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On generalized bi-Γ-ideals in Γ-semigroups

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Abstract. We study generalized bi-Γ-ideals, prime, semiprime and irreducible generalized bi-Γ-ideals in Γ-semigroups.

1. Introduction

Let S and Γ be two nonempty sets. Then a triple of the form $(S, \Gamma, \cdot)$ is called a Γ-*semigroup*, where $\cdot$ is a ternary operation $S \times \Gamma \times S \to S$ such that $(x \cdot \alpha \cdot y) \cdot \beta \cdot z = x \cdot \alpha \cdot (y \cdot \beta \cdot z)$ for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$.

We will denote $(S, \Gamma, \cdot)$ by S and $a \cdot \gamma \cdot b$ by $a\gamma b$.

Definition 1.1. A nonempty subset B of S is called

- a *sub*-Γ-*semigroup* of S if $a\gamma b \in B$, for all $a, b \in B$ and $\gamma \in \Gamma$,
- a *generalized bi*-Γ-*ideal* of S if $B\Gamma S\Gamma B \subseteq B$,
- a *bi*-Γ-*ideal* of S if $B\Gamma S\Gamma B \subseteq B$ and $B\Gamma B \subset B$.

A Γ-semigroup S is called a *gb-simple* if it does not contain the proper generalized bi-Γ-ideal.

Definition 1.2. A generalized bi-Γ-ideal B of a Γ-semigroup S is

- *prime* if $B_1\Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$,
- *strongly prime* if $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$,
- *irreducible* if $B_1 \cap B_2 = B$ implies $B_1 = B$ or $B_2 = B$,
- *strongly irreducible* if $B_1 \cap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$

for any generalized bi-Γ-ideals B_1 and B_2 of S.

A quasi Γ-ideal is prime if it is prime as a bi-Γ-ideal.

Definition 1.3. A generalized bi-Γ-ideal B of S is

- *semiprime* if $B_1\Gamma B_1 \subseteq B$ implies that $B_1 \subseteq B$

for any bi-Γ-ideal B_1 of S.

Other definition one can find in [1] and [2].

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2. Properties of generalized bi-Γ-ideals

Lemma 2.1. *The smallest generalized bi-Γ-ideal of a Γ-semigroup S containing a nonempty subset T of S has the form $T \cup T\Gamma S\Gamma T$*

Proof. Let $B = T \cup T\Gamma S\Gamma T$. Then $T \subseteq B$. So,

$$\begin{aligned}
B\Gamma S\Gamma B &= (T \cup T\Gamma S\Gamma T)\Gamma S\Gamma(T \cup T\Gamma S\Gamma T)\\
&\subseteq [T(\Gamma S\Gamma)(T \cup T\Gamma S\Gamma T)] \cup [T\Gamma S\Gamma T(\Gamma S\Gamma)(T \cup T\Gamma S\Gamma T)]\\
&\subseteq [T(\Gamma S\Gamma)T \cup T(\Gamma S\Gamma)T\Gamma S\Gamma T] \cup [T\Gamma S\Gamma T(\Gamma S\Gamma)T \cup T\Gamma S\Gamma T(\Gamma S\Gamma)T\Gamma S\Gamma T]\\
&\subseteq [T\Gamma S\Gamma T \cup T\Gamma S\Gamma T] \cup [T\Gamma S\Gamma T \cup T\Gamma S\Gamma T]\\
&= T\Gamma S\Gamma T \subseteq T \cup T\Gamma S\Gamma T = B.
\end{aligned}$$

Hence $B = T \cup T\Gamma S\Gamma T$ is a generalized bi-Γ-ideal of S.

To prove that B is the smallest generalized bi-Γ-ideal of S containing T suppose that G is a generalized bi-Γ-ideal of S containing T. Then $T\Gamma S\Gamma T \subseteq G\Gamma S\Gamma G \subseteq G$. Therefore, $B = T \cup T\Gamma S\Gamma T \subseteq G$. Hence B is the smallest generalized bi-Γ-ideal of S containing T. □

The smallest generalized bi-Γ-ideal of S containing T will be denoted by (T).

Lemma 2.2. *Suppose that A is a sub-Γ-semigroup of a Γ-semigroup S, $s \in S$ and $(s\Gamma A\Gamma s) \cap A \neq \emptyset$. Then $(s\Gamma A\Gamma s) \cap A$ is a generalized bi-Γ-ideal of A.*

Proof. Indeed,

$$\begin{aligned}
(s\Gamma A\Gamma s \cap A)\Gamma A\Gamma(s\Gamma A\Gamma s \cap A) &\subseteq [(s\Gamma A\Gamma s)\Gamma A \cap A\Gamma A]\Gamma(s\Gamma A\Gamma s \cap A)\\
&\subseteq [(s\Gamma A\Gamma s)\Gamma A \cap A]\Gamma(s\Gamma A\Gamma s \cap A)\\
&\subseteq [[(s\Gamma A\Gamma s\Gamma A)\Gamma(s\Gamma A\Gamma s)] \cap [A\Gamma(s\Gamma A\Gamma s \cap A)]]\\
&\subseteq [(s\Gamma A\Gamma s) \cap (A\Gamma s\Gamma A\Gamma s)] \cap A\\
&\subseteq (s\Gamma A\Gamma s) \cap A.
\end{aligned}$$

Hence $(s\Gamma A\Gamma s) \cap A$ is a generalized bi-Γ-ideal of A. □

Theorem 2.3. *For a Γ-semigroup S the following assertions are equivalent:*

(i) *S is a gb-simple Γ-semigroup,*

(ii) *$s\Gamma S\Gamma s = S$ for all $s \in S$,*

(iii) *$(s) = S$ for all $s \in S$.*

Proof. $(i) \Rightarrow (ii)$. Let S be a gb-simple Γ-semigroup and $s \in S$. Then $s\Gamma S\Gamma s$ is a generalized bi-Γ-ideal of S. As S is a gb-simple Γ-semigroup, $s\Gamma S\Gamma s = S$.

$(ii) \Rightarrow (iii)$. If $s\Gamma S\Gamma s = S$ for all s in S, then, $(s) = \{s\} \cup s\Gamma S\Gamma s = \{s\} \cup S = S$.

$(iii) \Rightarrow (i)$. Let $(s) = S$, for all $s \in S$, and assume B is a generalized bi-Γ-ideal of S and $s \in B$. Then $(s) \subseteq B$. By our hypothesis, we obtain $S = (s) \subseteq B \subseteq S$. So, $S = B$. Hence S is a gb-simple Γ-semigroup. □

Theorem 2.4. *A bi-*Γ*-ideal* B *of a* Γ*-semigroup* S *is a minimal generalized bi-*Γ*-ideal of* S *if and only if* B *is a gb-simple* Γ*-semigroup.*

Proof. Let B be a minimal generalized bi-Γ-ideal of S. By our hypothesis, B is a Γ-semigroup. Suppose D is a generalized bi-Γ-ideal of B. Then $D\Gamma B\Gamma D \subseteq D \subseteq B$. As B is a generalized bi-Γ-ideal of S, we obtain $D\Gamma B\Gamma D$ is a generalized bi-Γ-ideal of S. As B is a minimal generalized bi-Γ-ideal of S, we obtain $D\Gamma B\Gamma D = B$. So, we have $B = D$. Therefore, B is a *gb*-simple Γ-semigroup.

Conversely, let B be a *gb*-simple Γ-semigroup. Suppose D is a generalized bi-Γ-ideal of S so that $D \subseteq B$. Then $D\Gamma B\Gamma D \subseteq D\Gamma S\Gamma D \subseteq D$. So D is a generalized bi-Γ-ideal of B. As B is a *gb*-simple Γ-semigroup, we obtain $B = D$. Hence B is a minimal generalized bi-Γ-ideal of S. □

Theorem 2.5. *Every generalized bi-*Γ*-ideal of a* Γ*-semigroup* S *is a bi-*Γ*-ideal of* S *if and only if* $x\alpha y \in \{x, y\}\Gamma S\Gamma\{x, y\}$*, for every* $x, y \in S$ *and* $\alpha \in \Gamma$*.*

Proof. Suppose S is a Γ-semigroup in which every generalized bi-Γ-ideal is a bi-Γ-ideal. Then, for every $x, y \in S$, the generalized bi-Γ-ideal generated by subset $\{x, y\}$ is given by $\{x, y\} \cup \{x, y\}\Gamma S\Gamma\{x, y\}$ which is a bi-Γ-ideal of S, so we have $x\alpha y \in \{x, y\}\Gamma S\Gamma\{x, y\}$.

Conversely, if x, y are elements of a generalized bi-Γ-ideal B of S, then we have $x\alpha y \in B\Gamma S\Gamma B \subseteq B$. Hence B is a bi-Γ-ideal of S. □

3. Prime and irreducible generalized bi-Γ-ideals

Proposition 3.1. *A semiprime generalized bi-*Γ*-ideal of* S *is a quasi-*Γ*-ideal of* S*.*

Proof. Suppose that B is semiprime and let $x \in (S\Gamma B \cap B\Gamma S)$. Then $x\Gamma S\Gamma x \subseteq (B\Gamma S)\Gamma S\Gamma(S\Gamma B) = B\Gamma S\Gamma B \subseteq B$ and since B is semiprime, we obtain $x \in B$. Hence $B = S\Gamma B \cap B\Gamma S$. □

Proposition 3.2. *A* Γ*-semigroup* S *is regular if and only if every generalized bi-*Γ*-ideal of* S *is semiprime.*

Proof. Let S be regular and suppose that B is any generalized bi-Γ-ideal of S. If $b \notin B$, then $b \in s\Gamma S\Gamma s$, so we obtain $s\Gamma S\Gamma s \nsubseteq B$ and hence B is semiprime. Conversely, if every generalized bi-Γ-ideal of S is semiprime, then so is $B = s\Gamma S\Gamma s$ for any $s \in S$. As $s\Gamma S\Gamma s \subseteq B$, we obtain $b \in B$ and hence S is regular. □

Proposition 3.3. *The intersection of any nonempty family of prime generalized bi-*Γ*-ideals of a* Γ*-semigroup is a semiprime bi-*Γ*-ideal.*

Proof. Suppose that S is a Γ-semigroup and $\mathcal{P} = \{P \mid P$ is a prime generalized bi-Γ-ideal of $S\}$. As $0 \in P$, for all $P \in \mathcal{P}$, we obtain $0 \in \bigcap \mathcal{P}$. Thus $\bigcap \mathcal{P} \neq \emptyset$. Suppose $q \in (\bigcap \mathcal{P})\Gamma S\Gamma(\bigcap \mathcal{P})$. Then $q = q_1\alpha s\beta q_2$, for some $q_1, q_2 \in \bigcap \mathcal{P}, s \in S$

and $\alpha, \beta, \gamma \in \Gamma$. Thus $q = q_1 \alpha s \beta q_2 \in P\Gamma S\Gamma P \subseteq P$, for all $P \in \mathcal{P}$. Therefore, $q \in \bigcap \mathcal{P}$. So $(\bigcap \mathcal{P})\Gamma S\Gamma(\bigcap \mathcal{P}) \subseteq \bigcap \mathcal{P}$. Therefore, $\bigcap \mathcal{P}$ is a generalized bi-Γ-ideal of S. Suppose B be a generalized bi-Γ-ideal of S such that $B^2 \subseteq \bigcap \mathcal{P}$. We have $B^2 \subseteq$P, for all $P \in \mathcal{P}$. As P is a prime generalized bi-Γ-ideal of S, we obtain $B \subseteq P$, for all $P \in \mathcal{P}$. Thus $B \subseteq \bigcap \mathcal{P}$. Hence $\bigcap \mathcal{P}$ is a semiprime generalized bi-Γ-ideal of S. $\square$

Proposition 3.4. *A prime generalized bi-Γ-ideal is a prime one-sided Γ-ideal.*

Proof. Let $S\Gamma B \nsubseteq B$ and $B\Gamma S \nsubseteq B$. Since B is prime, it follows that $B\Gamma S\Gamma B = (B\Gamma S)\Gamma S\Gamma(S\Gamma B) \nsubseteq B$, which is a contradiction. Hence B is a prime one-sided Γ-ideal. $\square$

Corollary 3.5. *A quasi-Γ-ideal of S is a prime one-sided Γ-ideal of S.* $\square$

Proposition 3.6. *A generalized bi-Γ-ideal B of a Γ-semigroup S is prime if and only if $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$, where R and L are right and left Γ-ideal of S.*

Proof. If B is prime and $R\Gamma L \subseteq B$ with $R \nsubseteq B$, then for every $r \in R \setminus B$, $r\Gamma S\Gamma l \subseteq B$, for all $l \in L$, therefore $L \subseteq B$. Conversely, if B is not prime, there exists $a, b \notin B$ such that $a\Gamma S\Gamma b \subseteq B$. But then $(a\Gamma S)\Gamma(S\Gamma b) \subseteq B$ and $a\Gamma S, S\Gamma b \nsubseteq B$. $\square$

Proposition 3.7. *If a bi-Γ-ideal B of S is prime, then*

$$I(B) = \{s \in B \mid S\Gamma s\Gamma S \subseteq B\}$$

is a prime Γ-ideal of S.

Proof. Suppose B is prime and let $J_1\Gamma J_2 \subseteq I(B)$, for two-sided ideals J_1 and J_2. Then, since $J_1\Gamma J_2 \subseteq B$, by Proposition 3.6, $J_1 \subseteq B$ or $J_2 \subseteq B$. Now $I(B)$ is the largest Γ-ideal in B, it follows that $J_1 \subseteq I(B)$ or $J_2 \subseteq I(B)$. $\square$

Theorem 3.8. *Every strongly irreducible, semiprime generalized bi-Γ-ideal of a Γ-semigroup S is a strongly prime generalized bi-Γ-ideal.*

Proof. Let B be a strongly irreducible semiprime generalized bi-Γ-ideal of S. Suppose that B_1, B_2 are generalized bi-Γ-ideals of S such that $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$. As $(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2$ and $(B_1 \cap B_2)^2 \subseteq B_2\Gamma B_1$, it follows that $(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$. As B is semiprime, we obtain $B_1 \cap B_2 \subseteq B$ and since B is strongly irreducible, we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence B is a strongly prime generalized bi-Γ-ideal of S. $\square$

Theorem 3.9. *For any generalized bi-Γ-ideal B of a Γ-semigroup S and any $s \in S \setminus B$ there exists an irreducible generalized bi-Γ-ideal J of S such that $B \subseteq J$ and $s \notin J$.*

Proof. Suppose $GB_B = \{B_1 \mid B_1$ is a generalized bi-Γ-ideal of S and $B \subseteq B_1$ and $s \notin B_1\}$. Obviously, $B \in GB_B$ and so $GB_B \neq \emptyset$. We have GB_B is a partially ordered set under inclusion. Suppose C is a chain of GB_B. Suppose $c \in (\bigcup C)\Gamma S\Gamma(\bigcup C)$. Then $c = c^{'}\alpha s\beta c^{''}$, for some $c^{'}, c^{''} \in \bigcup C$, $s \in S$ and $\alpha, \beta \in \Gamma$. Therefore, $c^{'} \in B_1$ and $c^{''} \in B_2$, for some $B_1, B_2 \in C$. As C is a chain of GB_B, we obtain B_1 and B_2 are comparable. Thus $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$; so $c^{'}, c^{''} \in B_1$ or $c^{'}, c^{''} \in B_2$. As B_1 and B_2 are generalized bi-Γ-ideals of S, it follows that $c = c^{'}\alpha s\beta c^{''} \in B_1\Gamma S\Gamma B_1 \subseteq B_1 \subseteq \bigcup C$ or $c = c^{'}\alpha s\beta c^{''} \in B_2\Gamma S\Gamma B_2 \subseteq B_2 \subseteq \bigcup C$. Therefore, $c \in \bigcup C$, so $\bigcup C$ is a generalized bi-Γ-ideal of S. As $s \notin C$, for all $c \in C$, we obtain $s \notin \bigcup C$. Obviously, $B \subseteq \bigcup C$. Therefore, $\bigcup C \in GB_B$. We have $C \subseteq \bigcup C$, for any $c \in C$. Therefore $\bigcup C$ is an upper bound C in GB_B. By Zorn's Lemma, there exists a maximal element $J \in GB_B$. Therefore, J is a generalized bi-Γ-ideal of S such that $B \subseteq J$ and $b \notin J$. Suppose P and Q are generalized bi-Γ-ideals of S such that $P \cap Q = J$. Let $P \neq J$ and $Q \neq J$. Then $J = P \cap Q \subseteq P$ and $J = P \cap Q \subseteq Q$. So $B \subseteq J \subset P$ and $B \subseteq J \subset Q$. If $s \notin P$, then $C \in GB_B$. This is a contradiction since J is a maximal element of GB_B, therefore $s \in P$. In a similar fashion, we obtain $s \in Q$. Thus $s \in P \cap Q = J$ which is not possible. Therefore, $P = J$ or $Q = J$. Hence J is an irreducible generalized bi-Γ-ideal. □

Theorem 3.10. *For a Γ-semigroup S the following statements are equivalent:*

(*i*) *S is regular and intra-regular Γ-semigroup.*

(*ii*) *$B\Gamma B = B$ for every generalized bi-Γ-ideal B of S.*

(*iii*) *$B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$ for all generalized bi-Γ-ideals B_1 and B_2 of S.*

(*iv*) *Every generalized bi-Γ-ideal of S is semiprime.*

(*v*) *Every proper generalized bi-Γ-ideal B of S is the intersection of irreducible semiprime generalized bi-Γ-ideals of S containing B.*

Proof. It follows by Theorem 3.9 [3]. □

Theorem 3.11. *A generalized bi-Γ-ideal of a regular and intra-regular Γ-semigroup is strongly irreducible if and only if it is strongly prime.*

Proof. Follows by Proposition 3.10 [3]. □

Theorem 3.12. *In a Γ-semigroup S each generalized bi-Γ-ideal is strongly prime if and only if S is regular, intra-regular and the set of generalized bi-Γ-ideals of S is a totally ordered under inclusion.*

Proof. If each generalized bi-Γ-ideal of S be strongly prime, then each generalized bi-Γ-ideal of S is semiprime. Hence, by Theorem 3.10, S is a regular and intra-regular Γ-semigroup. Thus the set of all its generalized bi-Γ-ideals is partially ordered by inclusion. If B_1 and B_2 are generalized bi-Γ-ideals of S, then $B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$, by Theorem 3.10. As $B_1 \cap B_2$ is a strongly prime generalized bi-Γ-ideal, we obtain $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. If $B_1 \subseteq B_1 \cap B_2$, then $B_1 \subseteq B_2$.

If $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$. Thus the set of all generalized bi-Γ-ideals of S is totally ordered by inclusion.

The converse statement is a consequence of Theorem 3.12 in [3]. □

Theorem 3.13. *If the set of all generalized bi-Γ-ideals of a Γ-semigroup S is a totally ordered by inclusion, then S is both regular and intra-regular if and only if each generalized bi-Γ-ideal of S is prime.*

Proof. By Theorem 3.13 in [3], each generalized bi-Γ-ideal of S is prime.

Conversely, if each generalized bi-Γ-ideal of S is prime, then it is semiprime. Theorem 3.10 completes the proof. □

Theorem 3.14. *For a Γ-semigroup S the following statements are equivalent:*

(i) *The set of all generalized bi-Γ-ideals of S is totally ordered by inclusion.*

(ii) *Every generalized bi-Γ-ideal of S is strongly irreducible.*

(iii) *Every generalized bi-Γ-ideal of S is irreducible.*

Proof. $(i) \Rightarrow (ii)$. Let B, B_1, B_2 be generalized bi-Γ-ideals of S such that $B_1 \cap B_2 \subseteq B$. Then by (i) we obtain $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Therefore $B_1 = B_1 \cap B_2 \subseteq B$ or $B_2 = B_1 \cap B_2 \subseteq B$. Hence S is strongly irreducible.

$(ii) \Rightarrow (iii)$. Let B_1, B_2 be generalized bi-Γ-ideals of S such that $B_1 \cap B_2 = B$ for some strongly irreducible generalized bi-Γ-ideal B. Then $B \subseteq B_1$ and $B \subseteq B_2$. By the hypothesis, we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$. So $B_1 = B$ or $B_2 = B$. Hence B is irreducible.

$(iii) \Rightarrow (i)$. Suppose that B_1, B_2 are generalized bi-Γ-ideals of S. Then $B_1 \cap B_2$ also is a generalized bi-Γ-ideal of S and by the assumption, $B_1 = B_1 \cap B_2 \subseteq B_2$ or $B_2 = B_1 \cap B_2 \subseteq B_1$. Therefore $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. This proves (i). □

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