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On injective and subdirectly irreducible S-posets over left zero posemigroups

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Abstract. The notion of a Cauchy sequence in an S-poset is a useful tool to study algebraic concepts, specially the concept of injectivity. This paper is concerned with the relations between injectivity and Cauchy sequences in the category of S-posets in which S is a left zero posemigroup. We characterize subdirectly irreducible S-posets over this posemigroup and by Birkhof's Representation Theorem we get a description of such S-posets.

1. Introduction and preliminaries

The category of S-posets, as the ordered version of the category of S-acts, recently have captured the interest of some mathematicians [4, 5]. And it is always interesting to verify the counterpart results of S-acts in the category of S-posets (see [1, 4, 8]). Cauchy sequences in an S-act first introduced by E. Giuli in [3] for a particular class of acts, then generalized to S-acts, in [2]. Recently we generalized this concept to S-posets, [4, 5].

Left zero semigroups, all of whose elements are left zero, are an important class of semigroups, since every non-empty set S can be turned into a left zero semigroup by defining $st = s$ for all $s, t \in S$ also this semigroup is applied in automata theory, theory of computations, Boolean algebras.

Here we are going to use the notion of Cauchy sequences to study the dc-regular injectivity of S-posets over a left zero posemigroup, as we did in [7] for injectivity of S-acts. But the order here plays an important role and to get the counterpart results here we need to modify (some times strongly) the S-act version of the proofs. The aim of this paper is to determine the structure of dc-injective in the category of S-posets and characterize the subdiretly irreducible S-posets over a left zero semigroup. Therefore, throughout this article, we assume S to be a left zero posemigroup. Now let us briefly recall some necessary concepts.

A *partially ordered semigroup* (or simply, a *posemigoup*) is a semigroup which is also a poset whose partial order is compatible with its binary operation (that is $s \leqslant s'$ implies $st \leqslant s't$, for every $s, s', t \in S$).

For a posemigoup S, a (*right*) S-*poset* is a poset A equipped with a function $\alpha : A \times S \to A$, called the action of S on A, such that for $a, b \in A$, $s, t \in S$ (denoting $\alpha(a, s)$ by as): (1) $a(st) = (as)t$, (2) $a \leqslant b \Rightarrow as \leqslant bs$, (3) $s \leqslant t \Rightarrow as \leqslant at$.

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By an *S-poset morphism* $f : A \to B$, we mean a monotone map between S-posets which preserves the action (that is $f(as) = f(a)s$).

An element a of an S-poset A is called a *fixed* or *zero element* if $as = a$ for all $s \in S$. We denote the set of all fixed elements of an S-poset A by $FixA$, which is in fact a sub-S-poset of A that is $as \in FixA$ for all $a \in FixA$ and $s \in S$.

We define an S-poset A to be *separated* if it is separated as an S-act, that is any two points $a \neq b$ in A can be separated by at least one $s \in S$, by $sa \neq sb$.

We say that an S-poset A is *subseparated* if $a \leqslant b$ in A whenever $as \leqslant bs$ for all $s \in S$. It is clear that every subseparated S-poset is a separated one.

A *regular monomorphism* or an *embedding* is an S-poset morphism (that is, a monoton and action preserving map) $f : A \to B$ such that $a \leqslant b$ if and only if $f(a) \leqslant f(b)$, for each $a, b \in A$.

2. Cauchy sequences

Our central object of study in this paper is the notion of Cauchy sequences in S-posets [2, 3, 4].

First of all it is easy to check that:

- *If S is a left zero semigroup, then for every S-poset A, $AS \subseteq FixA$.*

Definition 2.1. A *Cauchy sequence* in an S-poset A is an S-poset morphism $f : S \to A$. More explicitly, $f : S \to A$ is a Cauchy sequence when it is order preserving and $f(st) = f(s)t$.

We denote a Cauchy sequence by $(a_s)_{s\in S}$, which expresses the fact that the element $s \in S$ is mapped to the element a_s in A. Since S is a left zero posemigroup, with this notation we have $a_s t = a_{st} = a_s$ and for $s, t \in S$ if $s \leqslant t$ then $a_s \leqslant a_t$.

It is worth noting that in an S-poset A (over the left zero posemigroup S) the terms of a Cauchy sequence are fixed elements of A. So if we denote the set of Cauchy sequences of A by $\mathcal{C}(A)$ then $\mathcal{C}(A) = (FixA)^S$ in which $(FixA)^S$ is the set of monotone mappings from S to $FixA$.

Definition 2.2. Let $(a_s)_{s\in S}$ be a Cauchy sequence in an S-poset A. An element b in an extension B of A is called a *limit* of $(a_s)_{s\in S}$ whenever $bs = a_s$ for each $s \in S$.

Lemma 2.3. *Given an S-poset A over a left zero posemigroup S, the set $\mathcal{C}(A)$ of all Cauchy sequences in A, is a subseparated S-poset.*

Proof. First we note that $\mathcal{C}(A)$ is an S-poset, by the action $\mathcal{C}(A) \times S \to \mathcal{C}(A)$ mapping each $((a_s)_{s\in S}, t) \in \mathcal{C}(A) \times S$ to $(a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S}$ which is obviously in $\mathcal{C}(A)$, for every $t \in S$. We should note that $\mathcal{C}(A)$ is a poset with point-wise order and $((a_s)_{s\in S} \cdot t) \cdot r = (a_s)_{s\in S} \cdot (tr)$. Indeed, $(a_s)_{s\in S} \cdot (tr) = (a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S} = (a_t)_{s\in S}$, namely $(a_s)_{s\in S} \cdot (tr)$ is the constant sequence $(a_t)_{s\in S}$, also we have $((a_s)_{s\in S} \cdot t) \cdot r = (a_{ts})_{s\in S} \cdot r = (a_t)_{s\in S} \cdot r = (a_t)_{s\in S}$; the last equality is true because $(a_t)_{s\in S}$ is a constant sequence. Now if $r \leqslant t$ in S, then $rs = r \leqslant t = ts$, for every $s \in S$ and since $(a_s)_{s\in S}$ is a Cauchy sequence, $a_{rs} = a_r \leqslant a_t = a_{ts}$. That is $(a_s)_{s\in S}\cdot r \leqslant (a_s)_{s\in S}\cdot t$. Finally if $(a_s)_{s\in S} \leqslant (b_s)_{s\in S}$, then $a_s \leqslant b_s$, for every $s \in S$. Hence $a_{ts} \leqslant b_{ts}$ for every $s, t \in S$. That is $(a_s)_{s\in S}\cdot t \leqslant (b_s)_{s\in S}\cdot t$, for every $t \in S$. To prove subseparatedness, let $(a_s)_{s\in S}\cdot t \leqslant (b_s)_{s\in S}\cdot t$, for every $t \in S$. Then $a_{ts} \leqslant b_{ts}$, for every $t, s \in S$. Now, since S is a left zero posemigroup, $a_t \leqslant b_t$, for every $t \in S$. That is $(a_s)_{s\in S} \leqslant (b_s)_{s\in S}$. □

Lemma 2.4. *Let A be an S-poset over a left zero posemigroup S and $(a_s)_{s\in S}$ be a sequence (indexed family of elements of A by $s \in S$). Then $(a_s)_{s\in S}$ has a limit in some extension B of A if and only if it is a Cauchy sequence.*

Proof. One way is clear. In fact the limit of the sequence $(a_s)_{s\in S}$ makes it to have the Cauchy property in Definition 2.1. For the converse, let $(a_s)_{s\in S}$ be a Cauchy sequence in A. Then take the extension B of A to be $A\dot{\cup}\{(a_s)_{s\in S}\}$ with the action $(a_s)_{s\in S}\cdot t = a_t$ for $t \in S$ and no order between $(a_s)_{s\in S}$ and the elements of A. The constructed B is an S-poset. This is because, for all $t, r \in S$, we have $((a_s)_{s\in S}\cdot t)\cdot r = a_t \cdot r = a_{tr} = (a_s)_{s\in S}\cdot(tr)$, and if $t \leqslant r$ then $a_t \leqslant a_r$ follows from this fact that $(a_s)_{s\in S}$ is a Cauchy sequence, and hence $(a_s)_{s\in S}\cdot t \leqslant (a_s)_{s\in S}\cdot r$. Now, by the defined action, we have that $(a_s)_{s\in S}$ is a limit of $(a_s)_{s\in S}$. □

Definition 2.5. An S-poset A is said to be *complete* if every Cauchy sequence over A has a limit in A.

For a given left zero posemigroup S and an S-poset A Lemma 2.3 shows that $\mathcal{C}(A)$ is an S-poset. In fact, $\mathcal{C}(A)$ is a complete S-poset.

Theorem 2.6. *Let A be an S-poset over a left zero posemigroup S. The S-poset $\mathcal{C}(A)$ is complete.*

Proof. Let $(f_s)_{s\in S}$ be a Cauchy sequence in $\mathcal{C}(A)$, in which $f_s = (a^s_r)_{r\in S}$ for each $s \in S$. Hence for each $s, t \in S$ we have $f_{st} = f_s t$. Since S is a left zero semigroup, $f_s = f_{st} = f_s t$, i.e., for each $s \in S$, f_s is a fixed element in $\mathcal{C}(A)$. Now, by the defined action of S over $\mathcal{C}(A)$ in Lemma 2.3, we have $f_s = f_s t = (a^s_{tr})_{r\in S} = (a^s_t)_{r\in S}$. So $(a^s_r)_{r\in S} = (a^s_t)_{r\in S}$ for each $r \in S$. Namely, for each $s \in S$, f_s is a constant sequence. Now we define the Cauchy sequence $(a_s)_{s\in S}$ to be $a_s = a^s_t$, for every $s \in S$ and claim that $(a_s)_{s\in S}$ is a limit of $(f_s)_{s\in S}$. This is because $(a_s)_{s\in S}\cdot r$ $= (a^s_t)_{s\in S}\cdot r = (a^s_{rt})_{s\in S} = (a^s_r)_{s\in S} = (a^s_t)_{r\in S} = f_s$. Indeed, the third equation follows from this fact that S is a left zero posemigroup. Also since f_s is a constant sequence and $(a^s_r)_{r\in S} = (a^s_t)_{r\in S}$, we have the fourth and fifth equations. □

3. dc-injective of S-posets

A sub-S-poset A of an S-poset B is called *down-closed* in B if $b \leqslant a$ for $a \in A$, $b \in B$ then $b \in A$. By a *down-closed embedding* or *dc-regular monomorphism*, we mean an embedding $f : A \to B$ such that $f(A)$ is a down-closed sub-S-poset of B.

An S-poset A is said to be *down-closed injective* or simply *dc-injective* if for every down-closed embedding $f : B \to C$ and each S-poset morphism $\varphi : B \to A$ there exists an S-poset morphism $\varphi^* : C \to A$ making the diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\varphi"'] & C \arrow[dl, dotted, "\varphi^*"] \\ A & \end{tikzcd}$$

commutative.

Theorem 3.1. *For a left zero posemigroup S every dc-injective S-poset is complete.*

Proof. Let $(a_s)_{s\in S}$ be a Cauchy sequence in a dc-injective S-poset A. Consider the extension $B = A\dot{\cup}\{(a_s)_{s\in S}\}$ of A with the action $(a_s)_{s\in S} \cdot t = a_t$ and no order relation between $(a_s)_{s\in S}$ and the elements of A as introduced in the proof of Lemma 2.4. It is clear that A is embedded in B, so the dc-injective property of A completes the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, equal] & B \arrow[dl, dotted, "\varphi"] \\ A & \end{tikzcd}$$

by an S-poset morphism φ. Now we claim that $\varphi((a_s)_{s\in S}) \in A$ is a limit of the Cauchy sequence $(a_s)_{s\in S}$. This is because $\varphi((a_s)_{s\in S})\cdot t = \varphi((a_s)_{s\in S}\cdot t) = \varphi(a_t) = a_t$, for every $t \in S$. $\square$

The converse of Theorem 3.1 is true if the S-poset has a "*good*" property. See the next theorem as the counterpart of Theorem 2.3 of [7] with the compeletly different method of proof.

Theorem 3.2. *If S is a left zero posemigroup S, then every complete subseparated S-poset A with a top fixed element is dc-injective.*

Proof. To prove, we show that A is a retract of each of its down-closed extensions (that is, to say A is an absolute down-closed retract) (see [8]). To do so, let B be a down-closed extension of A. Define $g : B \to A$ with $g|_A = id_A$ and for $b \in B \setminus A$ take $g(b) = a_b$ where a_b is a limit of the Cauchy sequence $(a_s)_{s\in S}$ with $a_s = bs$ for $bs \in A$, and $a_s = a_0$ for $bs \notin A$, where $a_0 \in FixA$ is the top fixed element in A mentioned in the hypotheses.

First we show that $(a_s)_{s\in S}$ is a Cauchy sequence. To do so, we note that $a_s t = a_{st}$. This is because, if $a_s = bs$, then $a_s t = (bs)t = b(st) = bs$ also $a_{st} = a_s = bs$, and if $a_s = a_0$, then $a_s t = a_0 t = a_0$ also $a_{st} = a_s = a_0$. Also if $s \leqslant t$, then $bs \leqslant bt$. This is because if $bt \in A$, then $bs \in A$, since A is down-closed in B, therefore $a_s \leqslant a_t$, and if $bt \notin A$, then $a_t = a_0$ but a_0 is a top fixed element and hence $bs \leqslant a_0$, that is $a_s \leqslant a_t$. Thus $(a_s)_{s\in S}$ is a Cauchy sequence.

Now we show that g is order preserving. To do so, let $b \leqslant b'$. Then $bs \leqslant b's$ for all $s \in S$. Therefore, by definition of $a_b, a_{b'}$, we have $a_b s \leqslant a_{b'} s$. But, since A is subseparated, $a_b \leqslant a_{b'}$. That is $g(b) \leqslant g(b')$. Finally g is equivariant on $B \setminus A$. Because $g(b)s = a_b s = a_s = bs = g(bs)$, if $bs \in A$, for every $b \in B \setminus A$ and $s \in S$. And if $bs \notin A$, then, since $(bs)t = bs$ for all $t \in S$, we get $g(bs) = a_{bs} = a_0 = a_0 s = a_b s = g(b)s$. □

As a corollary of Theorems 3.1 and 3.2 we get the following Theorem.

Theorem 3.3. *Let S be a left zero posemigroup S. Then a subseparated S-poset A with a top fixed element is dc-injective if and only if it is complete.*

Theorem 3.4. *For each S-poset A over a left zero posemigroup S with a top fixed element, $\mathcal{C}(A)$ is dc-injective.*

Proof. Let a_0 be a top fixed element in A. One can easily see that the constant sequence $(a_s = a_0)_{s \in S}$ is a Cauchy sequence and is a top fixed element in $\mathcal{C}(A)$. Now Theorems 3.3 and 2.6 give the result. □

Before the next definition it is worth noting that by a right down-closed ideal I of a posemigroup S we mean a non-empty subset I of S such that (i) $IS \subset I$ and (ii) $a \leqslant b \in I$ implies $a \in I$, for all $a, b \in S$.

Definition 3.5. An S-poset A is said to be

- *I-injective*, for a right down-closed ideal I of S, if each S-poset morphism $f : I \to A$ is of the form λ_a for some $a \in A$, that is $f(s) = as$ for $s \in I$.
- *S-injective*, if each S-poset morphism $f : S \to A$ is of the form λ_a for some $a \in A$, that is $f(s) = as$ for $s \in S$.

In the next theorem we compare the concept of completeness with the different types of injectivity for some special S-poset over a left zero posemigroup S, and we see that they are surprisingly equivalent to each other.

Theorem 3.6. *For a subseparated S-poset A with a top fixed element a_0, the following are equivalent:*

(1) *A is dc-injective;*
(2) *A is dc-absolutely retract;*
(3) *A is complete;*
(4) *A is I-injective, for each right down-closed ideal I of S;*
(5) *A is S-injective.*

Proof. (1)⇔(2). It is given in [8].

(1)⇔(3). See Theorem 3.3.

(3)⇒(4). Let A be complete and I be a right down-closed ideal of S and $f : I \to A$ be an S-poset morphism. Consider the sequence $(a_s)_{s \in S}$ to be $a_s = f(s)$ for $s \in I$, and $a_s = a_0$ for $s \in S - I$. The sequence $(a_s)_{s \in S}$ is a Cauchy sequence. This is because, if $s \leqslant t$ then four cases may occur:

∘ If $s, t \in I$ then $f(s) \leqslant f(t)$, since f is S-poset morphism, that is $a_s \leqslant a_t$.

∘ If $s, t \in S - I$, then $a_s = a_t = a_0$, that is $a_s \leqslant a_t$.

∘ It may $s \in S - I$, $t \in I$. But since $s \leqslant t$ and I is down-closed ideal, we must have $s \in I$ which is a contradiction. Hence this case is not possible.

∘ And finally if $s \in I$ and $t \in S - I$, then $f(s) = a_s$, $f(t) = a_0$. But a_0 is the top fixed element, hence $f(s) = a_s \leqslant a_0 = f(t)$.

Also let $s, t \in S$. Then if $s \in I$, we have $a_s t = f(s)t = f(st) = f(s) = a_s$ and if $s \in S - I$, then $f(s)t = a_0 t = a_0 = f(s)$.

Now since $(a_s)_{s \in S}$ is a Cauchy sequence, it has a limit a in A. So $a_s = as$, for all $s \in S$, which means $f(s) = a_s = \lambda_a(s)$. That is $f = \lambda_a$.

(4)$\Rightarrow$ (5). It is trivial.

(5)$\Rightarrow$ (3). Let A be S-injective and $(a_s)_{s \in S}$ be a Cauchy sequence over A. So $f : S \to A$ with $f(s) = a_s$ is an S-poset morphism. Now (5) gives $a \in A$ such that $f = \lambda_a$, hence $a_s = as$ for all $s \in S$, i.e., a is a limit of the given sequence. □

4. Subdirectly irreducible

By Birkhoff's Representation Theorem (see [6]) every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. This theorem, by analogous proof is established in the category of S-posets. In [7], characterization of subdirectly irreducible acts, respectively over the monoid $(\mathbb{N} \cup \{\infty\}, \min, \infty)$, and left zero semigroups can be seen. In this section we give a characterization of subdirectly irreducible S-posets over a left zero posemigroup.

Definition 4.1. (see [6]) An equivalence relation ρ on an S-act A is called a *congruence* on A, if $a\rho a'$ implies $(as)\rho(a's)$, for all $s \in S$. We denote the set of all congruences on A by $Con(A)$.

A congruence on an S-poset A is a congruence θ on the S-act A with the property that the S-act A/θ can be made into an S-poset in such a way that the natural map $A \to A/\theta$ is an S-poset map (see [1]).

For any relation θ on A, define the relation $\leqslant_\theta$ on A by

$$a \leqslant_\theta a' \quad \text{if and only if} \quad a \leqslant a_1 \theta a'_1 \leqslant a_2 \theta a'_2 \leqslant \ldots \leqslant a_n \theta a'_n \leqslant a',$$

where $a_i, a'_i \in A$ (such a sequence of elements is called a *θ-chain*). Then an S-act congruence θ on an S-poset A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leqslant_\theta a' \leqslant_\theta a$.

For $a, b \in A$, $\rho_{a,b}$ denotes the *smallest S- act congruence* on A containing (a, b). It is in fact, the equivalence relation generated by $\{(as, bs) : s \in S \cup \{1\}\}$. Its elements are as follows:

$$x\rho_{a,b}y \ \Leftrightarrow \ \exists s_1, s_2, ..., s_n \in S \cup \{1\} \ , p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in A,$$

$$\begin{array}{ccccccc} x = p_1 s_1 & & q_2 s_2 = p_3 s_3 & & \ldots & & q_n s_n = y \\ & q_1 s_1 = p_2 s_2 & & q_3 s_3 = p_4 s_4 & & \ldots & \end{array}$$

where $(p_i, q_i) = (as, bs)$ or $(p_i, q_i) = (bs, as)$ for some $s \in S \cup \{1\}$.

Lemma 4.2. *Let A be an S-act over a posemigroup S. Then $\rho_{x,y}$, for every distinct $x, y \in$ FixA, is an S-poset congruence.*

Proof. To prove we show the equivalence condition of an S-poset congruence. Namely, we show that if $a \leqslant_{\rho_{x,y}} a' \leqslant_{\rho_{x,y}} a$ then $a\rho_{x,y}a'$. But first we note that $\rho_{x,y} = \Delta \bigcup\{(x,y),(y,x)\}$ since $x, y \in FixA$. Now if $a \leqslant_{\rho_{x,y}} a' \leqslant_{\rho_{x,y}} a$ then two:

1) $a \leqslant x\rho_{x,y}y \leqslant y\rho_{x,y}x \leqslant a' \leqslant x\rho_{x,y}y \leqslant y\rho_{x,y}x \leqslant a$. Therefore $a \leqslant x \leqslant a' \leqslant x \leqslant a$ and hence $a = a'$ thus $a\rho_{x,y}a'$; or

2) $a \leqslant x\rho_{x,y}y \leqslant y\rho_{x,y}x \leqslant a' \leqslant y\rho_{x,y}x \leqslant x\rho_{x,y}y \leqslant a$. Therefore $a \leqslant x \leqslant a' \leqslant y \leqslant a$ and hence $x = y$ which is a contradiction. Hence this case is not possible. Thus we have $a = a'$, that is $a\rho_{x,y}a'$. □

Definition 4.3. *(see [6])*An S-poset A is called *subdirectly irreducible* if $\bigcap_{i\in I}\rho_i \neq \Delta$ for all congruences ρ_i on A with $\rho_i \neq \Delta$. If A is not subdirectly irreducible, then it is called *subdirectly reducible.*

It is worth noting that for each posemigroup S and an S-poset A with $|A| = 2$ there exist only two congruences Δ and $\bigtriangledown$ on A and therefore these S-posets are subdirectly irreducible.

Lemma 4.4. *Every S-poset A over a left zero posemigroup S with $|FixA| = 1$ or $|FixA| \geqslant 3$ is subdirectly reducible.*

Proof. It is clear that for a left zero semigroup S, every S-poset with only one fixed element is subirectly reducible. Also, let A be an S-poset with at least three distinct fixed elements a, b, c. Then we consider the S-poset congruences $\rho_{a,b}$ and $\rho_{b,c}$, by Lemma 4.2. Since $a, b, c \in \mathrm{Fix}A$ we obviously have $\rho_{a,b} = \Delta \bigcup\{(a,b),(b,a)\}$ and $\rho_{b,c} = \Delta \bigcup\{(b,c),(c,b)\}$. Therefore $\rho_{a,b} \cap \rho_{a,c} = \triangle$, and we are done. □

We give the following theorm as the counterpart of Theorem 3.2 of [7] in the category of S-posets over a left zero posemigroup.

Theorem 4.5. *An S-poset A over a left zero posemigroup S is subdirectly irreducible if and only if it is separated and $|FixA| = 2$.*

Proof. Let A be subdirectly irreducible. Then Lemma 4.4 ensures that $|FixA| = 2$ such as $\{a_0, b_0\}$. To show that A is separated, we suppose that there exists $x \neq y \in A$ such that $xs = ys$, for all $s \in S$, and find a contradiction. To do so, consider the S-act congruence $\rho_{x,y}$. Since $xs = ys$, for all $s \in S$, $\rho_{x,y} = \Delta \bigcup\{(x,y),(y,x)\}$. By the analogous method of the proof of Lemma 4.2 one can see that $\rho_{x,y}$ is an S-poset congruence on A. Also since $a_0, b_0 \in FixA$, by Lemma 4.2, we have the S-posset congruence ρ_{a_0,b_0} on A. But $\rho_{a_0,b_0} \cap \rho_{x,y} = \Delta$ which is a contradiction, therefore A is separated.

For the converse, let A be separated, $FixA = \{a_0, b_0\}$, and $\theta(\neq \Delta)$ be an S-poset congruence on A. Then there exists $x \neq y \in A$ such that $(x,y) \in \theta$. Thus $(xs, ys) \in \theta$ for every $s \in S$. But since $xs, ys \in \mathrm{Fix}A = \{a_0, b_0\}$ and A is separated, there exists $s \in S$ such that $xs \neq ys$. This means $(a_0, b_0), (b_0, a_0) \in \theta$. Therefore $\bigcap_{\theta \neq \Delta} \theta$ contains $\Delta \cup \{(a_0, b_0), (b_0, a_0)\}$, hence A is subdirectly irreducible. $\square$

Finally, by the above theorem, and Birkhoff's Representation Theorem we have:

Theorem 4.6. *Every S-poset over a left zero posemigroup S is isomorphic to a subdirect product of separated S-posets each of which has exactly two fixed elements.*

It is worth noting that every S-poset A over a left zero posemigroup S with one or two elements and $|FixA| = 1$ is dc-injective.

We close the paper by characterizing simple S-poset. Recall that an S-poset A is called *simple* if $ConA = \{\triangle, \nabla\}$. It is easy to check that every S-poset A with $|A| \leqslant 2$ is simple but no S-poset A with trivial action and $|A| > 2$ is simple.

Theorem 4.7. *For a left zero posemigroup S, there exists no simple S-poset A with $|A| > 2$.*

Proof. Let $a \neq b$ be elements of A. Then in the case where $a, b \in \mathrm{Fix}A$ we have $\rho_{a,b} \neq \nabla$, (where $\rho_{a,b}$ is an S-poset congruence that discussed in Lemma 4.2) since $|A| > 2$, hence there exists $(a, b \neq) x \in A$ and $(a, x) \notin \rho_{a,b}$. Therefore, $\rho_{a,b}$ is a nontrivial congruence on A. Also in the case that one of a, b is not fixed, taking $a \notin FixA$, then $\rho_{a,b} \neq \nabla$. Because otherwise, if $\rho_{a,b} = \nabla$ then for each $x \neq y \in A$, we have $(x, y) \in \rho_{a,b}$. Consequently there exist $s, t \in S$ such that $as = x,\ bt = y$. Hence $x, y \in FixA$. Thus $(a, x) \notin \rho_{x,y}$, and so $\rho_{x,y}$ is a nontrivial congruence. $\square$

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References

[1] **S. Bulman-Fleming and M. Mahmoudi**, *The category of S-posets*, Semigroup Forum **71** (2005), 443 – 461.

[2] **M.M. Ebrahimi and M. Mahmoudi**, *Baer criterion for injectivity of projection algebras*, Semigroup Forum **71** (2005), 332 – 335.

[3] **E. Giuli**, *On m-separated projection spaces*, Appl. Categ. Struct. **2** (1994), 91 – 99.

[4] **M. Haddadi**, *Nets and separated S-posets*, J. Algebraic Systems **1** (2013), 33 – 43.

[5] **M. Haddadi and Gh. Moghaddasi**, *Regualr sub-sequentially dense injectivity in the category of S-posets*, Italian J. Pure Appl. Math. **33** (2014), 149 – 160.

[6] **M. Kilp, U. Knauer and A. Mikhalev**, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York (2000).

[7] **Gh. Moghaddasi**, *On injective and subdirectly irreducible S-acts over left zero semigroups*, Turkish J. Math. **36** (2012), 359 – 365.

[8] **L. Shahbaz and M. Mahmoudi**, *Injectivity of S-posets with respect to down-closed regular monomorphism*, Semigroup Forum, DOI 10.1007/s00233-014-9676-y.

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