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K-loops from classical subgroups of $GL(\mathcal{H})$, $\mathcal{H}$ being a separable Hilbert space

Alper Bulut

Abstract. We study some examples of infinite dimensional K-loops from subgroups of invertible bounded linear operators $GL(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional separable Hilbert space. We use Kreuzer and Wefelscheid method given in [10] to show that if G is one of the classical complex Banach Lie group in $\{GL(\mathcal{H}), O(\mathcal{H}, J_{\mathbb{R}}), Sp(\mathcal{H}, J_{\mathbb{Q}})\}$, then the intersection of G and the set of positive self-adjoint operators form a K-loop with respect to a new binary operation induced by the group operation in G.

1. Introduction

A Bol loop satisfying the automorphic inverse property is called a K-loop. Karzel introduced the notion of near-domain $(F, \oplus, \cdot)$ in [4], [5] which is a generalization of a near-field where the additive structure of a near-domain is not necessarily associative. Kerby and Wefelsheid investigated the additive structure of a near-domain $(F, \oplus)$ with extra axioms, and then they called the new structure a K-loop, but according to [6], they used the term *K-loop* only in talks in 1970's and the beginning of 1980's. On the other hand, the first appearance of the term *K-loop* in literature goes back to A.A. Ungar's paper in [15].

The early history of the "K-loop" notion, with K named after Karzel, is unfolded in [12]. For different purposes, the term "K-loop" was already in use earlier by L.R. Soikis, in 1970 [13], and later but independently by A.S. Basarab, in 1992 [1]. The origin of the "K" in the term K-loop coined by Soikis and by Basarab, which certainly does not refer to "Karzel", is unclear.

Ungar investigated the Einsten's velocity addition binary operation $\oplus$ over $\mathbb{R}^3_c$. The elements of $\mathbb{R}^3_c$ are called relativistically admissible velocities that are vectors in $\mathbb{R}^3$ whose norms are strictly less than c, where c is the vacuum speed of light. The Einstein velocity addition of x and y in $\mathbb{R}^3_c$ is given by

$$x \oplus y = \frac{1}{1+\frac{x.y}{c^2}} \left\{ x + y + \frac{1}{c^2}\frac{\gamma_x}{1+\gamma_x}(x \times (x \times y)) \right\} \tag{1}$$

In (1) "." and "$\times$" stand for inner product and cross product respectively, and $\gamma_x = \frac{1}{\sqrt{1-(\frac{\|x\|}{c})^2}}$ is called *Lorentz factor* [15, 17].

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Ungar showed in [15] that Einstein's velocity addition over the $\mathbb{R}_c^3$ has unusual algebraic properties. For instance $(\mathbb{R}_c^3, \oplus)$ is a non-associative and non-commutative loop. Ungar stated that this loop can be placed in the context of K-loop, see [15], that was studied by Kerby and Wefelscheid. In literature, K-loops are also known as gyrocommutative gorogroups, see [16]. The non-associativity and non-commutativity of Einstein velocity addition in $\mathbb{R}_c^3$ can be upgraded to a weak form of associativity and commutativity by a linear map in $End(\mathbb{R}^3)$ that is called *Thomas Rotation*, see [14].

The weak forms of associativity and commutativity for the $x, y, z \in \mathbb{R}_c^3$ are given by

$$x \oplus (y \oplus z) = (x \oplus y) \oplus tom[x; y](z) \tag{2}$$

$$x \oplus y = tom[x; y](y \oplus x) \tag{3}$$

Thomas precession (or Thomas rotation) is also called *Thomas gyration* and denoted by $gyr[x, y]$ for $x, y \in \mathbb{R}_c^3$, and 2 and 3 are called gyroassociative and gyrocommutative laws respectively in [16]. It is quiet interesting that some of the properties of Thomas gyration are identical with the properties of a bijective map in the definition of a near-domain $(F, \oplus, .)$, namely $d_{a,b} : F \to F$ where $a, b \in F$, and $d_{a,b}$ sends the element x to $d_{a,b}.x$ such that $a \oplus (b \oplus x) = (a \oplus b) \oplus +d_{a,b}.x$ [9]. Ungar's example in physics motivated many people to investigate K-loop structures, hence many K-loop examples were derived. Kreuzer and Wefelscheid pioneered an abstract way to construct a K-loop from group transversals [10], and Kiechle in [7] gave many examples of K-loops derived from classical groups over ordered fields. Kiechle showed that

Theorem 1.1. *Let R be n-real, and $G \leq GL(n, K)$ with $G = L_G \Omega_G$, then there are $A \oplus B \in L_G$ and $d_{A,B} \in \Omega_G$ with $AB = (A \oplus B) d_{A,B}$ such that $(L_G, \oplus)$ is a K-loop.*

Here R is an ordered field and $K = R(i)$, where $i^2 = -1$. L is the set of positive definite hermitian $n \times n$ matrices over K and Ω is the unitary group as given below.

$$L = \left\{ A \in K^{n \times n} : A = A^*, \forall v \in K^n \backslash \{0\} : v^* A v > 0 \right\}, \tag{4}$$

$$\Omega = \left\{ U \in K^{n \times n} : UU^* = I_n \right\}. \tag{5}$$

Moreover, $L_G = L \cap G$ and $\Omega_G = \Omega \cap G$. Kiechle remarks in [7] that the construction of K-loops from classical groups over ordered fields can be generalized to K-loops from $GL(\mathcal{H})$ by using the polar decomposition theorem, where $GL(\mathcal{H})$ is the unit group of bounded linear operators over the Hilbert space $\mathcal{H}$.

In the second section we summarize Kerby and Wefelscheid's method in [10] to form K-loops from group transversals. This method enable us to extend the examples of K-loops not only from the purely algebraic groups, but also algebraic groups with additional structures such as groups with differentiable manifolds or topological groups.

In the third section we form infinite dimensional K-loops refer to Kiechle's remark not only from $GL(\mathcal{H})$, but also from some subgroups of $GL(\mathcal{H})$ such as symplectic and orthogonal classical Banach Lie groups.

2. Preliminaries

Let Q be a nonempty set and let $\oplus : Q \times Q \to Q$ be a binary operation. Consider the following axioms:

1. For all $a, b \in Q$ there exists a unique $x \in Q$ such that $a \oplus x = b$.
2. For all $a, b \in Q$ there exists a unique $y \in Q$ such that $y \oplus a = b$
3. There exists an $e \in Q$ satisfying $a \oplus e = e \oplus a = a$ for all $a \in Q$.

$(Q, \oplus)$ is called a ***right loop*** if (1) and (3) are satisfied, is called a ***left loop*** if (2) and (3) are satisfied. $(Q, \oplus)$ is a *loop* if (1), (2), and (3) are satisfied. A *K-loop*, $(Q, \oplus)$, is a loop which satisfies (6) (the left Bol identity) and (7) (the automorphic inverse property) for all a, b and c in Q.

$$a \oplus (b \oplus (a \oplus c)) = (a \oplus (b \oplus a)) \oplus c, \tag{6}$$

$$(a \oplus b)^{-1} = a^{-1} \oplus b^{-1}. \tag{7}$$

Kreuzer and Wefelscheid [10] undertook an axiomatic investigation and provided a new construction method for K-loops from the groups as follow:

Theorem 2.1. *Let G be a group. Let A be a subgroup of G and let K be a subset of G such that:*

1. *$G = KA$ is an exact decomposition, i.e., for every element $g \in G$ there are unique elements $k \in K$ and $a \in A$ such that $g = ka$.*
2. *If e is the neutral element of G, then $e \in K$.*
3. *For each $x \in K$, $xKx \subseteq K$.*
4. *For each $y \in A$, $yKy^{-1} \subseteq K$.*
5. *For each $k_1, k_2 \in K$ and $\alpha \in A$, if $k_1k_2\alpha \in K$, then there exists $\beta \in A$ such that $k_1k_2\alpha = \beta k_2k_1$.*

Then for all $a, b \in K$ there exists unique $a \oplus b \in K$ and $d_{a,b} \in A$ such that $ab = (a \oplus b)d_{a,b}$. Moreover, $(K, \oplus)$ is a K-loop.

2.1. Classical Banach-Lie Groups of bounded operators

In this section, we follow Pierre de la Harpe [3].

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space over $\mathbb{C}$. A semi-linear operator $J : \mathcal{H} \to \mathcal{H}$ is called *conjugation* if $\langle Jx, Jy\rangle = \overline{\langle x, y\rangle}$ and $J^2 = I$.

A semi-linear operator is called *anti-conjugation* if the last axiom is replaced by $J^2 = -I$. The conjugation and anti-conjugations will be denoted by $J_{\mathbb{R}}$ and $J_{\mathbb{Q}}$ respectively.

Examples of infinite dimensional classical complex Banach-Lie groups of bounded operators are given in [3]. Here we only focus $GL(\mathcal{H})$, $O(\mathcal{H}, J_{\mathbb{R}})$ and $Sp(\mathcal{H}, J_{\mathbb{Q}})$. Let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$, and let $GL(\mathcal{H})$ be the group of invertible operators in $\mathcal{L}(\mathcal{H})$. We use $Pos(\mathcal{H})$ and $U(\mathcal{H})$ to denote positive self-adjoint and unitary operators respectively. The Orthogonal and Symplectic Banach-Lie groups consist of those operators in $GL(\mathcal{H})$ that leave invariant the following bilinear forms respectively: $\mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$; $(x, y) \mapsto \langle x, J_{\mathbb{R}} y\rangle$ and $(x, y) \mapsto \langle x, J_{\mathbb{Q}} y\rangle$. Therefore the orthogonal and symplectic complex Banach-Lie groups can be defined by

1. $O(\mathcal{H}, J_{\mathbb{R}}) := \{T \in GL(\mathcal{H}) : \langle Tx, J_{\mathbb{R}} Ty\rangle = \langle x, J_{\mathbb{R}} y\rangle\}$,
2. $Sp(\mathcal{H}, J_{\mathbb{Q}}) := \{T \in GL(\mathcal{H}) : \langle Tx, J_{\mathbb{Q}} Ty\rangle = \langle x, J_{\mathbb{Q}} y\rangle\}$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called self-adjoint if $T = T^*$ i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{H}$. If T is self-adjoint, then $\langle Tx, x\rangle$ is real for each $x \in \mathcal{H}$. If T is a self-adjoint operator we say that T is positive, $T \geqslant 0$, if and only if $\langle Tx, x\rangle \geqslant 0$ for all $x \in \mathcal{H}$.

Theorem 2.2 ([11]). *Let $T \in L(\mathcal{H})$. Then there is a $U \in L(\mathcal{H})$ such that:*

1. *$T = UA$, where $A = \sqrt{TT^*}$,*
2. *$\|Ux\| = \|x\|$ for $x \in \overline{R(A)}$,*
3. *$Ux = 0$ for $x \in \overline{R(A)}^{\perp}$.*

Remark 2.3. The closure of the range of A is closed, so $\mathcal{H} = \overline{R(A)} \oplus \overline{R(A)}^{\perp}$. If T is invertible, then TT^* and its positive square root are both invertible, hence U as well. Therefore, the only solution of $Ux = 0$ is $x = 0$, i.e., $\overline{R(A)}^{\perp} = \{0\}$, hence $\mathcal{H} = \overline{R(A)}$. That is U is an isometry on $\mathcal{H}$ (or U is unitary). The polar decomposition theorem is unique if T is invertible. There is also reverse polar decomposition theorem, i.e., for any $T \in GL(\mathcal{H})$ there exists unique $Q \in Pos(\mathcal{H})$ and $R \in U(\mathcal{H})$ such that $T = QR$. In this paper we always use the reverse (or left) polar decomposition theorem.

Corollary 2.4. *Let $T \in Sp(\mathcal{H}, J_{\mathbb{Q}})$, then there exists unique $U \in U(\mathcal{H}) \cap Sp(\mathcal{H}, J_{\mathbb{Q}})$ and $P \in Pos(\mathcal{H}) \cap Sp(\mathcal{H}, J_{\mathbb{Q}})$ such that $T = PU$.*

Proof. Let $T \in Sp(\mathcal{H}, J_{\mathbb{Q}}) \subseteq GL(\mathcal{H})$, then the reverse polar decomposition theorem for invertible operators indicates that T has already a unique decomposition $T = PU$, where $P = \sqrt{TT^*} \in Pos(\mathcal{H})$ and $U \in U(\mathcal{H})$. We only need to check that P and U are also elements of Symplectic Banach Lie group. If $T \in Sp(\mathcal{H}, J_{\mathbb{Q}})$, then $\langle Tx, J_{\mathbb{Q}} Ty\rangle = \langle x, J_{\mathbb{Q}} y\rangle$ for all $x, y \in \mathcal{H}$. Letting $x = y$ and using the linearity of the inner product yield that $T^* J_{\mathbb{Q}} T = J_{\mathbb{Q}}$, and this is equivalent to $T = J_{\mathbb{Q}}^{-1}(T^*)^{-1} J_{\mathbb{Q}}$. Replacing T with PU gives that

$$T = J_{\mathbb{Q}}^{-1}((PU)^*)^{-1} J_{\mathbb{Q}} = J_{\mathbb{Q}}^{-1} P^{-1} (U^*)^{-1} J_{\mathbb{Q}} = [J_{\mathbb{Q}}^{-1} P^{-1} J_{\mathbb{Q}}][J_{\mathbb{Q}}{}^{-1} (U^*)^{-1} J_{\mathbb{Q}}].$$

It can be easily verified that $J_{\mathbb{Q}}^{-1}P^{-1}J_{\mathbb{Q}} \in Pos(\mathcal{H})$ and $J_{\mathbb{Q}}{}^{-1}(U^*)^{-1}J_{\mathbb{Q}} \in U(\mathcal{H})$ by using the facts that $J_{\mathbb{Q}}J_{\mathbb{Q}}^* = I$ and $J_{\mathbb{Q}}^* = -J_{\mathbb{Q}}$. Uniqueness of the polar decomposition theorem forces that $J_{\mathbb{Q}}^{-1}P^{-1}J_{\mathbb{Q}} = P$ and $J_{\mathbb{Q}}{}^{-1}(U^*)^{-1}J_{\mathbb{Q}} = U$, so P and U are in $Sp(\mathcal{H}, J_{\mathbb{Q}})$. □

Corollary 2.5. *Let $T \in O(\mathcal{H}, J_{\mathbb{R}})$, then there exists unique $U \in U(\mathcal{H}) \cap O(\mathcal{H}, J_{\mathbb{R}})$ and $P \in Pos(\mathcal{H}) \cap O(\mathcal{H}, J_{\mathbb{R}})$ and such that $T = PU$.* □

3. Main results

Theorem 3.1. *Let G be one of the classical complex Banach-Lie groups in $\{GL(\mathcal{H}), O(\mathcal{H}, J_{\mathbb{R}}), Sp(\mathcal{H}, J_{\mathbb{Q}})\}$, and let $Pos(\mathcal{H})$ and $U(\mathcal{H})$ are collection of positive self-adjoint operators and unitary operators respectively over $\mathbb{C}$. Let $P_G := G \cap Pos(\mathcal{H})$, and $U_G := G \cap U(\mathcal{H})$. Then for all $A, B \in P_G$ there exist unique $A \oplus B \in P_G$ and $d_{A,B} \in U_G$ such that $AB = (A \oplus B)d_{A,B}$. Moreover, $(P_G, \oplus)$ is a K-loop.*

Proof. Let $A, B \in P_G$, then $A, B \in G$. G is a group, so $AB \in G$. By polar decomposition theorem there exists unique $M \in P_G$ and $N \in U_G$ such that $AB = MN$. If we let $M := A \oplus B$ and $N := d_{A,B}$, then $AB = (A \oplus B)d_{A,B}$. This decomposition is exact due to uniqueness of M and N.

It is clear that $A \oplus B = (AB)d_{A,B}^{-1}$ for all $A, B \in P_G$, hence $\oplus$ is a new binary operation for P_G induced by the group operation in G. We use the Theorem 2.1 to see $(P_G, \oplus)$ is a K-loop.

1. $G = P_G U_G$ is an exact decomposition by Theorem 2.2, Corollary 2.4, and Corollary 2.5.
2. The identity operator $I \in G$ since G is a group, and $\langle Ix, x\rangle = \langle x, x\rangle = \|x\|^2 \geqslant 0$ for all $x \in \mathcal{H}$, so I is positive. On the other hand $\langle x, x\rangle = \langle Ix, x\rangle = \langle x, Ix\rangle = \langle x, I^*x\rangle$ for all $x \in \mathcal{H}$. The last equality indicates that $I = I^*$, thus I is self-adjoint, thus $I \in P_G$.
3. $\langle (PQP)(x), x\rangle = \langle Q(P(x)), P^*(x)\rangle = \langle Q(P(x)), P(x)\rangle \geqslant 0$ for $P, Q \in P_G$ since Q is positive. Moreover, $(PQP)^* = (P^*)(Q^*)(P^*) = PQP$. Therefore, $PP_GP \subseteq P_G$ for all $P \in P_G$.
4. Let $T \in U_G$ and let $P \in P_G$. $T \in U_G$ implies that $T^* = T^{-1}$. To see $TPT^{-1} \in P_G$, observe that $\langle (TPT^{-1})(x), x\rangle = \langle P(T^{-1}(x)), T^*(x)\rangle = \langle P(T^{-1}(x)), T^{-1}(x)\rangle \geqslant 0$ since P is positive operator, and $(TPT^{-1})^* = (T^{-1})^*P^*T^* = (T^*)^*PT^{-1} = TPT^{-1}$, thus TPT^{-1} is positive and self-adjoint. Therefore, $TP_GT^{-1} \subseteq P_G$ for all $T \in U_G$.
5. Let $P, Q \in P_G$ and let $U \in U_G$. Notice that $U^* = U^{-1} \in U_G$ since U is unitary and U_G is a group. We want to show that if $PQU \in P_G$, then there exist $\beta \in U_G$ such that $PQU = \beta QP$. Assume that $PQU \in P_G$, so $(PQU)^* = PQU = U^*Q^*P^* = U^*QP$ where $U^* \in U_G$.

We conclude that $(P_G, \oplus)$ is a K-loop. □

References

[1] **A.S. Basarab**, *K-loops*, Izv. Akad. Nauk Respub. Moldova Mathematics **1** (1992), 28 – 33.

[2] **C. Chevalley**, *Theory of Lie Groups*, Princeton University Press (1976).

[3] **P. de la Harpe**, *Classical Banach-Lie algebras and Banach-Lie groups of operators in Hibert space*, Lecture Notes in Math. **285** (1972).

[4] **H. Karzel**, *Inzidenzgruppen I*, Lecture notes by I. Pieper and K. Sorensen, Universitat Hamburg (1965).

[5] **H. Karzel**, *Zusammenhange zwischen Fastberichen, scharf und zweifach transitiven Permutationsgruppen und* 2*-Strukturen mit Rechtecksaxiom*, Abh. Math. Sem. Univ. Hamburg (32) (1968), 191 – 206.

[6] **H. Kiechle**, *Theory of K-loops*, Springer-Verlag,Berlin Heidelberg New York (2002).

[7] **H. Kiechle**, *K-loops from classical groups over ordered fields*, J. Geom. **113** (1998), 105 – 127.

[8] **A.W. Knapp**, *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton, New Jersey, (1986).

[9] **A. Kreuzer and H. Wefelscheid**, *The Maximal sub near-field of a near domain*, J. Algebra **28** (1974), 319 – 325.

[10] **A. Kreuzer and H. Wefelscheid**, *On K-loops of finite order*, Results in Math. **25** (1994), 79 – 102.

[11] **J.R. Retherford**, *Hilbert Space: Compact Operators and the Trace Theorem*, London Math. Soc. Student Texts, Cambridge University Press **27**, (1993).

[12] **R.U. Sexl and H.K. Urbantke**, *Relativity, groups, particles*, Springer Physics, Vienna, 2001. Special relativity and relativistic symmetry in field and particle physics, (1992).

[13] **L.R. Soikis**, *The special loops*, (Russian), Questions of the Theory of Quasigroups and Loops, Akad. Nauk Moldav. SSR, Kishinev (1970), 122 – 131.

[14] **A.A. Ungar**, *The Thomas rotation formalism underlying a nonassociative group structure for relativistic velocities*, Appl. Math Lett. **1** (1988), 403 – 405.

[15] **A.A. Ungar**, *The relativistic noncommutative nonassociative group of velocities and the Thomas rotation*, Results in Math. **16** (1989), 168 – 179.

[16] **A.A. Ungar**, *Thomas Precession: Its Underlying Gyrogroup Axioms and Their Use in hyperbolic Geometry and Relativistic Physics*, Foundations of Physics **27**, No. 6, (1997).

[17] **A.A. Ungar**, *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*, **117** Fumdamental Theories of Physics, Kluwer Academic publisher Group, Dordrecht (2001).

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Department of Mathematics, Western Michigan University, 1903 W Michigan Ave, Kalamazoo MI 49008-5248, USA

Department of Mathematics, American University of the Middle East, Block 3, Eqaila, Kuwait
E-mail: alper.bulut@aum.edu.kw