradiction. Therefore there exists $b \in P_{k+1} \cap \ldots . \cap P_n \setminus (P_1 \cup \ldots \cup P_k)$. Now, define $c = db \in I$ and note that $c \in P_{k+1} \cap \ldots \cap P_n \setminus (P_1 \cup \ldots \cup P_k)$. Since $a \in P_1 \cap \ldots \cap P_k \setminus (P_{k+1} \cup \ldots \cup P_n)$, it follows that $a + c \notin \bigcup\limits_{i=1}^{n} P_i$ (since $P_i' s$ are subtractive). □

Next theorem says that if I is a finitely generated ideal of S satisfying the assumption of the Prime Avoidance Theorem for semirings, then the linear combination of the generators of I also avoids $\bigcup\limits_{i=1}^{n} P_i$, where $P_i' s$, ($1 \leqslant i \leqslant n$) are subtractive prime ideals of S.

Theorem 2.4. *Let S be a semiring and $I = \langle a_1, a_2, \ldots, a_r \rangle$ be a finitely generated ideal of S. Let $P_1, P_2, \ldots, P_n$ be subtractive prime ideals of S such that $I \nsubseteq P_i$ for each i, $1 \leqslant i \leqslant n$. Then there exist $b_2, \ldots, b_r \in S$ such that $c = a_1 + b_2 a_2 + \ldots + b_r a_r \notin \bigcup\limits_{i=1}^{n} P_i$.*

Proof. We prove it by induction on n. Without loss of generality, assume that $P_i \nsubseteq P_j$ for all $i \neq j$. If $n = 1$, then clearly $c = a_1 + b_2a_2 + \ldots + b_ra_r \notin P_1$. Assume that the result is true for $(n-1)$ subtractive prime ideals of S. Then, there exist $c_2, c_3, \ldots, c_r \in S$ such that $d = a_1 + c_2a_2 + \ldots + c_ra_r \notin \bigcup_{i=1}^{n-1} P_i$. If $d \notin P_n$, then we are through. So assume that $d \in P_n$. If $a_2, \ldots, a_r \in P_n$, then from the expression for d, we have $a_1 \in P_n$, (since $d = a_1 + c_2a_2 + \ldots + c_ra_r$ and $d \in P_n$ implies $a_1 \in P_n$, since P_n is subtractive), which is a contradiction to $I \nsubseteq P_n$ (since, if $a_1 \in P_n$ and we have already assumed that $a_2, \ldots, a_r \in P_n$, we get $a_1, \ldots, a_r \in P_n$, this implies that $I \subseteq P_n$). So for some i, $a_i \notin P_n$. Without loss of generality, let $i = 2$. Since $P_i \nsubseteq P_j$ for all $i \neq j$, we can find $x \in \bigcap_{i=1}^{n-1} P_i$ such that $x \notin P_n$. Thus, $c = a_1 + (c_2 + x)a_2 + \ldots + c_ra_r \notin \bigcup_{i=1}^{n} P_i$. □

3. 2-absorbing ideals

The concept of 2-absorbing and weakly 2-absorbing ideals of a commutative ring with non-zero unity was first introduced by Badawi and Darani in [3], [4] which are generalizations of prime and weakly prime ideals in commutative ring, see [1]. After that Darani [7] and Kumar et. al [11], explored these concepts in commutative semiring and characterized many results in terms of 2-absorbing and weakly 2-absorbing ideals in commutative semiring. Most of the results of this section are inspired from [5] and [6].

Definition 3.1. A proper ideal I of a semiring S is said to be a *2-absorbing ideal* of S if $abc \in I$ implies $ab \in I$ or $bc \in I$ or $ac \in I$ for some $a, b, c \in S$.

Definition 3.2. A proper ideal I of a semiring S is said to be a *weakly 2-absorbing ideal* if whenever $a, b, c \in S$ such that $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Clearly, one can see that every 2-absorbing ideal of a semiring S is a weakly 2-absorbing ideal of S but converse need not be true. For more details of 2-absorbing and weakly 2-absorbing ideals in commutative semirings, we refer [7], [11].

Lemma 3.3. *Let I be a subtractive 2-absorbing ideal of S. Suppose that $abJ \subseteq I$ for some $a, b \in S$ and an ideal J of S. If $ab \notin I$, then either $aJ \subseteq I$ or $bJ \subseteq I$.*

Proof. Suppose that $aJ \nsubseteq I$ and $bJ \nsubseteq I$. Therefore, there are some $x, y \in J$ such that $ax \notin I$ and $by \notin I$. Since $abx \in I$ and $ab \notin I$ and $ax \notin I$, we have $bx \in I$. Since $aby \in I$ and $ab \notin I$ and $by \notin I$, we have $ay \in I$. Now, since $ab(x+y) \in I$ and $ab \notin I$, we have $a(x+y) \in I$ or $b(x+y) \in I$, since I is a 2-absorbing ideal of S. If $a(x+y) \in I$ and $ay \in I$, then $ax \in I$ (since I is subtractive), which is a contradiction. Similarly, if $b(x+y) \in I$ and $bx \in I$, we get $by \in I$ (since I is subtractive), which is again a contradiction. Hence, either $aJ \subseteq I$ or $bJ \subseteq I$. □

Theorem 3.4. *Let I be a proper subtractive ideal of S. Then I is a 2-absorbing ideal of S if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_3I_1 \subseteq I$.*

Proof. Let $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$. Then by definition, I is a 2-absorbing ideal of S. Conversely, let I be a 2-absorbing ideal of S and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, such that $I_1I_2 \nsubseteq I$. We show that $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. If possible, suppose that $I_1I_3 \nsubseteq I$ and $I_2I_3 \nsubseteq I$. Then there exist $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$. Also, $a_1a_2I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, we have $a_1a_2 \in I$ by above lemma. Since $I_1I_2 \nsubseteq I$, therefore for some $a \in I_1$, $b \in I_2$, $ab \notin I$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq I$ or $bI_3 \subseteq I$ by above lemma. Here three cases arise.

Case I: Suppose that $aI_3 \subseteq I$, but $bI_3 \nsubseteq I$. Since $a_1bI_3 \subseteq I$ and $bI_3 \nsubseteq I$ and $a_1I_3 \nsubseteq I$, by above lemma, we have $a_1b \in I$. Since $(a + a_1)bI_3 \subseteq I$ and $aI_3 \subseteq I$, but $a_1I_3 \nsubseteq I$, therefore $(a + a_1)I_3 \nsubseteq I$. Since $bI_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)b \in I$ by above lemma. Again, $(a + a_1)b = ab + a_1b \in I$ and $a_1b \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case II: Suppose that $bI_3 \subseteq I$, but $aI_3 \nsubseteq I$. Since $aa_2I_3 \subseteq I$ and $aI_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, by above lemma, we have $aa_2 \in I$. Again, $a(b + a_2)I_3 \subseteq I$ and $bI_3 \subseteq I$, but $a_2I_3 \nsubseteq I$, we have $(b + a_2)I_3 \nsubseteq I$. Since $aI_3 \nsubseteq I$ and $(b + a_2)I_3 \nsubseteq I$, we have $a(b + a_2) \in I$ by above lemma. Since $a(b + a_2) = ab + aa_2 \in I$ and $aa_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case III: Suppose that $aI_3 \subseteq I$ and $bI_3 \subseteq I$. Since $bI_3 \subseteq I$ and $a_2I_3 \nsubseteq I$, we have $(b+a_2)I_3 \nsubseteq I$. Since $a_1(b+a_2)I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $(b+a_2)I_3 \nsubseteq I$, we have $a_1(b + a_2) = a_1b + a_1a_2 \in I$ by lemma above. Since $a_1b + a_1a_2 \in I$ and $a_1a_2 \in I$, we have $ba_1 \in I$ (since I is subtractive). Since $aI_3 \subseteq I$ and $a_1I_3 \nsubseteq I$, we have $(a + a_1)I_3 \nsubseteq I$. Since $(a + a_1)a_2I_3 \subseteq I$ and $a_2I_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)a_2 = aa_2 + a_1a_2 \in I$ by above lemma. Since $a_1a_2 \in I$ and $aa_2 + a_1a_2 \in I$, we have $aa_2 \in I$ (since I is subtractive). Now, since $(a + a_1)(b + a_2)I_3 \subseteq I$ and $(a+a_1)I_3 \nsubseteq I$ and $(b+a_2)I_3 \nsubseteq I$, we have $(a+a_1)(b+a_2) = ab+aa_2+ba_1+a_1a_2 \in I$ by above lemma. Since $aa_2, ba_1, a_1a_2 \in I$, we have $aa_2 + ba_1 + a_1a_2 \in I$. Since $ab + aa_2 + ba_1 + a_1a_2 \in I$ and $aa_2 + ba_1 + a_1a_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction. Hence $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. □

Result 3.5 ([2], Lemma 2.1 (ii))**.** *If I is a subtractive ideal of S, then $(I : a)$ is a subtractive ideal of S, where $(I : a) = \{s \in S : sa \in I\}$.*

Proof. It is straight forward. □

Next theorem gives some characterizations of 2-absorbing ideals of semiring. Mostafanasab and Darani in [12], proved it for 2-absorbing primary ideals of rings.

Theorem 3.6. *Let S be a semiring and I be a proper subtractive ideal of S. Then the following are equivalent:*

(1) *I is a 2-absorbing ideal of S;*

(2) *For all* $a, b \in S$ *such that* $ab \notin I$, $(I : ab) \subseteq (I : a)$ *or* $(I : ab) \subseteq (I : b)$*;*

(3) *For all* $a \in S$ *and for all ideal* J *of* S *such that* $aJ \nsubseteq I$, $(I : aJ) \subseteq (I : J)$ *or* $(I : aJ) \subseteq (I : a)$*;*

(4) *For all ideals* J, K *of* S *such that* $JK \nsubseteq I$, $(I : JK) \subseteq (I : J)$ *or* $(I : JK) \subseteq (I : K)$*;*

(5) *For all ideals* J, K, L *of* S *such that* $JKL \subseteq I$, *either* $JK \subseteq I$ *or* $KL \subseteq I$ *or* $JL \subseteq I$.

Proof. (1) $\Rightarrow$ (2). Let $ab \notin I$ where $a, b \in S$ and $x \in (I : ab)$. Then $xab \in I$. Therefore, either $xa \in I$ or $xb \in I$ and hence either $x \in (I : a)$ or $x \in (I : b)$. Thus, $(I : ab) \subseteq (I : a) \cup (I : b)$. Then we have $(I : ab) \subseteq (I : a)$ or $(I : ab) \subseteq (I : b)$ (since if A, B are subtractive ideals of a semiring S such that $C \subseteq A \cup B$ where C is an ideal of S, then either $C \subseteq A$ or $C \subseteq B$).
(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) is similar as the proof of ([12], Theorem 2.1), by using the result (if A, B are subtractive ideals of a semiring S such that $C \subseteq A \cup B$ where C is an ideal of S, then either $C \subseteq A$ or $C \subseteq B$). □

Theorem 3.7. *Let* I *be a* 2*-absorbing ideal of* S *and* A *be a multiplicatively closed subset of* S *such that* $I \cap A = \Phi$. *Then* $A^{-1}I$ *is also a* 2*-absorbing ideal of* $A^{-1}S$.

Proof. Let $(a/s)(b/t)(c/k) \in A^{-1}I$ for some $a, b, c \in S$ and $s, t, k \in A$. Then there exists $u \in A$ such that $uabc \in I$. Therefore, we have $uab \in I$ or $bc \in I$ or $uac \in I$, since I is a 2-absorbing ideal of S. If $uab \in I$, then $(a/s)(b/t) = (uab/ust) \in A^{-1}I$. If $bc \in I$, then $(b/t)(c/k) \in A^{-1}I$. If $uac \in I$, then $(a/s)(c/k) = (uac/usk) \in A^{-1}I$. □

Lemma 3.8. *Let* S *be a semiring and* P_1 *and* P_2 *be distinct weakly prime ideals of* S. *Then* $P_1 \cap P_2$ *is also a weakly* 2*-absorbing ideal of* S.

Proof. Let $0 \neq abc \in P_1 \cap P_2$ for some $a, b, c \in S$. Suppose that $ab \notin P_1 \cap P_2$ and $ac \notin P_1 \cap P_2$. Assume that $ab \notin P_1$ and $ac \notin P_1$. Since $0 \neq abc \in P_1$ and P_1 is weakly prime, we get $c \in P_1$ and hence $ac \in P_1$, a contradiction. Similarly, if $ab \notin P_2$ and $ac \notin P_2$, we would get a contradiction. Therefore, either $ab \notin P_1$ and $ac \notin P_2$ or $ab \notin P_2$ and $ac \notin P_1$. First assume that, $ab \notin P_1$ and $ac \notin P_2$. Since $0 \neq abc \in P_1$, we get $c \in P_1$ and hence $bc \in P_1$. Similarly, since $0 \neq abc \in P_2$, we get $b \in P_2$ and hence $bc \in P_2$. Thus, $bc \in P_1 \cap P_2$. Hence $P_1 \cap P_2$ is a weakly 2-absorbing ideal of S. Likewise, we can prove for the second case when $ab \notin P_2$ and $ac \notin P_1$, we have $bc \in P_1 \cap P_2$. □

Definition 3.9. Let I be a weakly 2-absorbing ideal of S. We say that (a, b, c), where $a, b, c \in S$ is a *triple zero* of I if $abc = 0$, $ab \notin I$, $bc \notin I$ and $ac \notin I$.

Theorem 3.10. *Let* I *be a subtractive weakly* 2*-absorbing ideal of* S *and* (a, b, c) *be a triple zero of* I *for some* $a, b, c \in S$. *Then*

(1) $abI = bcI = acI = \{0\}$.

(2) $aI^2 = bI^2 = cI^2 = \{0\}$.

Proof. (1). Let $abI \neq 0$. Then there exists $x \in I$ such that $abx \neq 0$. Therefore, $ab(c+x) \neq 0$. Since I is a weakly 2-absorbing ideal of S and $ab \notin I$, we have $a(c+x) \in I$ or $b(c+x) \in I$ and hence $ac \in I$ or $bc \in I$ (since I is subtractive), which is a contradiction. Thus, $abI = 0$. Similarly, $bcI = acI = 0$.

(2). Let $aI^2 \neq 0$. Then there exist $x, y \in I$ such that $axy \neq 0$. Therefore (1) gives, $a(b+x)(c+y) = axy \neq 0$. Since I is a weakly 2-absorbing ideal of S, we have either $a(b+x) \in I$ or $a(c+y) \in I$ or $(b+x)(c+y) \in I$. Thus, $ab \in I$ or $ac \in I$ or $bc \in I$ (since I is subtractive), which is a contradiction. Hence $aI^2 = 0$. Similarly, $bI^2 = cI^2 = 0$. □

Definition 3.11. Let I be a weakly 2-absorbing ideal of S and let $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S. We say that I is a *free triple zero with respect to* $I_1I_2I_3$ if (a, b, c) is not a triple zero of I for every $a \in I_1$, $b \in I_2$, and $c \in I_3$.

Conjecture 3.12. *If I is a weakly 2-absorbing ideal of S with $0 \neq I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2, I_3 \in S$, then I is a free triple zero with respect to $I_1I_2I_3$.*

Lemma 3.13. *Let I be a subtractive weakly 2-absorbing ideal of S. Let $abJ \subseteq I$ for some $a, b \in S$ and some ideal J of S such that (a, b, c) is not a triple zero of I for every $c \in J$. If $ab \notin I$, then either $aJ \subseteq I$ or $bJ \subseteq I$.*

Proof. Suppose that $aJ \nsubseteq I$ and $bJ \nsubseteq I$. Then, there are some $x, y \in J$ such that $ax \notin I$ and $by \notin I$. Since (a, b, x) is not a triple zero of I and $abx \in I$ and $ab \notin I$ and $ax \notin I$, we have $bx \in I$. Since (a, b, y) is not a triple zero of I and $aby \in I$ and $ab \notin I$ and $by \notin I$, we have $ay \in I$. Again, $(a, b, x+y)$ is not a triple zero of I and $ab(x+y) \in I$ and $ab \notin I$, we have $a(x+y) \in I$ or $b(x+y) \in I$, since I is a weakly 2-absorbing ideal of S. If $a(x+y) \in I$ and $ay \in I$, then $ax \in I$ (since I is subtractive), which is a contradiction. Similarly, if $b(x+y) \in I$ and $bx \in I$, we get $by \in I$ (since I is subtractive), which is a contradiction. Hence, either $aJ \subseteq I$ or $bJ \subseteq I$. □

Remark 3.14. If I is a weakly 2-absorbing ideal of S and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S such that I is a free triple zero with respect to $I_1I_2I_3$. Then $ab \in I$ or $ac \in I$ or $bc \in I$ for all $a \in I_1$, $b \in I_2$ and $c \in I_3$.

Let I be a weakly 2-absorbing ideal of S. According to the following result, we see that Conjecture 3.12 is valid if and only if whenever $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_1I_3 \subseteq I$.

Theorem 3.15. *Let I be a subtractive weakly 2-absorbing ideal of S. If $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S such that I is a free triple zero with respect to $I_1I_2I_3$, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq I$ or $I_3I_1 \subseteq I$.*

Proof. Let I be a subtractive weakly 2-absorbing ideal of S and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S such that I is a free triple zero with respect to $I_1I_2I_3$. Let $I_1I_2 \nsubseteq I$. We show that $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. By using above remark 1 and lemma 3.13, it will proceed as the proof of theorem 3.4. If possible, suppose that $I_1I_3 \nsubseteq I$ and $I_2I_3 \nsubseteq I$. Then there exist $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$. Also, $a_1a_2I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, we have $a_1a_2 \in I$ by lemma 3.13. Since $I_1I_2 \nsubseteq I$, therefore for some $a \in I_1$, $b \in I_2$, $ab \notin I$. Since $abI_3 \subseteq I$ and $ab \notin I$, we have $aI_3 \subseteq I$ or $bI_3 \subseteq I$ by lemma 3.13 . Here three cases arise.

Case I: Suppose that $aI_3 \subseteq I$, but $bI_3 \nsubseteq I$. Since $a_1bI_3 \subseteq I$ and $bI_3 \nsubseteq I$ and $a_1I_3 \nsubseteq I$, by lemma 3.13, we have $a_1b \in I$. Since $(a + a_1)bI_3 \subseteq I$ and $aI_3 \subseteq I$, but $a_1I_3 \nsubseteq I$, therefore $(a + a_1)I_3 \nsubseteq I$. Since $bI_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)b \in I$ by lemma 3.13. Again, $(a + a_1)b = ab + a_1b \in I$ and $a_1b \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case II: Suppose that $bI_3 \subseteq I$, but $aI_3 \nsubseteq I$. Since $aa_2I_3 \subseteq I$ and $aI_3 \nsubseteq I$ and $a_2I_3 \nsubseteq I$, by lemma 3.13, we have $aa_2 \in I$. Again, $a(b + a_2)I_3 \subseteq I$ and $bI_3 \subseteq I$, but $a_2I_3 \nsubseteq I$, we have $(b + a_2)I_3 \nsubseteq I$. Since $aI_3 \nsubseteq I$ and $(b + a_2)I_3 \nsubseteq I$, we have $a(b + a_2) \in I$ by lemma 3.13. Since $a(b + a_2) = ab + aa_2 \in I$ and $aa_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction.

Case III: Suppose that $aI_3 \subseteq I$ and $bI_3 \subseteq I$. Since $bI_3 \subseteq I$ and $a_2I_3 \nsubseteq I$, we have $(b + a_2)I_3 \nsubseteq I$. Since $a_1(b + a_2)I_3 \subseteq I$ and $a_1I_3 \nsubseteq I$ and $(b + a_2)I_3 \nsubseteq I$, we have $a_1(b+a_2) = a_1b+a_1a_2 \in I$ by lemma 3.13. Since $a_1b+a_1a_2 \in I$ and $a_1a_2 \in I$, we have $ba_1 \in I$ (since I is subtractive). Since $aI_3 \subseteq I$ and $a_1I_3 \nsubseteq I$, we have $(a + a_1)I_3 \nsubseteq I$. Since $(a + a_1)a_2I_3 \subseteq I$ and $a_2I_3 \nsubseteq I$ and $(a + a_1)I_3 \nsubseteq I$, we have $(a + a_1)a_2 = aa_2 + a_1a_2 \in I$ by lemma 3.13. Since $a_1a_2 \in I$ and $aa_2 + a_1a_2 \in I$, we have $aa_2 \in I$ (since I is subtractive). Now, since $(a + a_1)(b + a_2)I_3 \subseteq I$ and $(a+a_1)I_3 \nsubseteq I$ and $(b+a_2)I_3 \nsubseteq I$, we have $(a+a_1)(b+a_2) = ab+aa_2+ba_1+a_1a_2 \in I$ by lemma 3.13. Since $aa_2, ba_1, a_1a_2 \in I$, we have $aa_2 + ba_1 + a_1a_2 \in I$. Since $ab + aa_2 + ba_1 + a_1a_2 \in I$ and $aa_2 + ba_1 + a_1a_2 \in I$, we conclude that $ab \in I$ (since I is subtractive), which is a contradiction. Hence $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. □

Proposition 3.16. *Let S be a semiring and I be a proper subtractive ideal of S. Then the following statements are equivalent:*

(1) *For any ideals I_1, I_2, I_3 of S, $0 \neq I_1I_2I_3 \subseteq I$ implies either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$;*

(2) *For any ideals I_1, I_2, I_3 of S such that $I \subseteq I_1$, $0 \neq I_1I_2I_3 \subseteq I$ implies either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Let $0 \neq JI_2I_3 \subseteq I$ for some ideals J, I_2, I_3 of S. Then obviously $0 \neq (J + I)I_2I_3 = (JI_2I_3) + (II_2I_3) \subseteq I$. Let $I_1 = J + I$. Then, either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$ by given hypothesis. Therefore, $(J + I)I_2 \subseteq I$ or $(J + I)I_3 \subseteq I$ or $I_2I_3 \subseteq I$. Thus, either $JI_2 \subseteq I$ or $JI_3 \subseteq I$ or $I_2I_3 \subseteq I$ (since I is subtractive). □

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