Finite 2-generated entropic quasigroups with a quasi-identity

Grzegorz Bińczak and Joanna Kaleta

Abstract. We describe all 2-generated entropic quasigroups with a quasi-identity.

1. Introduction

Entropic quasigroups with a quasi-identity are term equivalent to abelian groups with involution (i.e., every fundamental operation of abelian groups with involution is the composition of fundamental operations of corresponding entropic quasigroup with a quasi-identity and conversely).

Obviously every finite abelian group with involution is isomorphic to a finite product of directly indecomposable finite abelian groups with involution. This decomposition is unique up to reindexing and isomorphism of factors (cf. [6], Theorem 6.39).

Hence to obtain structural theorem describing finite abelian groups with involution it remains to find all finite directly indecomposable abelian groups with involution.

We have already described (in [2]) directly indecomposable finite one-generator abelian groups with involution.

There exists an infinite family of non-isomorphic two-generated abelian groups with involution which are directly indecomposable (see [3]). Exact describtion of finite abelian groups with involution by indecomposable finite abelian groups with involution is difficult.

In this paper we propose another method. First we give some fundamental definitions and facts. Next, we prove several technical results which will be used later. In the main theorems we characterize finite two-generated abelian groups with involution and finite two-generated quasigroups with a quasi-identity.

Finally using the equivalence between abelian groups with involution and entropic quasigroup with a quasi-identity we obtain characterization of 2-generated finite entropic quasigroups with a quasi-identity .

2010 Mathematics Subject Classification: 20N05.
Keywords: quasigroup, entropic quasigroup, abelian group, involution.
Research supported by the Warsaw University of Technology under grant number 504G/1120/0054/000.

Definition 1.1. An abelian group $(G,+,-,0)$ is called an *abelian group with involution* if there is an unary operation $^*: G \to G$ such that

$$\forall a,b \in G \quad 0^* = 0,\ a^{**} = a,\ (a+b)^* = a^* + b^*.$$

We denote the variety of all abelian groups with involution by AGI.

Definition 1.2. An algebra $(Q,\cdot,/,\backslash,1)$ is an *entropic quasigroup with a quasi-identity* if it satisfies the following axioms:

(1) $a \cdot (a \backslash b) = b, \quad (b/a) \cdot a = b,$

(2) $a \backslash (a \cdot b) = b, \quad (b \cdot a)/a = b,$

(3) $(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d),$

(4) $a \cdot 1 = a, \quad 1 \cdot (1 \cdot a) = a.$

One-generated entropic quasigroups with a quasi-identity are called *monogenic* or *cyclic*.

Let us observe that the identities (1), (2) and (3) define entropic quasigroups, whereas the identities (4) define the quasi-identity. We denote the variety of all entropic quasigroups with a quasi-identity by EQ1.

More information on entropic quasigroups may be found in [4], [5], [7] and [8]. In the paper [1], it is proved that abelian groups with involution are equivalent (in the sense of Theorems: 1.3 – 1.6) to entropic quasigroups with a quasi-identity.

Theorem 1.3. *If* $\mathcal{G} = (G,+,-,0,^*)$ *is an abelian group with involution, then* $\Psi(\mathcal{G}) = (G,\cdot,/,\backslash,1)$ *is an entropic quasigroup with a quasi-identity, where* $a \cdot b := a + (b^*)$, $a\backslash b := b^* + (-a^*)$, $a/b := a + (-b^*)$, $1 := 0$.

Theorem 1.4. *If* $\mathcal{Q} = (Q,\cdot,/,\backslash,1)$ *is an entropic quasigroup with a quasi-identity, then* $\Phi(\mathcal{Q}) = (Q,+,-,0,^*)$ *is an abelian group with involution, where* $a + b := a \cdot (1 \cdot b)$, $(-a) := 1/(1 \cdot a)$, $0 := 1$, $a^* := 1 \cdot a$.

Theorem 1.5. *If* $\mathcal{Q} = (Q,\cdot,/,\backslash,1)$ *is an entropic quasigroup with a quasi-identity, then* $\Psi(\Phi(\mathcal{Q})) = \mathcal{Q}$.

Theorem 1.6. *If* $\mathcal{G} = (G,+,-,0,^*)$ *is an abelian group with involution, then* $\Phi(\Psi(\mathcal{G})) = \mathcal{G}$.

Let $\mathcal{Q} = (Q,\cdot,/,\backslash,1)$ be a monogenic entropic quasigroup with a quasi-identity. Let $Q = \langle x \rangle$ and let $\Phi(\mathcal{Q}) = (Q,+,-,0,^*)$ be the abelian group with involution equivalent to $(Q,\cdot,/,\backslash,1)$.

We will consider three types of *rank* of the generator x:

$r_+(x) = \min\{n \in \mathbb{N} \mid nx = 0,\ n \geqslant 1\}$, (additive rank)

$r_*(x) = \min\{n \in \mathbb{N} \mid n \geqslant 1,\ \exists k \in \mathbb{Z}\ nx^* = kx\}$,

$$r_{*+}(x) = \min\{n \in \mathbb{N} \mid r_*(x)x^* = (r_*(x)+n)x\}.$$

Then we define

$$r_+(\mathcal{Q}) = r_+(x), \quad r_*(\mathcal{Q}) = r_*(x), \quad r_{*+}(\mathcal{Q}) = r_{*+}(x).$$

This definition does not depend on the choice of the generator x (see [1]).

Theorem 1.7. (cf. [1]) *If $\mathcal{Q} = (Q, \cdot, /, \backslash, 1)$ is a finite monogenic entropic quasigroup with a quasi-identity, then:*

(a) $r_*(\mathcal{Q})$ is a divisor of $r_+(\mathcal{Q})$,

(b) $r_*(\mathcal{Q})$ is a divisor of $r_{*+}(\mathcal{Q})$,

(c) $0 \leqslant r_{*+}(\mathcal{Q}) < r_+(\mathcal{Q})$,

(d) $r_+(\mathcal{Q})$ is a divisor of $2r_{*+}(\mathcal{Q}) + \frac{r_{*+}(\mathcal{Q})^2}{r_*(\mathcal{Q})}$.

Proposition 1.8. (cf. [1]) *Let $\mathcal{Q}$ =$(Q, \cdot, /.\backslash, 1)$ be a finite cyclic entropic quasigroup with a quasi-identity and $Q = \langle a \rangle$ for some $a \in Q$. If $c \in Z$ then $ca = 0 \Leftrightarrow r_+(Q)|c$.*

Let $E(a)$ be the integer part of a, $(a)_b$ – the remainder obtained after dividing a by b, $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.

Proposition 1.9. (cf. [1]) *Let $b, t, y \in \mathbb{Z}$ and $b \geq 1$. Then*

$$E\left(\frac{t}{b}\right) + E\left(\frac{y + (t)_b}{b}\right) = E\left(\frac{y+t}{b}\right), \tag{1}$$

$$(y + (t)_b)_b = (y+t)_b \ . \tag{2}$$

2. Auxiliary results

Abelian groups with involution generated by x can be described by three ranks: $r_+(x)$, $r_*(x)$ and $r_{*+}(x)$ (cf. [1]). Abelian groups with involution generated by two elements x_1, x_2 will be described by ten ranks defined below.

Definition 2.1. Let $\mathcal{Q} = (Q, \cdot, /, \backslash, 1) = \langle x_1, x_2 \rangle$ be a 2-generated finite entropic quasigroup with a quasi-identity. Let $A = \Phi(\mathcal{Q})$. Let

$a_1 = \min\{n \in \mathbb{N} \setminus \{0\} \mid nx_1 \in \langle x_2 \rangle\}$,
$b_1 = \min\{n \in \mathbb{N} \setminus \{0\} \mid \exists m \in \mathbb{Z} \quad nx_1^* - mx_1 \in \langle x_2 \rangle\}$,
$k_1 = \min\{n \in \mathbb{N} \cup \{0\} \mid b_1 x_1^* - (b_1 + n)x_1 \in \langle x_2 \rangle\}$,
$a_2 = \min\{n \in \mathbb{N} \setminus \{0\} \mid nx_2 = 0\}$,
$b_2 = \min\{n \in \mathbb{N} \setminus \{0\} \mid \exists k \in \mathbb{Z} \quad nx_2^* - kx_2 = 0\}$,
$k_2 = \min\{n \in \mathbb{N} \cup \{0\} \mid b_2 x_2^* - (b_2 + n)x_2 = 0\}$.

Then there exist $a_{12}, a'_{12} \in \mathbb{Z}_{a_2}$, $b_{12}, b'_{12} \in \mathbb{Z}_{b_2}$ such that

$$a_1x_1 = a_{12}x_2 + b_{12}x_2^* \in \langle x_2\rangle, \quad b_1x_1^* - (b_1+k_1)x_1 = a'_{12}x_2 + b'_{12}x_2^* \in \langle x_2\rangle.$$

So, to every finite entropic quasigroup with a quasi-identity generated by x_1, x_2 we assign ten parameters:

$$\psi(A, x_1, x_2) = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2) \in \mathbb{Z}^{10}.$$

Example 2.2. Let $n, m \in \mathbb{Z}$, $m > 1$ and $n \geqslant 1$ be fixed. Consider $W_{n,m} = (\mathbb{Z}_{2^m} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^n}, +, -, (0,0,0), ^*)$, where

$$(y_1,y_2,y_3) + (y'_1,y'_2,y'_3) = ((y_1+y'_1+E(\tfrac{y_2+y'_2}{2})2)_{2^m}, (y_2+y'_2)_2, (y_3+y'_3)_{2^n}),$$

$$-(y_1,y_2,y_3) = ((-y_1+E(\tfrac{-y_2}{2})2)_{2^m}, (-y_2)_2, (-y_3)_{2^n}),$$

$$(y_1,y_2,y_3)^* = \begin{cases} ((y_2+E(\frac{y_1}{2})2)_{2^m}, (y_1)_2, y_3) & for \quad 2|y_3, \\ ((y_2+2^{m-1}+E(\frac{y_1}{2})2)_{2^m}, (y_1)_2, y_3) & for \quad 2\nmid y_3. \end{cases}$$

Then, by Theorem 9 in [3], $W_{n,m} = \langle x_1, x_2\rangle = W_{n,(2^{m-1},0)}(Q^0_{2^m,2}) \in AGI$, where $x_1 = (1,0,0)$, $x_2 = (0,0,1)$. So $x_1^* = (0,1,0)$, $x_2^* = (2^{m-1},0,1)$ and

- $2^{m-1}x_1 = (2^n-1)x_2 + x_2^*$,
- $2x_1^* = 0$,
- $2^nx_2 = 0$,
- $2x_2^* - 2x_2 = 0$.

Thus $a_1 = 2^{m-1}$, $b_1 = 2$, $k_1 = 0$, $a_{12} = 2^n - 1$, $b_{12} = 1$, $a'_{12} = 0$, $b'_{12} = 0$, $a_2 = 2^n$, $b_2 = 2, k_2 = 0$. □

Definition 2.3. For $t = (a_1, b_1, k_1, a_{12}, b_{12}, , a'_{12}, b'_{12}, a_2, b_2, k_2) \in \mathbb{Z}^{10}$ let γ_t be the function $\mathbb{Z}^4 \to \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$ such that

$$\pi_1(\gamma_t(y)) = (\pi_1(y) + E(\tfrac{\pi_2(y)}{b_1})(b_1+k_1))_{a_1},$$

$$\pi_2(\gamma_t(y)) = (\pi_2(y))_{b_1},$$

$$\pi_3(\gamma_t(y)) = (\pi_3(y) + E(\tfrac{\pi_2(y)}{b_1})a'_{12} + \alpha a_{12} + E\left(\frac{\pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+\alpha b_{12}}{b_2}\right)(b_2+k_2))_{a_2},$$

$$\pi_4(\gamma_t(y)) = \left(\pi_4(y) + E(\tfrac{\pi_2(y)}{b_1})b'_{12} + \alpha b_{12}\right)_{b_2}$$

for every $y \in \mathbb{Z}^4$, where $\alpha = E\left(\frac{\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)}{a_1}\right)$ and $\pi_i(y_1,y_2,y_3,y_4) = y_i$ for $i = 1,2,3,4$ and $(y_1,y_2,y_3,y_4) \in \mathbb{Z}^4$.

Definition 2.4. For $t = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2) \in \mathbb{Z}^{10}$ we define $Q_t = (\mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}, +_t, -_t, \underline{0}, ^{*_t})$, and

- $(y_1, y_2, y_3, y_4) +_t (z_1, z_2, z_3, z_4) = \gamma_t(y_1 + z_1, y_2 + z_2, y_3 + z_3, y_4 + z_4)$,
- $-_t(y_1, y_2, y_3, y_4) = \gamma_t(-y_1, -y_2, -y_3, -y_4)$, $\quad \underline{0} = (0, 0, 0, 0)$,
- $(y_1, y_2, y_3, y_4)^* = (y_2, y_1, y_4, y_3)$,
- $(y_1, y_2, y_3, y_4)^{*_t} = \gamma_t(y_2, y_1, y_4, y_3)$, i.e., $y^{*_t} = \gamma_t(y^*)$.

Definition 2.5. Let D be the set of tuples $(a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2)$ such that:

$$b_1|a_1,\ b_1|k_1,\ a_1|\Big(2k_1 + \tfrac{k_1^2}{b_1}\Big),\ a_1 \geqslant 1,\ b_1 \geqslant 1,\ 0 \leqslant k_1 < a_1,$$
$$b_2|a_2,\ b_2|k_2,\ a_2|\Big(2k_2 + \tfrac{k_2^2}{b_2}\Big),\ a_2 \geqslant 1,\ b_2 \geqslant 1,\ 0 \leqslant k_2 < a_2,$$
$$b_2|\Big(a_{12} - (1 + \tfrac{k_1}{b_1})b_{12} - \tfrac{a_1}{b_1}b'_{12}\Big), \quad b_2|\Big(a'_{12} + (1 + \tfrac{k_1}{b_1})b'_{12} + \tfrac{2k_1 + \frac{k_1^2}{b_1}}{a_1}b_{12}\Big),$$
$$a_2|\Big(b_{12} - (1 + \tfrac{k_1}{b_1}))a_{12} - \tfrac{a_1}{b_1}a'_{12} + (1 + \tfrac{k_2}{b_2})(a_{12} - (1 + \tfrac{k_1}{b_1})b_{12} - \tfrac{a_1}{b_1}b'_{12}\Big),$$
$$a_2|\Big(b'_{12} + (1 + \tfrac{k_1}{b_1}))a'_{12} + \tfrac{2k_1 + \frac{k_1^2}{b_1}}{a_1}a_{12} + (1 + \tfrac{k_2}{b_2})(a'_{12} + (1 + \tfrac{k_1}{b_1})b'_{12} + \tfrac{2k_1 + \frac{k_1^2}{b_1}}{a_1}b_{12}\Big),$$
$$a_{12}, a'_{12} \in \mathbb{Z}_{a_2}, b_{12}, b'_{12} \in \mathbb{Z}_{b_2}.$$

Lemma 2.6. *If $G = \langle x \rangle$ is a finite abelian group with involution, $a = r_+(\langle x \rangle)$, $b = r_*(\langle x \rangle)$, $k = r_{*+}(\langle x \rangle)$, $m, n \in \mathbb{Z}$ and $mx + nx^* = 0$, then $b|n$ and $a|m + (1 + \frac{k}{b})n$.*

Proof. If $mx + nx^* = 0$, then $mx + (E(\frac{n}{b})b + (n)_b)x^* = 0$. Thus we have $(n)_b x^* = (-m - E(\frac{n}{b})b)x$ and $0 \leqslant (n)_b < b$. By definition of b we obtain $(n)_b = 0$ so $b|n$.

Moreover, $mx = -nx^* = -\frac{n}{b}bx^* = -n\frac{n}{b}(b + k)x$, so $(m + \frac{n}{b}(b + k))x = 0$ and by Proposition 1.8 we have $a|(m + \frac{n}{b}(b + k))$. Thus $a|m + (1 + \frac{k}{b})n$. □

Proposition 2.7. *If $G = \langle x_1, x_2 \rangle$ is a finite abelian group with involution, then $t = \psi(G, x_1, x_2) \in D$.*

Proof. Let $G = \langle x_1, x_2 \rangle$ be a finite abelian group with involution and $t = \psi(G, x_1, x_2) = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2)$. Then $a_2 = r_+(\langle x_2 \rangle)$, $b_2 = r_*(\langle x_2 \rangle)$ and $k_2 = r_{*+}(\langle x_2 \rangle)$, $a_1 = r_+(G/\langle x_2 \rangle)$, $b_1 = r_*(G/\langle x_2 \rangle)$, $k_1 = r_{*+}(G/\langle x_2 \rangle)$.

By Theorem 1.7 we have $b_1|a_1,\ b_1|k_1,\ a_1|(2k_1 + \frac{k_1^2}{b_1}),\ a_1 \geqslant 1,\ b_1 \geqslant 1,\ 0 \leqslant k_1 < a_1$, $b_2|a_2,\ b_2|k_2,\ a_2|(2k_2 + \frac{k_2^2}{b_2}),\ a_2 \geqslant 1,\ b_2 \geqslant 1,\ 0 \leqslant k_2 < a_2$.

Now we prove that

$$b_2|(a'_{12} + (1 + \tfrac{k_1}{b_1})b'_{12} + \tfrac{2k_1 + \frac{k_1^2}{b_1}}{a_1}b_{12}) \quad \text{and}$$
$$a_2|(b'_{12} + (1 + \tfrac{k_1}{b_1}))a'_{12} + \tfrac{2k_1 + \frac{k_1^2}{b_1}}{a_1}a_{12} + (1 + \tfrac{k_2}{b_2})(a'_{12} + (1 + \tfrac{k_1}{b_1})b'_{12} + \tfrac{2k_1 + \frac{k_1^2}{b_1}}{a_1}b_{12}).$$

By definition of t we obtain

$$a_1 x_1 = a_{12}x_2 + b_{12}x_2^* \in \langle x_2 \rangle, \tag{3}$$

$$b_1x_1^* - (b_1+k_1)x_1 = a'_{12}x_2 + b'_{12}x_2^* \in \langle x_2\rangle. \tag{4}$$

So, $(b_1+k_1)x_1^* = (1+\frac{k_1}{b_1})b_1x_1^* \overset{(4)}{=} (1+\frac{k_1}{b_1})((b_1+k_1)x_1 + a'_{12}x_2 + b'_{12}x_2^*)$ and $(b_1+k_1)x_1^* = ((b_1+k_1)x_1)^* \overset{(4)}{=} (b_1x_1^* - a'_{12}x_2 - b'_{12}x_2^*)^* = b_1x_1 - a'_{12}x_2^* - b'_{12}x_2$. Hence $0 = ((1+\frac{k_1}{b_1})(b_1+k_1)-b_1)x_1 + (b'_{12}+(1+\frac{k_1}{b_1})a'_{12})x_2 + (a'_{12}+(1+\frac{k_1}{b_1})b'_{12})x_2^*$ and $((1+\frac{k_1}{b_1})(b_1+k_1)-b_1)x_1 = (2k_1+\frac{k_1^2}{b_1})x_1 = \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}a_1x_1 \overset{(3)}{=} \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}(a_{12}x_2+b_{12}x_2^*)$. So, $0 = (b'_{12}+(1+\frac{k_1}{b_1})a'_{12} + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}a_{12})x_2 + (a'_{12}+(1+\frac{k_1}{b_1})b'_{12} + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})x_2^*$.

From this, applying Lemma 2.6, we obtain $b_2|(a'_{12}+(1+\frac{k_1}{b_1})b'_{12} + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})$ and $a_2|(b'_{12}+(1+\frac{k_1}{b_1}))a'_{12} + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}a_{12} + (1+\frac{k_2}{b_2})(a'_{12}+(1+\frac{k_1}{b_1})b'_{12} + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})$.

Now we prove that $b_2|(a_{12}-(1+\frac{k_1}{b_1})b_{12} - \frac{a_1}{b_1}b'_{12})$ and $a_2|(b_{12}-(1+\frac{k_1}{b_1}))a_{12} - \frac{a_1}{b_1}a'_{12} + (1+\frac{k_2}{b_2})(a_{12}-(1+\frac{k_1}{b_1})b_{12} - \frac{a_1}{b_1}b'_{12})$.

Let us observe that $a_{12}x_2^* + b_{12}x_2 = (a_{12}x_2+b_{12}x_2^*)^* \overset{(3)}{=} a_1x_1^* = \frac{a_1}{b_1}b_1x_1^* \overset{(4)}{=} \frac{a_1}{b_1}((b_1+k_1)x_1 + a'_{12}x_2 + b'_{12}x_2^*) = (1+\frac{k_1}{b_1})\,a_1x_1 + \frac{a_1}{b_1}\,a'_{12}x_2 + \frac{a_1}{b_1}\,b'_{12}x_2^* \overset{(3)}{=} (1+\frac{k_1}{b_1})(a_{12}x_2+b_{12}x_2^*) + \frac{a_1}{b_1}a'_{12}x_2 + \frac{a_1}{b_1}b'_{12}x_2^*$.

Thus $0 = (b_{12}-(1+\frac{k_1}{b_1})a_{12} - \frac{a_1}{b_1}a'_{12})x_2 + (a_{12}-(1+\frac{k_1}{b_1})b_{12} - \frac{a_1}{b_1}b'_{12})x_2^*$.

After applying Lemma 2.6 we conclude that $b_2|(a_{12}-(1+\frac{k_1}{b_1})b_{12} - \frac{a_1}{b_1}b'_{12})$ and $a_2|(b_{12}-(1+\frac{k_1}{b_1}))a_{12} - \frac{a_1}{b_1}a'_{12} + (1+\frac{k_2}{b_2})(a_{12}-(1+\frac{k_1}{b_1})b_{12} - \frac{a_1}{b_1}b'_{12})$. □

The following two lemmas and proposition serve as technical help to prove the Theorem 3.1.

Lemma 2.8. *If* $t = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2) \in D$, $x, y \in \mathbb{Z}^4$, *then*

$$\gamma_t(x+y) = \gamma_t(x+\gamma_t(y)). \tag{5}$$

If $x \in \mathbb{Z}_{a_1}\times\mathbb{Z}_{b_1}\times\mathbb{Z}_{a_2}\times\mathbb{Z}_{b_2}$, *then*

$$\gamma_t(x) = x. \tag{6}$$

Proof. Let $t = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2) \in D$, $x, y \in \mathbb{Z}^4$. We show that $\pi_i(\gamma_t(x+\gamma_t(y))) = \pi_i(\gamma_t(x+y))$ for $i = 1,2,3,4$.

We have

- $\pi_1(\gamma_t(x+\gamma_t(y))) = (\pi_1(x+\gamma_t(y)) + E(\frac{\pi_2(x+\gamma_t(y))}{b_1})(b_1+k_1))_{a_1}$
  $= (\pi_1(x) + \pi_1(\gamma_t(y)) + E(\frac{\pi_2(x)+\pi_2(\gamma_t(y))}{b_1})(b_1+k_1))_{a_1}$
  $= (\pi_1(x) + (\pi_1(y) + E(\frac{\pi_2(y)}{b_1})(b_1+k_1))_{a_1} + E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})(b_1+k_1))_{a_1}$
  $\overset{(1),(2)}{=} (\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)+(E(\frac{\pi_2(x)+\pi_2(y)}{b_1})-E(\frac{\pi_2(y)}{b_1}))(b_1+k_1))_{a_1}$
  $= (\pi_1(x)+\pi_1(y)+(E(\frac{\pi_2(x)+\pi_2(y)}{b_1}))_{a_1} = \pi_1(\gamma_t(x+y))$,

- $\pi_2(\gamma_t(x+\gamma_t(y)) = (\pi_2(x+\gamma_t(y)))_{b_1} = (\pi_2(x)+\pi_2(\gamma_t(y)))_{b_1}$
$= (\pi_2(x)+(\pi_2(y)_{b_1}))_{b_1} \overset{(2)}{=} (\pi_2(x)+\pi_2(y))_{b_1} = \pi_2(\gamma_t(x+y)).$

- Let us introduce the following abbreviations:
$\alpha_1 = \pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1),$
$\alpha_2 = E\left(\frac{\pi_1(x+\gamma_t(y))+E(\frac{\pi_2(x+\gamma_t(y))}{b_1})(b_1+k_1)}{a_1}\right)$
$= E\left(\frac{\pi_1(x)+(\alpha_1)_{a_1}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})(b_1+k_1)}{a_1}\right),$
$\alpha_3 = \pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+E(\frac{\alpha_1}{a_1})b_{12},$
$\alpha_4 = E\left(\frac{\pi_4(x)+(\alpha_3)_{b_2}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})b'_{12}+\alpha_2 b_{12}}{b_2}\right),$
$\alpha'_2 = E\left(\frac{\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(x)+\pi_2(y)}{b_1})(b_1+k_1)}{a_1}\right),$
$\alpha'_4 = E\left(\frac{\pi_4(x)+\pi_4(y)+\alpha'_2 b_{12}+E(\frac{\pi_2(x)+\pi_2(y)}{b_1})b'_{12}}{b_2}\right).$
By Proposition 1.9 we have
$E(\frac{\alpha_1}{a_1})+\alpha_2 = E(\frac{\alpha_1}{a_1})+E\left(\frac{\pi_1(x)+(\alpha_1)_{a_1}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})(b_1+k_1)}{a_1}\right)$
$\overset{(1)}{=} E\left(\frac{\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})(b_1+k_1)}{a_1}\right)$
$= E\left(\frac{\pi_1(x)+\pi_1(y)+\left(E(\frac{\pi_2(y)}{b_1})+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})\right)(b_1+k_1)}{a_1}\right)$
$\overset{(1)}{=} E\left(\frac{\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(x)+\pi_2(y)}{b_1})(b_1+k_1)}{a_1}\right) = \alpha'_2,$

so

$$E(\frac{\alpha_1}{a_1})+\alpha_2 = \alpha'_2. \tag{7}$$

Moreover,

$E(\frac{\alpha_3}{b_2})+\alpha_4 = E(\frac{\alpha_3}{b_2})+E\left(\frac{\pi_4(x)+(\alpha_3)_{b_2}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})b'_{12}+\alpha_2 b_{12}}{b_2}\right)$
$\overset{(1)}{=} E\left(\frac{\pi_4(x)+\alpha_3+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})b'_{12}+\alpha_2 b_{12}}{b_2}\right)$
$+E\left(\frac{\pi_4(x)+\pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+E(\frac{\alpha_1}{a_1})b_{12}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})b'_{12}+\alpha_2 b_{12}}{b_2}\right)$
$\overset{(7)}{=} E\left(\frac{\pi_4(x)+\pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+\alpha'_2 b_{12}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})b'_{12}}{b_2}\right)$

$$\overset{(1)}{=} E\left(\frac{\pi_4(x)+\pi_4(y)+\alpha_2' b_{12}+E(\frac{\pi_2(x)+\pi_2(y)}{b_1})b_{12}'}{b_2}\right) = \alpha_4',$$

so

$$E(\frac{\alpha_3}{b_2}) + \alpha_4 = \alpha_4'. \tag{8}$$

Hence,

$$\pi_3(\gamma_t(x+\gamma_t(y))) =$$

$$\left(\pi_3(x+\gamma_t(y))+E(\frac{\pi_2(x+\gamma_t(y))}{b_1})a_{12}'+\alpha_2 a_{12}+E\left(\frac{\pi_4(x+\gamma_t(y))+E(\frac{\pi_2(x+\gamma_t(y))}{b_1})b_{12}'+\alpha_2 b_{12}}{b_2}\right)(b_2+k_2)\right)_{a_2}$$

$$= \left(\pi_3(x)+\left(\pi_3(y)+E(\frac{\pi_2(y)}{b_1})a_{12}'+E(\frac{\alpha_1}{a_1})a_{12}+E(\frac{\alpha_3}{b_2})(b_2+k_2)\right)_{a_2}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})a_{12}'\right.$$

$$\left.+\alpha_2 a_{12}+E(\frac{\pi_4(x)+(\alpha_3)_{b_2}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})b_{12}'+\alpha_2 b_{12}}{b_2})(b_2+k_2)\right)_{a_2}$$

$$\overset{(2)}{=} \left(\pi_3(x)+\pi_3(y)+a_{12}'\left(E(\frac{\pi_2(y)}{b_1})+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})\right)+a_{12}(E(\frac{\alpha_1}{a_1})+\alpha_2)+\right.$$

$$\left.(b_2+k_2)(\alpha_4+E(\frac{\alpha_3}{b_2}))\right)_{a_2}$$

$$\overset{(1),(7),(8)}{=} \left(\pi_3(x)+\pi_3(y)+a_{12}'E(\frac{\pi_2(x)+\pi_2(y)}{b_1})+a_{12}\alpha_2'+(b_2+k_2)\alpha_4'\right)_{a_2}$$

$$= \left(\pi_3(x+y)+a_{12}'E(\frac{\pi_2(x+y)}{b_1})+a_{12}\alpha_2'+(b_2+k_2)E\left(\frac{\pi_4(x+y)+\alpha_2' b_{12}+E(\frac{\pi_2(x+y)}{b_1})b_{12}'}{b_2}\right)\right)_{a_2}$$

$$= \pi_3(\gamma_t(x+y))$$

- Let $\beta_1 = \pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)$, $\beta_2 = E\left(\frac{\pi_1(x+\gamma_t(y))+E(\frac{\pi_2(x+\gamma_t(y))}{b_1})(b_1+k_1)}{a_1}\right)$ $= E\left(\frac{\pi_1(x)+(\beta_1)_{a_1}+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})(b_1+k_1)}{a_1}\right)$.

Then

$$E(\frac{\beta_1}{a_1})+\beta_2 \overset{(1)}{=} E\left(\frac{\pi_1(x)+\beta_1+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})(b_1+k_1)}{a_1}\right)$$

$$= E\left(\frac{\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)+E(\frac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})(b_1+k_1)}{a_1}\right)$$

$$\overset{(1)}{=} E\left(\frac{\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(x)+\pi_2(y)}{b_1})(b_1+k_1)}{a_1}\right).$$

Therefore

$$E(\frac{\beta_1}{a_1})+\beta_2 = E\left(\frac{\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(x)+\pi_2(y)}{b_1})(b_1+k_1)}{a_1}\right). \tag{9}$$

So we conclude that

$$\begin{aligned}
&\pi_4(\gamma_t(x+\gamma_t(y))) \\
&= \left(\pi_4(x+\gamma_t(y)) + E(\tfrac{\pi_2(x+\gamma_t(y))}{b_1})b'_{12} + E(\tfrac{\pi_1(x+\gamma_t(y))+E(\frac{\pi_2(x+\gamma_t(y)}{b_1})(b_1+k_1)}{a_1})b_{12}\right)_{b_2} \\
&= \Bigg(\pi_4(x) + \left(\pi_4(y) + E(\tfrac{\pi_2(y)}{b_1})b'_{12} + E(\tfrac{\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)}{a_1})b_{12}\right)_{b_2} + \\
&\qquad\qquad E(\tfrac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1})b'_{12} + \beta_2 b_{12}\Bigg)_{b_2} \\
&\overset{(2)}{=} \left(\pi_4(x) + \pi_4(y) + (E(\tfrac{\pi_2(y)}{b_1}) + E(\tfrac{\pi_2(x)+(\pi_2(y))_{b_1}}{b_1}))b'_{12} + (E(\tfrac{\beta_1}{a_1}) + \beta_2)b_{12}\right)_{b_2} \\
&\overset{(1)}{=} \left(\pi_4(x) + \pi_4(y) + (E(\tfrac{\pi_2(x)+\pi_2(y)}{b_1}))b'_{12} + (E(\tfrac{\beta_1}{a_1}) + \beta_2)b_{12}\right)_{b_2} \\
&\overset{(9)}{=} \left(\pi_4(x+y) + (E(\tfrac{\pi_2(x+y)}{b_1}))b'_{12} + E\left(\tfrac{\pi_1(x)+\pi_1(y)+E(\frac{\pi_2(x)+\pi_2(y)}{b_1})(b_1+k_1)}{a_1}\right)b_{12}\right)_{b_2} \\
&= \pi_4(\gamma_t(x+y)).
\end{aligned}$$

Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$. It is clear that if $y \in \mathbb{Z}_n$, then $(y)_n = y$ and $E(\frac{y}{n}) = 0$. So,

- $\pi_1(\gamma_t(x)) = (x_1 + E(\frac{x_2}{b_1})(b_1 + k_1))_{a_1} = (x_1)_{a_1} = x_1$,
- $\pi_2(\gamma_t(x)) = (x_2)_{b_1} = x_2$,
- $\pi_3(\gamma_t(x)) = (x_3 + E(\frac{x_2}{b_1})a'_{12} + \alpha a_{12} + E\left(\frac{x_4+E(\frac{x_2}{b_1})b'_{12}+\alpha b_{12}}{b_2}\right)(b_2 + k_2))_{a_2}$
  $= (x_3)_{a_2} = x_3$ since $\alpha = E\left(\frac{x_1+E(\frac{x_2}{b_1})(b_1+k_1)}{a_1}\right) = E(\frac{x_1}{a_1}) = 0$,
- $\pi_4(\gamma_t(x)) = (x_4 + E(\frac{x_2}{b_1})b'_{12} + \alpha b_{12})_{b_2} = (x_4)_{b_2} = x_4$.

Hence $\gamma_t(x) = x$. □

Proposition 2.9. *If $x, y, b \in \mathbb{Z}$ and $b \geqslant 1$ is a divisor of $x - y$, then*

$$E\left(\frac{x}{b}\right) - E\left(\frac{y}{b}\right) = \frac{x-y}{b}\,.$$

Proof. It is obvious. □

Lemma 2.10. *For $t \in D$ and $y \in \mathbb{Z}^4$ we have*

$$\gamma_t(y^*) = \gamma_t((\gamma_t(y))^*), \tag{10}$$

where $y^* = (y_1, y_2, y_3, y_4)^* = (y_2, y_1, y_4, y_3)$.

Proof. Let $y \in \mathbb{Z}^4$, $t = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2) \in D$ and

- $\pi_1(\gamma_t(y^*)) = (L_1)_{a_1}, \quad \pi_2(\gamma_t(y^*)) = (L_2)_{b_1}$,

- $\pi_3(\gamma_t(y^*)) = (L_3)_{a_2}, \quad \pi_4(\gamma_t(y^*)) = (L_4)_{b_2}$,
- $\pi_1(\gamma_t((\gamma_t(y))^*)) = (R_1)_{a_1}, \quad \pi_2(\gamma_t((\gamma_t(y))^*)) = (R_2)_{b_1}$,
- $\pi_3(\gamma_t((\gamma_t(y))^*)) = (R_3)_{a_2}, \quad \pi_4(\gamma_t((\gamma_t(y))^*)) = (R_4)_{b_2}$.

We show that $a_1|R_1 - L_1$, $b_1|R_2 - L_2$, $a_2|R_3 - L_3$ and $b_2|R_4 - L_4$. Let

$\beta_1 = \pi_1(y) + E(\frac{\pi_2(y)}{b_1})(b_1 + k_1)$,

$\beta_2 = E(\frac{(\pi_2(y))_{b_1} + E(\frac{(\beta_1)_{a_1}}{b_1})(b_1+k_1)}{a_1})$,

$\beta_3 = E(\frac{\pi_4(y) + E(\frac{\pi_2(y)}{b_1})b'_{12} + E(\frac{\beta_1}{a_1})b_{12}}{b_2})$,

$\beta_4 = \pi_3(y) + E(\frac{\pi_2(y)}{b_1})a'_{12} + E(\frac{\beta_1}{a_1})a_{12} + \beta_3(b_2 + k_2)$,

$\beta'_1 = \pi_2(y) + E(\frac{\pi_1(y)}{b_1})(b_1 + k_1)$.

First we show that

$$E\left(\frac{(\beta_1)_{a_1}}{b_1}\right) - E\left(\frac{\pi_1(y)}{b_1}\right) = (1 + \frac{k_1}{b_1})E\left(\frac{\pi_2(y)}{b_1}\right) - \frac{a_1}{b_1}E\left(\frac{\beta_1}{a_1}\right), \qquad (11)$$

$$\beta_2 - E\left(\frac{\beta'_1}{a_1}\right) = \frac{2k_1 + \frac{k_1^2}{b_1}}{a_1}E\left(\frac{\pi_2(y)}{b_1}\right) - (1 + \frac{k_1}{b_1})E\left(\frac{\beta_1}{a_1}\right). \qquad (12)$$

$$\left.\begin{aligned} &E\left(\frac{(\beta_4)_{a_2} + E(\frac{(\beta_1)_{a_1}}{b_1})b'_{12} + \beta_2 b_{12}}{b_2}\right) - E\left(\frac{\pi_3(y) + E(\frac{\pi_1(y)}{b_1})b'_{12} + E(\frac{\beta'_1}{a_1})b_{12}}{b_2}\right) \\ &= \Bigg(E(\tfrac{\pi_2(y)}{b_1})(a'_{12} + (1 + \tfrac{k_1}{b_1})b'_{12} + \tfrac{2k_1 + \frac{k_1^2}{b_1}}{a_1}b_{12}) + \\ &E(\tfrac{\beta_1}{a_1})(a_{12} - (1 + \tfrac{k_1}{b_1})b_{12} - \tfrac{a_1}{b_1}b'_{12}) + (b_2 + k_2)\beta_3 - E(\tfrac{\beta_4}{a_2})a_2\Bigg)\tfrac{1}{b_2}. \end{aligned}\right\} \qquad (13)$$

- Let us observe that $(\beta_1)_{a_1} - \pi_1(y) = \beta_1 - E(\frac{\beta_1}{a_1})a_1 - \pi_1(y) = \pi_1(y) + E(\frac{\pi_2(y)}{b_1})(b_1 + k_1) - E(\frac{\beta_1}{a_1})a_1 - \pi_1(y)$ $= E(\frac{\pi_2(y)}{b_1})(b_1 + k_1) - E(\frac{\beta_1}{a_1})a_1$ so $b_1|(\beta_1)_{a_1} - \pi_1(y)$, and by Proposition 2.9 we conclude $E\left(\frac{(\beta_1)_{a_1}}{b_1}\right) - E\left(\frac{\pi_1(y)}{b_1}\right) = \frac{(\beta_1)_{a_1} - \pi_1(y)}{b_1} = \frac{E(\frac{\pi_2(y)}{b_1})(b_1+k_1) - E(\frac{\beta_1}{a_1})a_1}{b_1} =$ $(1 + \frac{k_1}{b_1})E\left(\frac{\pi_2(y)}{b_1}\right) - \frac{a_1}{b_1}E\left(\frac{\beta_1}{a_1}\right)$. So, we obtain (11).

- $(\pi_2(y))_{b_1} + E(\frac{(\beta_1)_{a_1}}{b_1})(b_1 + k_1) - (\pi_2(y) + E(\frac{\pi_1(y)}{b_1})(b_1 + k_1))$
  $= -E(\frac{\pi_2(y)}{b_1}) + (b_1 + k_1)(E\left(\frac{(\beta_1)_{a_1}}{b_1}\right) - E\left(\frac{\pi_1(y)}{b_1}\right))$
  $\overset{(11)}{=} -E(\frac{\pi_2(y)}{b_1}) + (b_1 + k_1)((1 + \frac{k_1}{b_1})E\left(\frac{\pi_2(y)}{b_1}\right) - \frac{a_1}{b_1}E\left(\frac{\beta_1}{a_1}\right))$
  $= E(\frac{\pi_2(y)}{b_1})(2k_1 + \frac{k_1^2}{b_1}) - a_1(1 + \frac{k_1}{b_1})E\left(\frac{\beta_1}{a_1}\right)$
  is divided by a_1 since $a_1|(2k_1 + \frac{k_1^2}{b_1})$.

By Proposition 2.9 we have

$$\beta_2 - E\left(\frac{\beta_1'}{a_1}\right) = \frac{(\pi_2(y))_{b_1} + E(\frac{(\beta_1)_{a_1}}{b_1})(b_1+k_1) - (\pi_2(y) + E(\frac{\pi_1(y)}{b_1})(b_1+k_1))}{a_1} =$$
$$\frac{E(\frac{\pi_2(y)}{b_1})(2k_1+\frac{k_1^2}{b_1}) - a_1(1+\frac{k_1}{b_1})E\left(\frac{\beta_1}{a_1}\right)}{a_1} = \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}E\left(\frac{\pi_2(y)}{b_1}\right) - (1+\frac{k_1}{b_1})E\left(\frac{\beta_1}{a_1}\right),$$

so we obtain (12).

- $(\beta_4)_{a_2} + E(\frac{(\beta_1)_{a_1}}{b_1})b_{12}' + \beta_2 b_{12} - \left(\pi_3(y) + E(\frac{\pi_1(y)}{b_1})b_{12}' + E(\frac{\beta_1'}{a_1})b_{12}\right)$

$$= (\beta_4)_{a_2} - \pi_3(y) + b_{12}'(E(\frac{(\beta_1)_{a_1}}{b_1}) - E(\frac{\pi_1(y)}{b_1})) + b_{12}(\beta_2 - E(\frac{\beta_1'}{a_1}))$$
$$\overset{(11),(12)}{=} \beta_4 - E(\frac{\beta_4}{a_2})a_2 - \pi_3(y) + b_{12}'\left((1+\frac{k_1}{b_1})E(\frac{\pi_2(y)}{b_1}) - \frac{a_1}{b_1}E(\frac{\beta_1}{a_1})\right) +$$
$$b_{12}\left(\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}E(\frac{\pi_2(y)}{b_1}) - (1+\frac{k_1}{b_1})E(\frac{\beta_1}{a_1})\right)$$
$$= \pi_3(y) + E(\frac{\pi_2(y)}{b_1})a_{12}' + E(\frac{\beta_1}{a_1})a_{12} + \beta_3(b_2+k_2) - \pi_3(y) - E(\frac{\beta_4}{a_2})a_2 +$$
$$b_{12}'\left((1+\frac{k_1}{b_1})E(\frac{\pi_2(y)}{b_1}) - \frac{a_1}{b_1}E(\frac{\beta_1}{a_1})\right) + b_{12}\left(\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}E(\frac{\pi_2(y)}{b_1}) - (1+\frac{k_1}{b_1})E(\frac{\beta_1}{a_1})\right)$$
$$= E(\frac{\pi_2(y)}{b_1})(a_{12}' + (1+\frac{k_1}{b_1})b_{12}' + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12}) +$$
$$E(\frac{\beta_1}{a_1})(a_{12} - (1+\frac{k_1}{b_1})b_{12} - \frac{a_1}{b_1}b_{12}') + (b_2+k_2)\beta_3 - E(\frac{\beta_4}{a_2})a_2)$$

is divided by b_2 because $b_2|a_2$, $b_2|(b_2+k_2)$, $b_2|(a_{12}'+(1+\frac{k_1}{b_1})b_{12}'+\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})$, $b_2|(a_{12}-(1+\frac{k_1}{b_1})b_{12}-\frac{a_1}{b_1}b_{12}')$, whereas last two divisions are consequence of the assumption that $t \in D$.

By Lemma 2.9 we have

$$E\left(\frac{(\beta_4)_{a_2} + E(\frac{(\beta_1)_{a_1}}{b_1})b_{12}' + \beta_2 b_{12}}{b_2}\right) - E\left(\frac{\pi_3(y) + E(\frac{\pi_1(y)}{b_1})b_{12}' + E(\frac{\beta_1'}{a_1})b_{12}}{b_2}\right)$$
$$= \frac{1}{b_2}\left((\beta_4)_{a_2} + E(\frac{(\beta_1)_{a_1}}{b_1})b_{12}' + \beta_2 b_{12} - (\pi_3(y) + E(\frac{\pi_1(y)}{b_1})b_{12}' + E(\frac{\beta_1'}{a_1})b_{12})\right)$$
$$= \left(E(\frac{\pi_2(y)}{b_1})(a_{12}' + (1+\frac{k_1}{b_1})b_{12}' + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12}) + E(\frac{\beta_1}{a_1})(a_{12} - (1+\frac{k_1}{b_1})b_{12}\right.$$
$$\left. - \frac{a_1}{b_1}b_{12}') + (b_2+k_2)\beta_3 - E(\frac{\beta_4}{a_2})a_2\right)\frac{1}{b_2}.$$

So we obtain (13).

In this part of the proof we show that $a_1|(R_1 - L_1)$. Indeed:

$$R_1 - L_1 = (\pi_1(\gamma_t(y)^*) + E(\frac{\pi_2(\gamma_t(y)^*)}{b_1})(b_1+k_1)) - \pi_2(y) + E(\frac{\pi_1(y)}{b_1})(b_1+k_1)$$
$$= ((\pi_2(y))_{b_1} + E(\frac{(\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1))_{a_1}}{b_1})(b_1+k_1)) - \pi_2(y) - E(\frac{\pi_1(y)}{b_1})(b_1+k_1)$$
$$= -E(\frac{\pi_2(y)}{b_1})b_1 + E(\frac{(\beta_1)_{a_1}}{b_1})(b_1+k_1) - E(\frac{\pi_1(y)}{b_1})(b_1+k_1)$$

$$\overset{(11)}{=} -E(\tfrac{\pi_2(y)}{b_1})b_1 + (b_1+k_1)\left((1+\tfrac{k_1}{b_1})E\left(\tfrac{\pi_2(y)}{b_1}\right) - \tfrac{a_1}{b_1}E\left(\tfrac{\beta_1}{a_1}\right)\right)$$
$$= E(\tfrac{\pi_2(y)}{b_1})(2k_1+\tfrac{k_1^2}{b_1}) - (1+\tfrac{k_1}{b_1})a_1E\left(\tfrac{\beta_1}{a_1}\right) \text{ is divided by } a_1 \text{ since } a_1|(2k_1+\tfrac{k_1^2}{b_1}).$$

Now we show that $b_1|(R_2 - L_2)$:

$$R_2 - L_2 = \pi_2(\gamma_t(y)^*) - \pi_2(y^*) = (\pi_1(y) + E(\tfrac{\pi_2(y)}{b_1})(b_1+k_1))_{a_1} - \pi_1(y)$$
$$= \pi_1(y) + E(\tfrac{\pi_2(y)}{b_1})(b_1+k_1) - E(\tfrac{\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)}{a_1})a_1 - \pi_1(y)$$
$$= E(\tfrac{\pi_2(y)}{b_1})(b_1+k_1) - E(\tfrac{\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)}{a_1})a_1$$

is divided by b_1 since $b_1|a_1$ and $b_1|k_1$.

The third our task is to show that $a_2|(R_3 - L_3)$:

$$L_3 = \pi_3(y^*) + E(\tfrac{\pi_2(y^*)}{b_1})a'_{12} + \alpha a_{12} + E\left(\tfrac{\pi_4(y^*)+E(\frac{\pi_2(y^*)}{b_1})b'_{12}+\alpha b_{12}}{b_2}\right)(b_2+k_2),$$

where $\alpha = E\left(\tfrac{\pi_1(y^*)+E(\frac{\pi_2(y^*)}{b_1})(b_1+k_1)}{a_1}\right) = E\left(\tfrac{\pi_2(y)+E(\frac{\pi_1(y)}{b_1})(b_1+k_1)}{a_1}\right) = E(\tfrac{\beta'_1}{a_1})$. So,

$$L_3 = \pi_4(y) + E(\tfrac{\pi_1(y)}{b_1})a'_{12} + E(\tfrac{\beta'_1}{a_1})a_{12} + E\left(\tfrac{\pi_3(y)+E(\frac{\pi_1(y)}{b_1})b'_{12}+E(\frac{\beta'_1}{a_1})b_{12}}{b_2}\right)(b_2+k_2).$$

$$R_3 = \pi_3(\gamma_t(y)^*) + E(\tfrac{\pi_2(\gamma_t(y)^*)}{b_1})a'_{12} + \alpha' a_{12} + E\left(\tfrac{\pi_4(\gamma_t(y)^*)+E(\frac{\pi_2(\gamma_t(y)^*)}{b_1})b'_{12}+\alpha' b_{12}}{b_2}\right)(b_2+k_2),$$

where $\alpha' = E\left(\tfrac{\pi_1(\gamma_t(y)^*)+E(\frac{\pi_2(\gamma_t(y)^*)}{b_1})(b_1+k_1)}{a_1}\right) = E\left(\tfrac{\pi_2(\gamma_t(y))+E(\frac{\pi_1(\gamma_t(y))}{b_1})(b_1+k_1)}{a_1}\right) = E\left(\tfrac{(\pi_2(y))_{b_1}+E(\frac{(\beta_1)_{a_1}}{b_1})(b_1+k_1)}{a_1}\right) = \beta_2$.

Moreover,

$$\left.\begin{array}{l}\pi_3(\gamma_t(y)) = \\ \left(\pi_3(y) + E(\tfrac{\pi_2(y)}{b_1})a'_{12} + \alpha'' a_{12} + E\left(\tfrac{\pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+\alpha'' b_{12}}{b_2}\right)(b_2+k_2)\right)_{a_2} \\ = (\beta_4)_{a_2}\end{array}\right\} \quad (14)$$

where

$$\alpha'' = E\left(\tfrac{\pi_1(y)+E(\frac{\pi_2(y)}{b_1})(b_1+k_1)}{a_1}\right) = E(\tfrac{\beta_1}{a_1}).$$

Hence

$$R_3 = \pi_4(\gamma_t(y)) + E(\tfrac{\pi_1(\gamma_t(y))}{b_1})a'_{12} + \beta_2 a_{12} + E\left(\tfrac{\pi_3(\gamma_t(y))+E(\frac{\pi_1(\gamma_t(y))}{b_1})b'_{12}+\beta_2 b_{12}}{b_2}\right)(b_2+k_2)$$
$$= \left(\pi_4(y) + E(\tfrac{\pi_2(y)}{b_1})b'_{12} + E(\tfrac{\beta_1}{a_1})b_{12}\right)_{b_2} + E(\tfrac{(\beta_1)_{a_1}}{b_1})a'_{12} + \beta_2 a_{12} +$$
$$E\left(\tfrac{\pi_3(\gamma_t(y))+E(\frac{(\beta_1)_{a_1}}{b_1})b'_{12}+\beta_2 b_{12}}{b_2}\right)(b_2+k_2)$$
$$\overset{(14)}{=} \left(\pi_4(y) + E(\tfrac{\pi_2(y)}{b_1})b'_{12} + E(\tfrac{\beta_1}{a_1})b_{12}\right)_{b_2} + E(\tfrac{(\beta_1)_{a_1}}{b_1})a'_{12} + \beta_2 a_{12} +$$

$$E\left(\frac{(\beta_4)_{a_2}+E(\frac{(\beta_1)_{a_1}}{b_1})b'_{12}+\beta_2 b_{12}}{b_2}\right)(b_2+k_2).$$

Thus

$R_3 - L_3\left(\pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+E(\frac{\beta_1}{a_1})b_{12}\right)_{b_2} + E(\frac{(\beta_1)_{a_1}}{b_1})a'_{12}+\beta_2 a_{12}+ E\left(\frac{(\beta_4)_{a_2}+E(\frac{(\beta_1)_{a_1}}{b_1})b'_{12}+\beta_2 b_{12}}{b_2}\right)(b_2+k_2) - \left(\pi_4(y)+E(\frac{\pi_1(y)}{b_1})a'_{12}+E(\frac{\beta'_1}{a_1})a_{12}+ E\left(\frac{\pi_3(y)+E(\frac{\pi_1(y)}{b_1})b'_{12}+E(\frac{\beta'_1}{a_1})b_{12}}{b_2}\right)(b_2+k_2)\right) = \pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+E(\frac{\beta_1}{a_1})b_{12} - E\left(\frac{\pi_4(y)+E(\frac{\pi_2(y)}{b_1})b'_{12}+E(\frac{\beta_1}{a_1})b_{12}}{b_2}\right)b_2 + a'_{12}(E(\frac{(\beta_1)_{a_1}}{b_1})-E(\frac{\pi_1(y)}{b_1}))+a_{12}(\beta_2-E(\frac{\beta'_1}{a_1}))+ (b_2+k_2)(E\left(\frac{(\beta_4)_{a_2}+E(\frac{(\beta_1)_{a_1}}{b_1})b'_{12}+\beta_2 b_{12}}{b_2}\right)-E\left(\frac{\pi_3(y)+E(\frac{\pi_1(y)}{b_1})b'_{12}+E(\frac{\beta'_1}{a_1})b_{12}}{b_2}\right))-\pi_4(y) \overset{(11),(12)}{=} E(\frac{\pi_2(y)}{b_1})b'_{12}+E(\frac{\beta_1}{a_1})b_{12}-\beta_3 b_2+a'_{12}((1+\frac{k_1}{b_1})E\left(\frac{\pi_2(y)}{b_1}\right)-\frac{a_1}{b_1}E\left(\frac{\beta_1}{a_1}\right))+ a_{12}(\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}E\left(\frac{\pi_2(y)}{b_1}\right)-(1+\frac{k_1}{b_1})E\left(\frac{\beta_1}{a_1}\right))+(b_2+k_2)(E\left(\frac{(\beta_4)_{a_2}+E(\frac{(\beta_1)_{a_1}}{b_1})b'_{12}+\beta_2 b_{12}}{b_2}\right)- E\left(\frac{\pi_3(y)+E(\frac{\pi_1(y)}{b_1})b'_{12}+E(\frac{\beta'_1}{a_1})b_{12}}{b_2}\right)) \overset{(13)}{=} a'_{12}((1+\frac{k_1}{b_1})E\left(\frac{\pi_2(y)}{b_1}\right)-\frac{a_1}{b_1}E\left(\frac{\beta_1}{a_1}\right))+ a_{12}(\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}E\left(\frac{\pi_2(y)}{b_1}\right)-(1+\frac{k_1}{b_1})E\left(\frac{\beta_1}{a_1}\right))+(b_2+k_2)\left(E(\frac{\pi_2(y)}{b_1})(a'_{12}+(1+\frac{k_1}{b_1})b'_{12}+ \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})+E(\frac{\beta_1}{a_1})(a_{12}-(1+\frac{k_1}{b_1})b_{12}-\frac{a_1}{b_1}b'_{12})+(b_2+k_2)\beta_3-E(\frac{\beta_4}{a_2})a_2\right)\frac{1}{b_2}+ E(\frac{\pi_2(y)}{b_1})b'_{12}+E(\frac{\beta_1}{a_1})b_{12}-\beta_3 b_2 = E(\frac{\pi_2(y)}{b_1})\left(b'_{12}+(1+\frac{k_1}{b_1})a'_{12}+\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}a_{12}+ (1+\frac{k_2}{b_2})(a'_{12}+(1+\frac{k_1}{b_1})b'_{12}+\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})\right)+E(\frac{\beta_1}{a_1})\left(b_{12}-(1+\frac{k_1}{b_1})+a_{12}-\frac{a_1}{b_1}a'_{12}+ (1+\frac{k_2}{b_2})(a_{12}-(1+\frac{k_1}{b_1})b_{12}-\frac{a_1}{b_1}b'_{12})\right)+\beta_3((b_2+k_2)^2\frac{1}{b_2}-b_2)-\frac{b_2+k_2}{b_2}E(\frac{\beta_4}{a_2})a_2 = E(\frac{\pi_2(y)}{b_1})\left(b'_{12}+(1+\frac{k_1}{b_1})a'_{12}+\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}a_{12}+(1+\frac{k_2}{b_2})(a'_{12}(1+\frac{k_1}{b_1})b'_{12}+\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})\right)+ E(\frac{\beta_1}{a_1})\left(b_{12}-(1+\frac{k_1}{b_1})a_{12}-\frac{a_1}{b_1}a'_{12}+(1+\frac{k_2}{b_2})(a_{12}-(1+\frac{k_1}{b_1})b_{12}-\frac{a_1}{b_1}b'_{12})\right)+\beta_3(2k_2+ \frac{k_2^2}{b_2})-(1+\frac{k_2}{b_2})E(\frac{\beta_4}{a_2})a_2$ is divided by a_2 because we have $a_2|(2k_2+\frac{k_2^2}{b_2})$ and $a_2|\left(b'_{12}+(1+\frac{k_1}{b_1})a'_{12}+\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}a_{12}+(1+\frac{k_2}{b_2})(a'_{12}+(1+\frac{k_1}{b_1})b'_{12}+\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})\right)$, $a_2|\left((b_{12}-(1+\frac{k_1}{b_1})a_{12}-\frac{a_1}{b_1}a'_{12}+(1+\frac{k_2}{b_2})(a_{12}-(1+\frac{k_1}{b_1})b_{12}-\frac{a_1}{b_1}b'_{12})\right)$, where two last divisions are consequence of the assumption on $t \in D$.

It remains to show that $b_2|(R_4 - L_4)$:

$$\begin{aligned} R_4 &= \pi_4(\gamma_t(y)^*) + E(\tfrac{\pi_2(\gamma_t(y)^*)}{b_1})b'_{12} + E\left(\frac{\pi_1(\gamma_t(y)^*)+E(\frac{\pi_2(\gamma_t(y)^*)}{b_1})(b_1+k_1)}{a_1}\right)b_{12} \\ &= \pi_3(\gamma_t(y)) + E(\tfrac{\pi_1(\gamma_t(y))}{b_1})b'_{12} + E\left(\frac{\pi_2(\gamma_t(y))+E(\frac{\pi_1(\gamma_t(y))}{b_1})(b_1+k_1)}{a_1}\right)b_{12} \\ &\overset{(14)}{=} (\beta_4)_{a_2} + E(\tfrac{(\beta_1)_{a_1}}{b_1})b'_{12} + E\left(\frac{(\pi_2(y))_{b_1}+E(\frac{(\beta_1)_{a_1}}{b_1})(b_1+k_1)}{a_1}\right)b_{12} \\ &= (\beta_4)_{a_2} + E(\tfrac{(\beta_1)_{a_1}}{b_1})b'_{12} + \beta_2 b_{12}. \end{aligned}$$

$$\begin{aligned} L_4 &= \pi_4(y^*) + E(\tfrac{\pi_2(y^*)}{b_1})b'_{12} + E\left(\frac{\pi_1(y^*)+E(\frac{\pi_2(y^*)}{b_1})(b_1+k_1)}{a_1}\right)b_{12} \\ &= \pi_3(y) + E(\tfrac{\pi_1(y)}{b_1})b'_{12} + E\left(\frac{\pi_2(y)+E(\frac{\pi_1(y)}{b_1})(b_1+k_1)}{a_1}\right)b_{12} \\ &= \pi_3(y) + E(\tfrac{\pi_1(y)}{b_1})b'_{12} + E\left(\frac{\beta'_1}{a_1}\right)b_{12}. \end{aligned}$$

Hence

$$\begin{aligned} R_4 - L_4 &= (\beta_4)_{a_2} + E(\tfrac{(\beta_1)_{a_1}}{b_1})b'_{12} + \beta_2 b_{12} - (\pi_3(y) + E(\tfrac{\pi_1(y)}{b_1})b'_{12} + E\left(\frac{\beta'_1}{a_1}\right)b_{12}) \\ &= \beta_4 - E(\tfrac{\beta_4}{a_2})a_2 - \pi_3(y) + b'_{12}(E(\tfrac{(\beta_1)_{a_1}}{b_1}) - E(\tfrac{\pi_1(y)}{b_1})) + b_{12}(\beta_2 - E(\tfrac{\beta'_1}{a_1})) \\ &\overset{(11),(12)}{=} -E(\tfrac{\beta_4}{a_2})a_2 + \pi_3(y) + E(\tfrac{\pi_2(y)}{b_1})a'_{12} + E(\tfrac{\beta_1}{a_1})a_{12} + \beta_3(b_2+k_2) - \pi_3(y) \\ &\quad + b'_{12}\left((1+\tfrac{k_1}{b_1})E(\tfrac{\pi_2(y)}{b_1}) - \tfrac{a_1}{b_1}E(\tfrac{\beta_1}{a_1})\right) + b_{12}\left(\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}E(\tfrac{\pi_2(y)}{b_1}) - (1+\tfrac{k_1}{b_1})E(\tfrac{\beta_1}{a_1})\right) \\ &= -E(\tfrac{\beta_4}{a_2})a_2 + \beta_3(b_2+k_2) + E(\tfrac{\pi_2(y)}{b_1})(a'_{12} + b'_{12}(1+\tfrac{k_1}{b_1}) + b_{12}\frac{2k_1+\frac{k_1^2}{b_1}}{a_1}) \\ &\quad + E(\tfrac{\beta_1}{a_1})(a_{12} - b'_{12}\tfrac{a_1}{b_1} - b_{12}(1+\tfrac{k_1}{b_1})) \end{aligned}$$

is divided by b_2 because $b_2|a_2$, $b_2|k_2$ and $b_2|(a'_{12} + (1+\frac{k_1}{b_1})b'_{12} + \frac{2k_1+\frac{k_1^2}{b_1}}{a_1}b_{12})$, $b_2|(a_{12} - (1+\frac{k_1}{b_1})b_{12} - \frac{a_1}{b_1}b'_{12})$, where last two divisions are consequence of the assumption on $t \in D$.

Therefore $\gamma_t(y^*) = \gamma_t(\gamma_t(y)^*)$ and the proof of Lemma 2.10 is finished. □

3. Main results

Theorem 3.1. *If $t \in D$, then Q_t is a 2-generated abelian group with involution.*

Proof. Obviously the operation $+_t$ is commutative. We show that $+_t$ is associative:

$$\begin{aligned} x +_t (y +_t z) &= x +_t \gamma_t(y+z) = \gamma_t(x + \gamma_t(y+z)) \overset{(5)}{=} \gamma_t(x + (y+z)) \\ &= \gamma_t((x+y)+z) \overset{(5)}{=} \gamma_t(\gamma_t(x+y)+z)) = \gamma_t((x +_t y)+z) = (x +_t y) +_t z). \end{aligned}$$

If $x \in \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$ then $\gamma_t(x) \overset{(6)}{=} x$, so $x +_t \underline{0} = \gamma_t(x + \underline{0}) = \gamma_t(x) = x$.

Moreover, $x +_t (-_t x) = \gamma_t(x + (-_t x)) = \gamma_t(x + \gamma_t(-x)) \overset{(5)}{=} \gamma_t(x + (-x)) = \gamma_t(\underline{0}) = \underline{0}$.

Hence the group reduct of Q_t is an abelian group.

If $x \in \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$ then $(x^{*_t})^{*_t} = \gamma_t((x^{*_t})^*) = \gamma_t(\gamma_t(x^*)^*) \overset{(10)}{=} \gamma_t((x^*)^*) = \gamma_t(x) \overset{(6)}{=} x$.

If $x, y \in \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$ then $(x +_t y)^{*_t} = \gamma_t((x +_t y)^*) = \gamma_t((\gamma_t(x+y))^*) \overset{(10)}{=} \gamma_t((x+y)^*) = \gamma_t(x^* + y^*) \overset{(5)}{=} \gamma_t(\gamma_t(x^*) + \gamma_t(y^*)) = \gamma_t(x^{*_t} + y^{*_t}) = (x^{*_t} +_t y^{*_t})$. Also $\underline{0}^{*_t} = \gamma_t(\underline{0}^*) = \gamma_t(\underline{0}) = \underline{0}$. Thus Q_t is an abelian group with involution.

Let $x_1 = (1, 0, 0, 0)$ and $x_2 = (0, 0, 1, 0)$, then $Q_t = \langle x_1, x_2 \rangle$ since $(y_1, y_2, y_3, y_4) = y_1 x_1 + y_2 x_1^* + y_3 x_2 + y_4 x_2^*$ for every $(y_1, y_2, y_3, y_4) \in \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$. Therefore Q_t is 2-generated. □

Proposition 3.2. *If $G = \langle x_1, x_2 \rangle$ is a finite abelian group with involution and $\phi\colon \mathbb{Z}^4 \to G$ is such that $\phi(y_1, y_2, y_3, y_4) = y_1 x_1 + y_2 x_1^* + y_3 x_2 + y_4 x_2^*$, then*

$$\gamma_t \phi = \phi$$

for $t = \psi(G, x_1, x_2) = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2)$.

In other words, *if $\underline{Z} = \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$, then the following diagram*

$$\begin{array}{ccc} \mathbb{Z}^4 & \xrightarrow{\ \phi\ } & G \\ \downarrow{\scriptstyle \gamma_t} & & \uparrow{\scriptstyle \phi} \\ \underline{Z} & \hookrightarrow & \mathbb{Z}^4 \end{array}$$

is commutative.

Proof. Let $G = \langle x_1, x_2 \rangle$ be a finite abelian group with involution and $t = \psi(G, x_1, x_2) = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2)$. Then

$$a_1 x_1 = a_{12} x_2 + b_{12} x_2^* \in \langle x_2 \rangle, \tag{15}$$

$$b_1 x_1^* - (b_1 + k_1) x_1 = a'_{12} x_2 + b'_{12} x_2^* \in \langle x_2 \rangle, \tag{16}$$

$$a_2 x_2 = 0, \tag{17}$$

$$b_2 x_2^* = (b_2 + k_2) x_2. \tag{18}$$

For $\alpha_1 = y_1 + E(\frac{y_2}{b_1})(b_1 + k_1)$ and $\beta = y_4 + E(\frac{y_2}{b_1}) b'_{12} + E(\frac{\alpha_1}{a_1}) b_{12}$, where $y = (y_1, y_2, y_3, y_4) \in \mathbb{Z}^4$, we obtain

$$\begin{aligned} \phi(y) &= y_1 x_1 + y_2 x_1^* + y_3 x_2 + y_4 x_2^* = y_1 x_1 + (E(\tfrac{y_2}{b_1}) b_1 + (y_2)_{b_1}) x_1^* + y_3 x_2 + y_4 x_2^* \\ &\overset{(16)}{=} y_1 x_1 + E(\tfrac{y_2}{b_1})((b_1 + k_1) x_1 + a'_{12} x_2 + b'_{12} x_2^*) + (y_2)_{b_1} x_1^* + y_3 x_2 + y_4 x_2^* \\ &= (y_1 + E(\tfrac{y_2}{b_1})(b_1 + k_1)) x_1 + (y_2)_{b_1} x_1^* + (y_3 + E(\tfrac{y_2}{b_1}) a'_{12}) x_2 + (y_4 + E(\tfrac{y_2}{b_1}) b'_{12}) x_2^* \\ &= \alpha_1 x_1 + (y_2)_{b_1} x_1^* + (y_3 + E(\tfrac{y_2}{b_1}) a'_{12}) x_2 + (y_4 + E(\tfrac{y_2}{b_1}) b'_{12}) x_2^* \\ &= (E(\tfrac{\alpha_1}{a_1}) a_1 + (\alpha_1)_{a_1}) x_1 + (y_2)_{b_1} x_1^* + (y_3 + E(\tfrac{y_2}{b_1}) a'_{12}) x_2 + (y_4 + E(\tfrac{y_2}{b_1}) b'_{12}) x_2^* \end{aligned}$$

$$\overset{(15)}{=} E(\tfrac{\alpha_1}{a_1})(a_{12}x_2+b_{12}x_2^*)+(\alpha_1)_{a_1}x_1+(y_2)_{b_1}x_1^*+(y_3+E(\tfrac{y_2}{b_1})a'_{12})x_2+(y_4+E(\tfrac{y_2}{b_1})b'_{12})x_2^*$$
$$= (\alpha_1)_{a_1}x_1+(y_2)_{b_1}x_1^*+(y_3+E(\tfrac{y_2}{b_1})a'_{12}+E(\tfrac{\alpha_1}{a_1})a_{12})x_2+(y_4+E(\tfrac{y_2}{b_1})b'_{12}+E(\tfrac{\alpha_1}{a_1})b_{12})x_2^*$$
$$= (\alpha_1)_{a_1}x_1 + (y_2)_{b_1}x_1^* + (y_3 + E(\tfrac{y_2}{b_1})a'_{12} + E(\tfrac{\alpha_1}{a_1})a_{12})x_2 + \beta x_2^*$$
$$= (\alpha_1)_{a_1}x_1 + (y_2)_{b_1}x_1^* + (y_3 + E(\tfrac{y_2}{b_1})a'_{12} + E(\tfrac{\alpha_1}{a_1})a_{12})x_2 + (E(\tfrac{\beta}{b_2})b_2 + (\beta)_{b_2})x_2^*$$
$$\overset{(18)}{=}(\alpha_1)_{a_1}x_1+(y_2)_{b_1}x_1^*+(y_3+E(\tfrac{y_2}{b_1})a'_{12}+E(\tfrac{\alpha_1}{a_1})a_{12})x_2+E(\tfrac{\beta}{b_2})(b_2+k_2)x_2+(\beta)_{b_2}x_2^*$$
$$= (\alpha_1)_{a_1}x_1 + (y_2)_{b_1}x_1^* + \Big(y_3 + E(\tfrac{y_2}{b_1})a'_{12} + E(\tfrac{\alpha_1}{a_1})a_{12} + E(\tfrac{\beta}{b_2})(b_2+k_2)\Big)x_2 + (\beta)_{b_2}x_2^*$$
$$\overset{(17)}{=}(\alpha_1)_{a_1}x_1+(y_2)_{b_1}x_1^*+\Big(y_3+E(\tfrac{y_2}{b_1})a'_{12}+E(\tfrac{\alpha_1}{a_1})a_{12}+E(\tfrac{\beta}{b_2})(b_2+k_2)\Big)_{a_2}x_2+(\beta)_{b_2}x_2^*$$
$$= \phi(\gamma_t(y)). \qquad \square$$

The main theorem of this paper is formulated in the following way:

Theorem 3.3. *If $G = \langle x_1, x_2\rangle$ is a finite abelian group with involution, then $G \cong Q_t$ for $t = \psi(G, x_1, x_2)$.*

Proof. Indeed, let $t = \psi(G, x_1, x_2) = (a_1, b_1, k_1, a_{12}, b_{12}, a'_{12}, b'_{12}, a_2, b_2, k_2) \in \mathbb{Z}^{10}$ and $\underline{\mathbb{Z}} = \mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1} \times \mathbb{Z}_{a_2} \times \mathbb{Z}_{b_2}$. Consider the map $\phi\colon \mathbb{Z}^4 \to G$ defined by

$$\phi(y_1, y_2, y_3, y_4) = y_1x_1 + y_2x_1^* + y_3x_2 + y_4x_2^*.$$

Then $\phi|_{\underline{\mathbb{Z}}}$ is an isomorphism of Q_t and G.

In fact,

- $\phi|_{\underline{\mathbb{Z}}}$ is onto: if $g \in G$ then there exist $y_1, y_2, y_3, y_4 \in \mathbb{Z}$ such that
  $g = y_1x_1 + y_2x_1^* + y_3x_2 + y_4x_2^* = \phi(y_1, y_2, y_3, y_4) = \phi(\gamma_t(y_1, y_2, y_3, y_4))$ and $\gamma_t(y_1, y_2, y_3, y_4) \in \underline{\mathbb{Z}}$.
- $\phi|_{\underline{\mathbb{Z}}}$ is injective: if $\phi(y_1, y_2, y_3, y_4) = \phi(y'_1, y'_2, y'_3, y'_4)$, then
  $$y_1x_1 + y_2x_1^* + y_3x_2 + y_4x_2^* = y'_1x_1 + y'_2x_1^* + y'_3x_2 + y'_4x_2^*.$$
  Thus $(y_2-y'_2)x_1^* - (y'_1-y_1)x_1 \in \langle x_2\rangle$ and $|y_2-y'_2| \in \mathbb{Z}_{b_1}$, which by definition of b_1, implies $y_2 = y'_2$. Hence $(y'_1 - y_1)x_1 \in \langle x_2\rangle$ and $|y_1 - y'_1| \in \mathbb{Z}_{a_1}$, and by definition of a_1, we have $y_1 = y'_1$. So $y_3x_2 + y_4x_2^* = y'_3x_2 + y'_4x_2^*$ and $(y_4 - y'_4)x_2^* = (y'_3 - y_3)x_2$ and $|y_4 - y'_4| \in \mathbb{Z}_{b_2}$. This by definition of b_2 gives $y_4 = y'_4$. Therefore, $(y'_3 - y_3)x_2 = 0$ and $|y'_3 - y_3| \in \mathbb{Z}_{a_2}$, which by definition of a_2 implies $y_3 = y'_3$. This shows that $\phi|_{\underline{\mathbb{Z}}}$ is injective.
- $\phi|_{\underline{\mathbb{Z}}}$ is a homomorphism since for all $y, y' \in \underline{\mathbb{Z}}$ we have
  $$\phi(y +_t y') = \phi(\gamma_t(y + y')) = \phi(y + y') = \phi(y) + \phi(y').$$

Moreover, for all $(y_1, y_2, y_3, y_4) \in \underline{Z}$ we also have:

$$\begin{aligned}\phi((y_1, y_2, y_3, y_4)^{*_t}) &= \phi(\gamma_t(y_2, y_1, y_4, y_3)) = \phi(y_2, y_1, y_4, y_3) \\ &= y_2x_1 + y_1x_1^* + y_4x_2 + y_3x_2^* = (y_1x_1 + y_2x_1^* + y_3x_2 + y_4x_2^*)^* \\ &= (\phi(y_1, y_2, y_3, y_4))^*. \qquad \square\end{aligned}$$

Corollary 3.4. *G is a 2-generated finite abelian group with involution if and only if $G \cong Q_t$ for some $t \in D$.*

Proof. If G is a 2-generated finite abelian group with involution, then by Theorem 3.3 we have $G \cong Q_t$, where $t = \psi(G, x_1, x_2)$ and $t \in D$ by Proposition 2.7.

The converse statement is a consequence of Theorem 3.1. □

Corollary 3.5. *Q is a 2-generated finite entropic quasigroup with a quasi-identity if and only if $G \cong \Psi(Q_t)$ for some $t \in D$.*

References

[1] **G. Bińczak and J. Kaleta**, *Cyclic entropic quasigroups*, Demonstratio Math. **42** (2009), 269 – 281.

[2] **G. Bińczak and J. Kaleta**, *Finite directly indecomposable monogenic entropic quasigroups with a quasi-identity*, Demonstratio Math. **45** (2012), 519 – 532.

[3] **G. Bińczak and J. Kaleta**, *Some finite directly indecomposable non-monogenic entropic quasigroups with a quasi-identity*, Discuss. Math. General Algebra and Applications **34** (2014), 5 – 26.

[4] **O. Chein, H.O. Pflugfelder, and J.D.H. Smith**, *Quasigroups and Loops: Theory and Applications*, Heldermann Verlag, Berlin, 1990.

[5] **V.J. Havel and A. Vanžurová**, *Medial Quasigroups and Geometry*, Olomouc, 2006.

[6] **J.J. Rotman**, *An Introduction to the Theory of Groups*, Springer-Verlag, New York, 1994.

[7] **J.D.H. Smith**, *Representation Theory of Infinite Groups and Finite Quasigroups*, Université de Montréal, 1986.

[8] **J.D.H. Smith**, *An Introduction to Quasigroups and Their Representations*, Chapman and Hall/CRC, Boca Raton, FL, 2007.

Received July 4, 2014
Revised September 25, 2014

G. Bińczak
Faculty of Mathematics and Information Sciences, Warsaw University of Technology, 00-661 Warsaw, Poland
E-mail: binczak@mini.pw.edu.pl

J. Kaleta
Department of Applied Mathematics, Warsaw University of Agriculture, 02-787 Warsaw, Poland
E-mail: joanna_kaleta@sggw.pl