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Dynamic groups

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Abstract. In this essay we introduce a class of groups which any member of it has a dynamic product. We prove that any subgroup of a dynamic group is a dynamic group and the product of two dynamic groups is a dynamic group. We deduce a new equivalency on dynamical systems via Rees matrix semigroups.

1. Introduction

Groups with dynamic products are a class of groups which have important role in topological cocycles [5]. Cocycles [1, 2, 3] are time-dependent dynamical systems and they can describe by these kind of groups [5]. To present the definition of a dynamic group we first recall the definition of a dynamical system. We assume that $(T, +)$ is the group of real numbers or the group of integer numbers. The binary operation $+$ can be any group operation on this set. If Y is a non-empty set, then a family $\xi = \{\xi^t : t \in T\}$ of the maps $\xi^t : Y \to Y$ is called a *dynamical system* if

(i) $\xi^0 = id_Y$;

(ii) $\xi^{t+s} = \xi^t \circ \xi^s$ for all $t, s \in T$.

$(T, +)$ is called the *time group* of ξ, and T is called the *time set* of ξ. If $(T, +)$ is a semigroup, and ξ satisfies the condition (ii), then it is called a *semi-dynamical system.*

Definition 1.1. Suppose ξ is a dynamical system (semi-dynamical system) on Y, and G is a group (semigroup). (G, ξ) is called a *dynamic group* (*dynamic semigroup*) if there is a one-to-one map $h : G \to \xi$ such that $h(b) \circ h(c) = h(cb)$.

If G is a group with the identity e, then the above definition implies that $h(e) = \xi^0$. One must pay attention to this point that: in the above definition if h is an onto map, and T is a commutative group, then h is a group isomorphism.

Example 1.2. We define a self map η on the circle S^1 by $\eta(e^{2\pi i\theta}) = e^{2\pi i(\theta+\frac{1}{4})}$, and we take $\xi = \{\eta^n : n \in Z\}$, where

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$$\eta^n = \begin{cases} \underbrace{\eta \circ \eta \circ \eta \circ \cdots \circ \eta}_{n\ times} & if\ \ n \in N, \\ id & if\ \ n = 0, \\ \underbrace{\eta^{-1} \circ \eta^{-1} \circ \eta^{-1} \circ \cdots \circ \eta^{-1}}_{-n\ times} & if\ \ -n \in N. \end{cases}$$

Let G be the additive group modulo 4. Then (G, ξ) is a dynamic group. □

In the next section we present an example of a non trivial dynamic group.

Example 1.3. The set of integer numbers with the product $m * n = n|m|$ is a semigroup. Z with the product $m \times n = m|n|$ is also a semigroup. If for given $n \in Z$, we define $\eta^n : Z \to Z$ by

$$\eta^n(m) = \begin{cases} n^3|m| & if\ \ m^{\frac{1}{3}} \in Z, \\ m & if\ \ m^{\frac{1}{3}} \notin Z, \end{cases}$$

then $\xi = \{\eta^n : n \in (Z, \times)\}$ is a semi-dynamical system, and $((Z, *), \xi)$ is a dynamic semigroup. In fact, if we define $h : Z \to \xi$ by $h(n) = \eta^n$, then $h(n) \circ h(m) = h(m*n)$ and h is one-to-one. □

In the next section we present two methods for constructing dynamic groups (dynamic semigroups), and we show that a dynamic product is an algebraic property. We associate a completely simple semigroup to a dynamic group. By using of completely simple semigroups or Rees matrix semigroups we present an equivalence relation on dynamical systems.

2. Structural consideration

We begin this section by presenting a nontrivial example of a dynamic group.

Example 2.1. Let

$$Y = \{y : R^2 \to R : y(t, x) = 2x + g(t)\ where\ g(t)\ is\ a\ continuous\ function\},$$

and for given $s \in R$ let $\xi^t : Y \to Y$ be defined by $\xi^s(y)(t, x) = y(t + s, x)$. Then $\xi = \{\xi^s\ :\ s \in R\}$ is a dynamical system on Y. For given $x, t, s \in R\ and\ y \in Y$ we take $\varphi^t(y(t + s, x), x) = e^{2t}[x + \int_0^t e^{-2u} g(u + s) du]$. Suppose $G = \{\varphi^t(y(t, .), .) : t \in R\ and\ y \in Y\}$. We define a product on G by the following form

$$\varphi^t(y(t, .), .)\varphi^s(z(s, .), .) = \varphi^t(y(t + s, .), .) o \varphi^s(z(s, .), .).$$

Then G with this product is a group and (G, ξ) is a dynamic group. The map $h : G \to \xi$ defined by $h(\varphi^t(y(t, .), .)) = \xi^t$ has the properties of Definition 1.1. □

Dynamic group is a kind of groups which it's product is look alike to an evolution operator up to a one-to-one map. To see this let (G, ξ) be a dynamic group with a one-to-one mapping $h : G \to \xi$. Any member of ξ is called an *evolution operator.* If $P : G \times G \to G$ is the product of G, and if

$$A = \{P_b = P(., b) : G \to G : b \in G\},$$

then there is a bijection

$$\phi : G \longrightarrow A, \quad b \mapsto P_b.$$

Under the map $h \circ \phi^{-1}$ any given P_b is look alike to the evolution operator $(h \circ \phi^{-1})(P_b)$. So there is a dynamics on the product of G.

Theorem 2.2. *If H is a subgroup of a dynamic group (G, ξ), then H is a dynamic group.*

Proof. Suppose $h : G \to \xi$ is a one-to-one map with the properties of Definition 1.1, then $h|_H : H \to \xi$ has the properties of Definition 1.1 for (H, ξ). □

Theorem 2.3. *If (G_1, ξ_1) and (G_2, ξ_2) are two dynamic groups with a common time set T, then $G_1 \times G_2$ is a dynamic group.*

Proof. Suppose $h_1 : G_1 \longrightarrow \xi_1,\ g \mapsto \xi_1^{t_g}$ and $h_2 : G_2 \longrightarrow \xi_1,\ g \mapsto \xi_2^{t_g}$ are the one-to-one maps which satisfy the conditions of Definition 1.1. We know that $G_1 \times G_2$ with the multiplication $(g_1, g_2)(j_1, j_2) = (g_1 j_1, g_2 j_2)$ is a group, and we know that there is a bijection $\sigma : T \times T \to T$, where $T = R$ or $T = Z$. We define the following binary operation on T:

$$+_\sigma : T \times T \longrightarrow T, \quad (t, s) \mapsto \sigma(t + s).$$

Clearly $(T, +_\sigma)$ is a group. We assume that ξ_i is a dynamical system on Y_i for $i \in \{1, 2\}$. If $t \in T$, then we define

$$\xi^t : Y_1 \times Y_2 \longrightarrow Y_1 \times Y_2, \quad (y_1, y_2) \mapsto (\xi_1^{t_1}(y_1), \xi_2^{t_2}(y_2)),$$

where $t = \sigma(t_1, t_2)$. The straightforward calculations imply that

$$\xi = \{\xi^t : t \in T \textit{ and the operation of } T \textit{ is } +_\sigma\}$$

is a dynamical system on $Y_1 \times Y_2$. Now we define $h : G_1 \times G_2 \to \xi$ by $h(g_1, g_2) = \xi^{\sigma(t_{g_1}, t_{g_2})}$. Since σ, h_1, h_2 are one-to-one, then h is one-to-one.

For given $(g_1, g_2), (l_1, l_2) \in G_1 \times G_2$, we have

$$h(g_1, g_2) \circ h(l_1, l_2) = \xi^{\sigma(t_{g_1}, t_{g_2})} \circ \xi^{\sigma(t_{l_1}, t_{l_2})} = \xi^{\sigma(t_{g_1} + t_{l_1}, t_{g_2} + t_{l_2})}$$

$$= (h_1(g_1) \circ h_1(l_1), h_2(g_2) \circ h_2(l_2)) = (h_1(l_1 g_1), h_2(l_2 g_2)) = (\xi_1^{t_{l_1} + t_{g_1}}, \xi_2^{t_{l_2} + t_{g_2}})$$

$$= \xi^{\sigma(t_{l_1} + t_{g_1}, t_{l_2} + t_{g_2})} = h(l_1 g_1, l_2 g_2) = h((l_1, l_2)(g_1, g_2)).$$

Thus $(G_1 \times G_2, \xi)$ is a dynamic group. □

We say that a property is an *algebraic property* if it preserves under algebraic isomorphisms. The next theorem show that the concept of dynamic product is an algebraic concept.

Theorem 2.4. *If (G,ξ) is a dynamic group, and if $f : G \to H$ is a group isomorphism, then (H,ξ) is a dynamic group.*

Proof. Suppose $h : G \to \xi$ has the properties of Definition 1.1. We define $\widetilde{h} : H \to \xi$ by $\widetilde{h}(a) = h(f^{-1}(a))$. Clearly $\widetilde{h}$ is one-to-one. If $a, b \in H$, then

$$\widetilde{h}(ab) = h(f^{-1}(ab)) = h(f^{-1}(a)f^{-1}(b)) = h(f^{-1}(b)) \circ h(f^{-1}(a)) = \widetilde{h}(b) \circ \widetilde{h}(a).$$

We also have $\widetilde{h}(e_H) = h(e_G) = id$. Thus (H,ξ) is a dynamic group. □

3. Dynamic mappings

We begin this section by definition of a Rees matrix semigroup which is defined first in [6]. Suppose that G is a group and Λ and I are two sets. If $p : \Lambda \times I \to G$ is a mapping then $I \times G \times \Lambda$ with the product $(i,a,\lambda)(j,b,\mu) = (i, ap(\lambda,j)b, \mu)$ is a completely simple semigroup [4]. $I \times G \times \Lambda$ with this product is denoted by $M(G,I,\Lambda,p)$ and it is called a *Rees matrix semigroup*. Rees proved in [6] that any completely simple semigroup is isomorphic to a Rees matrix semigroup.

Now we are going to associate a Rees matrix semigroup to a dynamic group.

We assume that (G,ξ) is a dynamic group with the mapping $h : G \to \xi$, and ξ is a dynamical system on Y, then the mapping $p : Y \times Y \to G$ defined by

$$p(y,z) = \begin{cases} h^{-1}(\xi^{t_0}) & if\ A = \{|t| : \xi^t(y) = z\} \neq \emptyset\ and\ t_0 = infA, \\ e & if\ A = \emptyset \end{cases}$$

is a well defined map. In this case the Rees matrix $M(G,Y,Y,p)$ is associated to (G,ξ).

Definition 3.1. If (G,ξ) and (H,η) are two dynamic groups, and ξ and η are dynamical systems on Y and X respectively, then we say that (G,ξ) and (H,η) *are equivalent* if their associated Rees matrices $M(G,Y,Y,p)$ and $M(H,X,X,q)$ are isomorphic semigroups.

Theorem 3.2. *Suppose (G,ξ) and (H,η) are two dynamic groups with the time set T and one-to-one maps $h : G \to \xi$ and $g : H \to \eta$. If there exists a bijection $f : Y \to X$ such that $f \circ \xi^t = \eta^t \circ f$ for all $t \in T$, then (G,ξ) is equivalent to (H,η).*

Proof. We define $w : h(G) \to g(H)$ by $w(\xi^t) = \eta^t$. The condition $f \circ \xi^t = \eta^t \circ f$ implies that w is a bijection. If $l = g^{-1} \circ w \circ h$, then $l : G \to H$ is an isomorphism. Because if $a, b \in G$, then

$$l(ab) = (g^{-1} \circ w)(h(ab)) = (g^{-1} \circ w)(h(b) \circ h(a)) = g^{-1}(w(h(b)) \circ w(h(a)))$$

$$= (g^{-1}(w(h(a))))(g^{-1}(w(h(b)))) = l(a)l(b).$$

Since l is a bijection, by similar method we can show that l^{-1} is a homomorphism. So it is an isomorphism.

Now we show that the mapping $\psi : M(G,Y,Y,p) \to M(H,X,X,q)$ defined by $\psi(y,s,z) = (f(y), l(s), f(z))$ is a semigroup isomorphism. If $(y_1,s_1,z_1), (y_2,s_2,z_2)$ are in $M(G,Y,Y,p)$, then

$$\psi((y_1,s_1,z_1),(y_2,s_2,z_2)) = \psi(y_1, s_1p(z_1,y_2)s_2, z_2)$$

$$= (f(y_1, l(s_1)l(p(z_1),y_2))l(s_2), f(z_2))$$

and

$$(\psi(y_1,s_1,z_1))(\psi(y_2,s_2,z_2)) = (f(y_1), l(s_1), f(z_1))((f(y_2), l(s_2), f(z_2))$$

$$= (f(y_1), l(s_1)q(f(z_1), f(z_2))l(s_2), f(z_2)).$$

So ψ is a homomorphism if we prove that $l(p(z_1,y_2)) = q(f(z_1), f(y_2))$. To prove this we have the following two cases.

Case 1. If $p(z_1,y_2) = e_G$, then $\xi^t(z_1) \neq y_2$ for all $t \in T$. So $f^{-1} \circ \eta^t \circ f(z_1) \neq y_2$ for all $t \in T$. Thus $\eta^t(f(z_1)) \neq f(y_2)$ for all $t \in T$. Hence $q(f(z_1), f(y_2)) = e_H$. Thus $l(p(z_1,y_2)) = l(e_G) = e_H = q(f(z_1), f(y_2))$.

Case 2. If there is $t \in T$ such that $\xi^t(z_1) = y_2$, then $(f^{-1} \circ \eta^t \circ f)(z_1) = y_2$. So $\eta^t(f(z_1)) = f(y_2)$. Thus

$$A = \{|t| : \xi^t(z_1) = y_2\} = \{|t| : \eta^t(f(z_1)) = f(y_2)\}.$$

Hence $p(z_1,y_2) = h^{-1}(\xi^{t_0})$ and $q(f(z_1), f(y_2)) = g^{-1}(\eta^{t_0})$, where $t_0 = infA$. Thus $l(p(z_1,y_2)) = l(h^{-1}(\xi^{t_0})) = g^{-1}(\eta^{t_0}) = q(f(z_1), f(y_2))$. So ψ is a homomorphism.

Since ψ is one-to-one and onto, then by similar method we can show that ψ^{-1} is a homomorphism. Hence it is an isomorphism. □

If a finite set Y and a finite group G are given and if a is the cardinality of the set $\{p : p : Y \times Y \to G \textit{ is a mapping}\}$, then the number of non-equivalent dynamical systems on Y which can make G a dynamic group is at most a.

One must attention to this point that there exist completely simple semigroups which are not associated to any dynamic group. For example, if Y and G have more than two elements, and if $p : Y \times Y \to G$ is the constant mapping $p(y,z) = e$, then there is no any dynamical system on Y such that $M(G,Y,Y,p)$ can be associated to it. Because if there is a ξ and a one-to-one mapping $h : G \to \xi$, then the condition $p(y,z) = e$ implies that ξ can not have more than one element, and it's element is the identity mapping on Y. Since h is one-to-one, then the order of G is 1, and this is a contradiction.

To determine dynamical systems on Y which can prove a group G is a dynamic group is basically related to the number of $M(G,Y,Y,.)$. In fact when we determine $M(G,Y,Y,p)$, then we must check the existence of h.

4. Conclusion

We introduce dynamic groups, and we consider their properties. We show that if (G_1, ξ_1) and (G_2, ξ_2) are two dynamic groups, then there is a dynamical system ξ such that $(G_1 \times G_2, \xi)$ is a dynamic group. In Theorem 2.2 time sets of ξ_1 and ξ_2 can be different groups. Now let us to pose a problem.

Problem. *Suppose that the time groups of ξ_1 and ξ_2 are equal, and it is a group $(T, +)$. Is it possible to find a dynamical system ξ with the time group $(T, +)$ such that $(G_1 \times G_2, \xi)$ be a dynamic group?*

We present an equivalence relation on a set of dynamical systems. The characterization of dynamical systems via this kind of equivalency can be a topic for further research.

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