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Free semiabelian n-ary groups

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Abstract. Free n-ary groups in the class of semiabelian n-ary groups are described.

1. Introduction

The non-empty set G together with an n-ary operation $f : G^n \to G$ is called an *n-ary groupoid* (or an *n-ary operative* – in the Gluskin terminology, cf. [10]) and is denoted by $\langle G, f\rangle$. We will assume that $n \geqslant 2$.

According to the general convention used in the theory of such groupoids we will use the following abbreviated notation:

$$f(x_1, \dots, x_i, \underbrace{x, \dots, x}_{t}, x_{i+t+1}, \dots, x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n),$$

$$f_{(k)}(x_1^{k(n-1)+1}) = \underbrace{f(f(\dots, f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), \dots), x_{(k-1)(n-1)+2}^{k(n-1)+1}),$$

where $\overset{(0)}{x}$ and x_i^j for $i > j$ are empty symbols. In certain situations, when the arity of the operation $f_{(k)}$ does not play a crucial role or when it will differ depending on additional assumptions, we will write $f_{(.)}$ instead of $f_{(k)}$.

The algebra $\langle G, f\rangle$ is called an *n-ary group* if it satisfies the *generalized associative law*:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \tag{1}$$

and for all $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n, b \in G$ the equation

$$f(a_1, \dots, a_{j-1}x_j, a_{j+1}, \dots, a_n) = b$$

is uniquely solvable for each $j = 1, \dots, n$. Other equivalent definitions of n-ary groups one can find in [4] and [5].

For $n = 2$ we obtain usual (binary) groups. Thus n-ary groups are a generalization of groups.

Initial investigations of n-ary groups were presented in [2], [15] and [20]. The necessity for such research is explained in the Kurosh's book [13].

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The theory of n-ary groups differs from the theory of ordinary groups. This is stipulated, for example, by absence of neutral elements. Therefore the invertibility is also absent. Instead of this in n-ary groups is considered the *skew element* defined as a solution of the equation $f(a,\ldots,a,x)=a$. It is denoted by $\bar{a}$ and is called the *skew element* for a. Since for each $a\in G$ it is uniquely defined we have the map $^{-}\colon x\to\bar{x}$. Thus any n-ary group $\langle G,f\rangle$ may be considered as an algebra $\langle G,f,^{-}\rangle$ in which the generalized associative law (1) and the identities

$$f(y,x,\ldots,x,\bar{x},x)=f(x,\bar{x},x,\ldots,x,y)=y. \tag{2}$$

are fulfilled (for details see [3], [4] and [5]).

An n-ary group $\langle G,f\rangle$ is called *semiabelian* if

$$f(x_1,x_2\ldots,x_{n-1},x_n)=f(x_n,x_2\ldots,x_{n-1},x_1)$$

and *abelian* or *commutative* if $f(x_1,\ldots,x_n)=f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ is valid for all $x_1,\ldots,x_n\in G$ and all $\sigma\in S_n$. An n-ary group is semiabelian $\langle G,f\rangle$ if and only if $f(x,a,\ldots,a,y)=f(y,a,\ldots,a,x)$ for some $a\in G$ and all $x,y\in G$ (cf. [3]).

Note that in semiabelian n-ary groups the map $^{-}\colon x\to\bar{x}$ is an endomorphism (cf. [4]), but it is an endomorphism also in some n-ary groups which are not semiabelian (cf. [8]).

The class of all n-ary groups considered as algebras of the form $\langle G,f,^{-}\rangle$ forms a variety determined by (1) and (2). Free n-ary groups in this class are described in [1]. Free n-ary groups in the class of all abelian n-ary groups were investigated in [19], [17] and [18]. The description of structure of free n-ary groups in class of abelian semicyclic (precyclic) n-ary groups one can find in [14].

In this paper we describe the structure of free n-ary groups in class of semiabelian n-ary groups.

2. Some facts on semiabelian n-ary groups

There is a close relationship between binary (i.e., classical) and n-ary groups. For example, on any semiabelian n-ary group $\langle G,f\rangle$ the abelian group $\langle G,+\rangle$ may be defined by putting $a+b=f(a,c,\ldots,c,\bar{c},b)$ for fixed element c from G. Then (cf. [10], [11]) for the element $d=f(c,\ldots,c)$ and for the map $\varphi(x)=f(c,x,c,\ldots,c,\bar{c})$, which is an automorphism of the group $\langle G,+\rangle$, we obtain

$$\varphi(d)=d,\qquad \varphi^{n-1}(x)=x\ \text{ for any } x\in G,\ \text{ and} \tag{3}$$

$$f(a_1,\ldots,a_n)=a_1+\varphi(a_2)+\ldots+\varphi^{n-2}(a_{n-1})+a_n+d. \tag{4}$$

It is easily to see that c is a zero of the group $\langle G,+\rangle$, and $-a=f(c,a,\ldots,a,\bar{a},c)$. Moreover,

$$\varphi^s(x)=f(\overset{(s)}{c},x,\overset{(n-2-s)}{c},\bar{c}). \tag{5}$$

Since $f(x,\dots,x,\bar{x}) = x$, from (4) we get $x+\varphi(x)+\ldots+\varphi^{n-2}(x)+\bar{x}+d = x$. Thus, $\bar{x} = -\varphi(x)-\ldots-\varphi^{n-2}(x)-d$, and consequently,

$$\begin{aligned}\varphi^s(\bar{x}) &= \varphi^s(-\varphi(x)-\ldots-\varphi^{n-2}(x)-d) = -\varphi^s(\varphi(x))-\ldots-\varphi^s(\varphi^{n-2}(x))-\varphi^s(d)\\ &= -\varphi^{s+1}(x)-\ldots-\varphi^{n-2}(x)-x-\varphi(x)-\ldots-\varphi^{s-1}(x)-d\\ &= -\varphi^{s+1}(x)-\ldots-\varphi^{n-2}(x)-\varphi(x)-\ldots-\varphi^{s-1}(x)-\varphi^s(x)-d-x+\varphi^s(x)\\ &= \bar{x}-x+\varphi^s(x).\end{aligned}$$

Hence

$$\varphi^s(\bar{x}) = \bar{x}-x+\varphi^s(x). \tag{6}$$

The group $\langle G,+\rangle$ is called the *retract* of an n-ary group $\langle G,f\rangle$ and is denoted by $ret_c\langle G,f\rangle$. Two retracts of the same n-ary group are isomorphic (cf. [6]). For an abelian n-ary group $\langle G,f\rangle$ the automorphism φ is the identity map.

The converse is also true: if $\langle G,+\rangle$ is an abelian group, φ its automorphism such that for some $d\in G$ the conditions (3) are satisfied, then $\langle G,f\rangle$ with the operation defined by (4) is a semiabelian n-ary group. Such obtained an n-ary group $\langle G,f\rangle$ is called (φ,d)-*derived* from the group $\langle G,+\rangle$ and is denoted by $der_{\varphi,d}\langle G,+\rangle$. In the case $\varphi = 1_G$, $d=0$ we say that an n-ary group $der_{\varphi,d}\langle G,+\rangle$ is *derived* from the group $\langle G,+\rangle$.

One can prove (cf. [6]) that

$$\langle G,+\rangle = ret_c der_{\varphi,d}\langle G,+\rangle, \qquad \langle G,f\rangle = der_{\varphi,d} ret_c\langle G,f\rangle. \tag{7}$$

An n-ary group with a cyclic retract is called *semicyclic* [16] or *precyclic* [8]. A semicyclic n-ary group $\langle (a),f\rangle$ which is (φ,d)-derived from the cyclic group $\langle (a),+\rangle$ of order k has the form $der_{m,la}\langle (a),+\rangle$, i.e.,

$$f(s_1a,\dots,s_na) = (s_1+s_2m+s_3m^2\ldots+s_{n-1}m^{n-2}+s_n+l)a,$$

where $0\leqslant m,l<k$, m and k are relatively prime, $lm\equiv l \pmod{k}$ and $m|n-1$. Any finite semicyclic n-ary group of order k is isomorphic to an n-ary group $\langle (a),f\rangle = der_{m,la}\langle (a),+\rangle$, where $l|\gcd(1+m+m^2+\ldots+m^{n-2},k)$ (see [16]).

Using the basic idea of [7], the structure of homomorphisms of n-ary groups was investigated in [12]. We need the special case of Theorem 1.2 from [12].

Theorem 2.1. *Let $\langle G,f\rangle = der_{\varphi,d}\langle G,+\rangle$ and $\langle H,h\rangle = der_{\mu,b}\langle H,\oplus\rangle$ be two semiabelian n-ary groups and $\psi:\langle G,f\rangle\to\langle H,h\rangle$ be a homomorphism. Then there exists $a\in H$ and a group homomorphism $\sigma:\langle G,+\rangle\to\langle H,\oplus\rangle$ such that $\psi(x) = \sigma(x)\oplus a$ for any $x\in G$. In this case*

$$h(a,\dots,a) = \sigma(d)\oplus a \quad \textit{and} \quad \sigma\circ\varphi = \mu\circ\sigma.$$

Moreover, if a and σ satisfy these two conditions, then $\psi(x) = \sigma(x)\oplus a$ is a homomorphism $\langle G,f\rangle\to\langle H,h\rangle$.

3. Generating sets of semiabelian n-ary groups

It is not difficult to see that

$$\overline{f(x_1^n)} = f_{(\cdot)}(\underbrace{\overset{(n-3)}{x_n}, \bar{x}_n, \ldots, \overset{(n-3)}{x_1}, \bar{x}_1}_{n-2}), \quad \bar{\bar{x}} = f_{(n-3)}(x, \ldots, x).$$

Theorem 3.1. *If a semiabelian n-ary group $\langle G, f\rangle$ is generated by the set $X = \{x_\alpha | \alpha \in I\}$, then its retract $ret_{x_\beta}\langle G, f\rangle$ is generated by the set*

$$Y = \left\{ f(\overset{(i-1)}{x_\beta}, x_\alpha, \overset{(n-i-1)}{x_\beta}, \bar{x}_\beta) \,|\, x_\alpha \in X \backslash \{x_\beta\},\ i = 1, \ldots, n-1 \right\} \cup \left\{ f(\overset{(n)}{x_\beta}) \right\}.$$

Proof. Let $\langle G, +\rangle = ret_{x_\beta}\langle G, f\rangle$ for some $x_\beta \in X$. Then x_β is a zero of $\langle G, +\rangle$ and (3) is valid for $d = f(\overset{(n)}{x_\beta})$ and $\varphi(x) = f(x_\beta, x, \overset{(n-3)}{x_\beta}, \bar{x}_\beta)$.

Denote by B the subset of $\langle G, +\rangle$ generated (in $\langle G, +\rangle$) by Y. Obviously $d \in Y$. So, also $-d, x_\beta \in B$. We will show that $B = G$.

First observe that $\varphi^s(x_\alpha) \in Y \subseteq B$ for every $s = 1, \ldots, n-1$ and $x_\alpha \in X \backslash \{x_\beta\}$. For x_β we obtain $\varphi^s(x_\beta) = x_\beta \in B$. Since $\bar{x}_\alpha = -\varphi(x_\alpha) - \ldots - \varphi^{n-2}(x_\alpha) - d$, by (2) and (4), we see that $\varphi^s(\bar{x}_\alpha) \in B$ for every $x_\alpha \in X$.

Now consider an arbitrary element $g \in G$. By our assumption each element $g \in G$ has the form $g = f_{(k)}(y_{\alpha_1}, \ldots, y_{\alpha_m})$, where $m = k(n-1)+1$, $y_{\alpha_i} = x_{\alpha_i}$ or $y_{\alpha_i} = \bar{x}_{\alpha_i}$, $x_{\alpha_i} \in X$, $i = 1, \ldots, m$. This, according to (4), means that each element g of G can be written in the form

$$g = y_{\alpha_1} + h_0 + \varphi(h_1) + \varphi^2(h_2) + \ldots + \varphi^{n-2}(h_{n-2}) + h_{n-1} + d,$$

where

$$h_j = y_{\alpha_{j(n-1)+2}} + \varphi(y_{\alpha_{j(n-1)+3}}) + \ldots + \varphi^{n-2}(y_{\alpha_{j(n-1)+n-1}}) + y_{\alpha_{j(n-1)+n}} + d$$

and $j = 0, 1, \ldots, k-1$. Since x_β is a zero of $\langle G, +\rangle$, each y_j, and consequently, g depends only on $x_\alpha, \bar{x}_\alpha$ and $\bar{x}_\beta$. But $\bar{x}_\beta = -d$ and $d \in Y$. Thus g depends only on d and $x_\alpha \in X \backslash \{x_\beta\}$. Therefore Y generates G, so $B = G$. □

Corollary 3.2. *If a semiabelian n-ary group $\langle G, f\rangle$ is generated by the set $X = \{x_\alpha | \alpha \in I\}$, then the retract $ret_c\langle G, f\rangle$ is generated by the set*

$$\left\{ f(f(\overset{(i-1)}{x_\beta}, x_\alpha, \overset{(n-i-1)}{x_\beta}, \bar{x}_\beta), \overset{(n-3)}{c}, \bar{c}, x_\beta) \,|\, x_\alpha \in X \backslash \{x_\beta\}, 1 \leqslant i \leqslant n-1 \right\} \cup \left\{ f(f(\overset{(n)}{x_\beta}), \overset{(n-3)}{c}, \bar{c}, x_\beta) \right\},$$

where x_β is an arbitrary fixed element of X.

Proof. It is not difficult to see that the map $\sigma : ret_{x_\beta}\langle G,f\rangle \to ret_c\langle G,f\rangle$ defined by $\sigma(x) = f(x, \overset{(n-3)}{c}, \bar{c}, x_\beta)$ is an isomorphism of retracts which transfers generators of $red_{x_\beta}\langle G,f\rangle$ onto generators of $red_c\langle G,f\rangle$. □

Theorem 3.3. *If an abelian group $\langle G,+\rangle$ is generated by the set $Z = \{z_\alpha | \alpha \in I\}$, then a semiabelian n-ary group $\langle G,f\rangle = der_{\varphi,d}\langle G,+\rangle$ is generated by the set $X = \{-d + z_\alpha \mid \alpha \in I\} \cup \{0\}$.*

Proof. In this case $a + b = f(a, \overset{(n-3)}{0}, \bar{0}, b)$, $\varphi(x) = f(0, x, \overset{(n-3)}{0}, \bar{0})$, $d = f(\overset{(n)}{0})$ and $-d = \bar{0}$. Thus $z_{\alpha_i} = f(\overset{(n-1)}{0}, -d + z_{\alpha_i})$. Moreover, for $n_i > 0$ we obtain

$$\begin{aligned} n_i z_{\alpha_i} &= f_{(n_i-1)}(z_{\alpha_i}, \overset{(n-3)}{0}, \bar{0}, z_{\alpha_i}, \overset{(n-3)}{0}, \bar{0}, \ldots, z_{\alpha_i}, \overset{(n-3)}{0}, \bar{0}, z_{\alpha_i}) \\ &= f_{(n_i-1)}(f(\overset{(n-1)}{0}, -d + z_{\alpha_i}), \overset{(n-3)}{0}, \bar{0}, \ldots, \overset{(n-3)}{0}, \bar{0}, f(\overset{(n-1)}{0}, -d + z_{\alpha_i})). \end{aligned}$$

Since

$$\begin{aligned} -d+z_{\alpha_i} &= f(-d + z_{\alpha_i}, \overline{-d + z_{\alpha_i}}, \overset{(n-2)}{-d + z_{\alpha_i}}) \\ &= -d+z_{\alpha_i}+\varphi(\overline{-d+z_{\alpha_i}})+\varphi^2(-d+z_{\alpha_i})+\ldots+\varphi^{n-2}(-d+z_{\alpha_i})-d+z_{\alpha_i}+d, \end{aligned}$$

we have

$$\begin{aligned} -z_{\alpha_i} &= \varphi(\overline{-d + z_{\alpha_i}}) + \varphi^2(-d + z_{\alpha_i}) + \ldots + \varphi^{n-2}(-d + z_{\alpha_i}) \\ &= -d + \varphi(\overline{-d + z_{\alpha_i}}) + \varphi^2(-d + z_{\alpha_i}) + \ldots + \varphi^{n-2}(-d + z_{\alpha_i}) + 0 + d \\ &= f(-d, \overline{-d + z_{\alpha_i}}, \overset{(n-3)}{-d + z_{\alpha_i}}, 0) = f(\bar{0}, \overline{-d + z_{\alpha_i}}, \overset{(n-3)}{-d + z_{\alpha_i}}, 0). \end{aligned}$$

Thus, for $n_i < 0$ we have

$$\begin{aligned} n_i z_{\alpha_i} &= (-n_i)(-z_{\alpha_i}) = f_{(-n_i-1)}(-z_{\alpha_i}, \overset{(n-3)}{0}, \bar{0}, -z_{\alpha_i}, \overset{(n-3)}{0}, \bar{0}, \ldots, -z_{\alpha_i}, \overset{(n-3)}{0}, \bar{0}, -z_{\alpha_i}) \\ &= f_{(-n_i-1)}(f(\bar{0}, \overline{-d+z_{\alpha_i}}, \overset{(n-3)}{-d+z_{\alpha_i}}, 0), \overset{(n-3)}{0}, \bar{0}, f(\bar{0}, \overline{-d+z_{\alpha_i}}, \overset{(n-3)}{-d+z_{\alpha_i}}, 0), \overset{(n-3)}{0}, \bar{0}, \\ &\qquad \ldots, f(\bar{0}, \overline{-d+z_{\alpha_i}}, \overset{(n-3)}{-d+z_{\alpha_i}}, 0), \overset{(n-3)}{0}, \bar{0}, f(\bar{0}, \overline{-d+z_{\alpha_i}}, \overset{(n-3)}{-d+z_{\alpha_i}}, 0)). \end{aligned}$$

Hence, in any case $n_i z_{\alpha_i}$ can be expressed in $\langle G,f\rangle = der_{\varphi,d}\langle G,+\rangle$ by elements of X. Since each element of G has the form $g = n_1 z_{\alpha_1} + \ldots + n_k z_{\alpha_k}$, the above means that an n-group $\langle G,f\rangle = der_{\varphi,d}\langle G,+\rangle$is generated by X. □

4. Structure of free semiabelian n-ary groups

Let $\mathfrak{K}$ be the class of n-ary groups. An n-ary group $\langle F,f\rangle$ from $\mathfrak{K}$ is *free* in $\mathfrak{K}$ with the set X of *free generators* if any map ψ_0 of X to any n-ary group $\langle B,f\rangle$ from the class $\mathfrak{K}$ can be uniquely extended to a homomorphism $\psi : \langle F,f\rangle \to \langle B,f\rangle$.

Denote by C_l, where $0 \leqslant l \leqslant [\frac{n-1}{2}]$, the class of all abelian semicyclic n-ary groups $(1, la)$-derived from cyclic groups. An abelian semicyclic n-ary group constructed on a cyclic group $\langle (a), +\rangle$ of order k has the form $der_{1,l_1 a}\langle (a), +\rangle$, where $0 \leqslant l_1 \leqslant k-1$. By Lemma 1 in [9], n-ary groups $der_{1,l_1 a}\langle (a), +\rangle$ and $der_{1,l_2 a}\langle (a), +\rangle$ of order k are isomorphic if and only if $\gcd(l_1, n-1, k) = \gcd(l_2, n-1, k)$. So, an n-ary group $der_{1,l_1 a}\langle (a), +\rangle$ is isomorphic to an n-ary group $der_{1,l_2 a}\langle (a), +\rangle$, where $\gcd(l_1, n-1, k) = l_2$. If $l_2 = n-1$, then $der_{1,l_2 a}\langle (a), +\rangle \cong der_{1,0}\langle (a), +\rangle$. Thus, $der_{1,l_1 a}\langle (a), +\rangle \in C_0$. If $l_2 < n-1$, then $1 \leqslant l_2 \leqslant [\frac{n-1}{2}]$ which means that $der_{1,l_1 a}\langle (a), +\rangle \in C_{l_2}$. If $der_{1,l_1 a}\langle (a), +\rangle$ is an infinite abelian semicyclic n-ary group, then $der_{1,l_1 a}\langle (a), +\rangle \cong der_{1,la}\langle (a), +\rangle$, where $0 \leqslant l \leqslant [\frac{n-1}{2}]$ (see Theorem 3 in [16]). Thus $der_{1,,l_1 a}\langle (a), +\rangle \in C_l$. This shows that each abelian semicyclic n-ary group belongs only to one class C_l, where $0 \leqslant l \leqslant [\frac{n-1}{2}]$.

Each class C_l has only one (up to isomorphism) free n-ary group. It has the form $der_{1,l}Z$, where Z is the additive group of integers (see [14]).

Free n-ary groups in the class of all abelian n-ary groups are described in the following theorem proved in [18].

Theorem 4.1. *An n-ary group is free in the class of abelian n-ary groups if and only if it is an infinite cyclic n-ary group or a direct product of an infinite cyclic n-ary group and an n-ary group derived from a free abelian group.*

To describe all free n-ary groups in the class of all semiabelian n-ary groups consider the set $\{x_\alpha \,|\, \alpha \in I\}$. For each element x_α we determine the direct sum $\langle A_\alpha, +\rangle = \sum_{j=1}^{n-1}\langle (x_{\alpha j}), +\rangle$ of infinite cyclic groups $\langle (x_{\alpha j}), +\rangle$ and the direct sum $\langle F, +\rangle = \langle (a), +\rangle + \sum_{\alpha\in I}\langle A_\alpha, +\rangle$, where $\langle (a), +\rangle$ is an infinite cyclic group. On each group $\langle A_\alpha, +\rangle$ we select an automorphism φ_α such that

$$\varphi_\alpha(t_1 x_{\alpha 1} + t_2 x_{\alpha 2} + \ldots + t_{n-1} x_{\alpha n-1}) = t_{n-1} x_{\alpha 1} + t_1 x_{\alpha 2} + \ldots + t_{n-2} x_{\alpha n-1}$$

for any $t_1 x_{\alpha 1} + t_2 x_{\alpha 2} + \ldots + t_{n-1} x_{\alpha n-1} \in A_\alpha$.

Then φ defined by $\varphi(sa + \sum_{i=1}^{k} z_{\alpha_i}) = sa + \sum_{i=1}^{k}\varphi_{\alpha_i}(z_{\alpha_i})$ is an automorphism of the group $\langle F, +\rangle$. Since $d = a$ and φ satisfy (3), on the group $\langle F, +\rangle$ we can construct the semiabelian n-ary group $\langle F, f\rangle = der_{\varphi,a}\langle F, +\rangle$ with the operation f defined by (4).

Proposition 4.2. *The n-ary group $\langle F, f\rangle$ is generated by the set*

$$X = \{-a + x_{\alpha 1} \,|\, \alpha \in I\} \cup \{0\}.$$

Proof. The abelian group $\langle F, +\rangle$ is generated by the set

$$Z = \{a\} \cup \{x_{\alpha 1} \,|\, \alpha \in I\} \cup \{x_{\alpha 2} \,|\, \alpha \in I\} \cup \ldots \cup \{x_{\alpha n-1} \,|\, \alpha \in I\}.$$

Thus, according to Theorem 3.3, the n-ary group $\langle F, f\rangle$ is generated by the set

$$T = \{0\} \cup \{-a + x_{\alpha 1} \,|\, \alpha \in I\} \cup \{-a + x_{\alpha 2} \,|\, \alpha \in I\} \cup \ldots \cup \{-a + x_{\alpha n-1} \,|\, \alpha \in I\}.$$

Note that for $\alpha \in I$ and $j = 2, \ldots, n-1$ we have

$$-a + x_{\alpha j} = -a + \varphi_\alpha^{j-1}(x_{\alpha 1}) = \varphi^{j-1}(-a + x_{\alpha 1}) = f(\overset{(i-1)}{0}, -a + x_{\alpha 1}, \overset{(n-i-1)}{0}, \bar{0}),$$

$$\overline{-a + x_{\alpha j}} = \overline{f(\overset{(i-1)}{0}, -a + x_{\alpha 1}, \overset{(n-i-1)}{0}, \bar{0})} = f(\overset{(i-1)}{\bar{0}}, \overline{-a + x_{\alpha 1}}, \overset{(n-i-1)}{\bar{0}}, \bar{\bar{0}})$$
$$= f(\overset{(i-1)}{\bar{0}}, \overline{-a + x_{\alpha 1}}, \overset{(n-i-1)}{\bar{0}}, f_{(n-3)}(\overset{((n-2)^2)}{0})).$$

This completes the proof. □

Theorem 4.3. *The n-ary group $\langle F, f\rangle$ is free in the class of semiabelian n-ary groups.*

Proof. Let $\langle B, f'\rangle$ be an arbitrary semiabelian n-ary group and ψ_0 be a map of the set X into B. Let $\psi_0(0) = c$ and $\psi_0(-a + x_{\alpha 1}) = y_\alpha$ for all $\alpha \in I$. Choose in $\langle B, f'\rangle$ an n-ary subgroup $\langle G, f'\rangle$ generated by the set $Y = \{c\} \cup \{y_\alpha \mid \alpha \in I\}$ and consider the retract $\langle G, +\rangle = ret_c\langle G, f'\rangle$. By Theorem 3.1, this retract is generated by the set

$$U = \{f'(\overset{(i-1)}{c}, y_\alpha, \overset{(n-i-1)}{c}, \bar{c}) \mid \alpha \in I,\ i = 1, \ldots, n-1\} \cup \{f'(\overset{(n)}{c})\}.$$

Since $d' = f'(\overset{(n)}{c})$ and $\varphi'(x) = f'(c, x, \overset{(n-3)}{c}, \bar{c})$ satisfy (3), we see that $\langle G, f'\rangle = der_{\varphi', d'}\langle G, +\rangle$.

Moreover, the map $\sigma_0 : Z \to U$ such that $\sigma_0(a) = d'$ and

$$\sigma_0(x_{\alpha j}) = f'(\overset{(j-1)}{c}, y_\alpha, \overset{(n-j-1)}{c}, \bar{c}) + d' = {\varphi'}^{j-1}(y_\alpha) + d'$$

for all $\alpha \in I$, $j = 1, \ldots, n-1$, can be extended to the homomorphism

$$\sigma : \langle F, +\rangle \to \langle G, +\rangle$$

with the property $\sigma(0) = c$ and $\sigma(-a + x_{\alpha 1}) = -\sigma(a) + \sigma(x_{\alpha 1}) = y_\alpha$.

Let us show that σ is a homomorphism of an n-ary group $\langle F, f\rangle$ into an n-ary group $\langle B, f'\rangle$. For this consider $x = sa + \sum_{i=1}^k z_{\alpha_i} \in \langle F, +\rangle$, where $z_{\alpha_i} = t_{i1}x_{\alpha_i 1} + t_{i2}x_{\alpha_i 2} + \ldots + t_{in-1}x_{\alpha_i n-1}$. Then

$$\varphi' \circ \sigma(x) = \varphi'(s\sigma(a) + \textstyle\sum_{i=1}^k (t_{i1}\sigma(x_{\alpha_i 1}) + t_{i2}\sigma(x_{\alpha_i 2}) + \ldots + t_{in-1}\sigma(x_{\alpha_i n-1})))$$
$$= \varphi'(sd' + \textstyle\sum_{i=1}^k (t_{i1}(y_{\alpha_i} + d') + t_{i2}(\varphi'(y_{\alpha_i}) + d') + \ldots + t_{in-1}({\varphi'}^{n-2}(y_{\alpha_i}) + d')))$$
$$= sd' + \textstyle\sum_{i=1}^k (t_{in-1}(y_{\alpha_i} + d') + t_{i1}(\varphi'(y_{\alpha_i}) + d') + \ldots + t_{in-2}({\varphi'}^{n-2}(y_{\alpha_i}) + d')).$$

On the other hand

$$\sigma \circ \varphi(x) = \sigma(sa + \textstyle\sum_{i=1}^k \varphi_{\alpha_i}(t_{i1}x_{\alpha_i 1} + t_{i2}x_{\alpha_i 2} + \ldots + t_{in-1}x_{\alpha_i n-1}))$$
$$= \sigma(sa + \textstyle\sum_{i=1}^k (t_{in-1}x_{\alpha_i 1} + t_{i1}x_{\alpha_i 2} + \ldots + t_{in-2}x_{\alpha_i n-1}))$$

$= s\sigma(a) + \sum_{i=1}^{k}(t_{in-1}\sigma(x_{\alpha_i 1}) + t_{i1}\sigma(x_{\alpha_i 2}) + \ldots + t_{in-2}\sigma(x_{\alpha_i n-1})))$
$= sd' + \sum_{i=1}^{k}(t_{in-1}(y_{\alpha_i} + d') + t_{i1}(\varphi'(y_{\alpha_i}) + d') + \ldots + t_{in-2}({\varphi'}^{n-2}(y_{\alpha_i}) + d')))$.

Thus $\sigma \circ \varphi(x) = \varphi' \circ \sigma(x)$ for any $x \in F$.

Since the neutral element of $\langle G, +\rangle$ (i.e., the element c) and σ satisfy the conditions of Theorem 2.1, σ is the homomorphism of an n-ary group $\langle F, f\rangle$ into an n-ary group $\langle B, f'\rangle$. Obviously, σ is the extension of the map $\psi_0 : X \to B$. □

Theorem 4.4. *In the class of semiabelian n-ary groups a free n-ary group freely generated by the set $W = \{x_\alpha \mid \alpha \in I\} \cup \{c\}$ is isomorphic to an n-ary group $\langle F, f\rangle$.*

Proof. Let $\langle H, h\rangle$ be a free n-ary group generated by W. Then there is a homomorphism ψ from $\langle H, h\rangle$ into an n-ary group $\langle F, f\rangle$, which is the extension of the map $c \to 0$, $x_\alpha \to -a + x_{\alpha 1}$, $\alpha \in I$. On the other hand, by Theorem 4.3, there exists a homomorphism $\tau : \langle F, f\rangle \to \langle H, h\rangle$, which is the extension of the map $0 \to c$, $-a + x_{\alpha 1} \to x_\alpha$, $\alpha \in I$. This means that $\tau \circ \psi(w) = w$ for all $w \in W$. Also, $\tau \circ \psi(\bar{w}) = \overline{\tau \circ \psi(w)} = \bar{w}$ for all $w \in W$.

Now if $\psi(u) = \psi(v)$ for some $u, v \in H$, then

$$u = h_{(k)}(y_{\alpha_1}, \ldots, y_{\alpha_{k(n-1)+1}}), \qquad v = h_{(l)}(z_{\alpha_1}, \ldots, z_{\alpha_{l(n-1)+1}}),$$

where y_{α_i} and z_{α_j} are elements of W or are skew to some elements of W. Thus

$$\begin{aligned} \tau \circ \psi(u) &= \tau \circ \psi(h_{(k)}(y_{\alpha_1}, \ldots, y_{\alpha_{k(n-1)+1}})) \\ &= \tau(f_{(k)}(\psi(y_{\alpha_1}), \ldots, \psi(y_{\alpha_{k(n-1)+1}}))) \\ &= h_{(k)}(\tau(\psi(y_{\alpha_1})), \ldots, \tau(\psi(y_{\alpha_{k(n-1)+1}}))) \\ &= h_{(k)}(y_{\alpha_1}, \ldots, y_{\alpha_{k(n-1)+1}}) = u. \end{aligned}$$

In a similar way we obtain $\tau \circ \psi(v) = v$, whence, by the uniqueness of τ, we conclude $u = v$. So, ψ is one-to-one. It is surjective too. Indeed, each $g \in F$ has the form

$$g = f_{(k)}(y_{\alpha_1}, \ldots, y_{\alpha_{k(n-1)+1}}),$$

where $y_{\alpha_i} \in X$ or is skew to some element from X. For each $y_{\alpha_i} \in X$ there exists $z_{\alpha_i} \in W$ such that $\psi(z_{\alpha_i}) = y_{\alpha_i}$. If y_{α_i} is skew to some element from X, then also there exists $z_{\alpha_i} \in W$ which is skew to some element from W and such that $\psi(z_{\alpha_i}) = y_{\alpha_i}$. This means that for each $u = h_{(k)}(z_{\alpha_1}, \ldots, z_{\alpha_{k(n-1)+1}})$ we have

$$\begin{aligned} \psi(u) &= \psi(h_{(k)}(z_{\alpha_1}, \ldots, z_{\alpha_{k(n-1)+1}})) \\ &= f_{(k)}(\psi(z_{\alpha_1}), \ldots, \psi(z_{\alpha_{k(n-1)+1}})) \\ &= f_{(k)}(y_{\alpha_1}, \ldots, y_{\alpha_{k(n-1)+1}}) = g. \end{aligned}$$

So ψ is surjective. Therefore ψ is a bijection. This completes the proof. □

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