Quasigroups and Related Systems **23** (2015), 175 − 180

The Coxeter group $G^{5,5,12}$

Muhammad Ashiq and *Tahir Imran*

Abstract. The groups $G^{l,m,n}$ are studied extensively by Coxeter. Higman has posed the question that how small l, m, n can be made while maintaining the property that all but finitely many alternating and symmetric groups are quotients of $G^{l,m,n}$. In this paper, by using the coset diagrams, we have proved that for all but finitely many positive integers n either A_n or S_n are quotients of $G^{5,5,12}$.

1. Introduction

The groups $G^{l,m,n}$ are studied by Coxeter. He has defined it as

$$\langle R, S, T : R^l = S^m = T^n = (RS)^2 = (ST)^2 = (TR)^2 = (RST)^2 = 1\rangle$$

in his paper [4] published in 1939. This group can take another presentation

$$\langle x, y, t : x^2 = y^l = t^2 = (xt)^2 = (yt)^2 = (xy)^m = (xyt)^n = 1\rangle$$

by replacing $x = RS$, $y = R$ and $t = ST$. He has revealed that these groups are infinite and insoluble if $\frac{1}{l} + \frac{1}{m} + \frac{1}{m} \leqslant 1$ and are finite or Euclidean triangle group if $\frac{1}{l} + \frac{1}{m} + \frac{1}{m} > 1$, which are soluble. Conder [3] has used coset diagrams to prove the fact that A_n is a Hurwitz group for all $n \geqslant 168$, and for all but 64 integers n in the range $3 \leqslant n \leqslant 167$. He has shown that "all but finitely many positive integers n the alternating group A_n and the symmetric group S_n are homomorphic images of the group $G^{6,6,6}$ having the presentation

$$\langle R, S, T : R^6 = S^6 = T^6 = (RS)^2 = (ST)^2 = (TR)^2 = (RST)^2 = 1\rangle \quad \text{or}$$

$$\langle x, y, t : x^2 = y^6 = t^2 = (xt)^2 = (yt)^2 = (xy)^6 = (xyt)^6 = 1\rangle$$

by replacing $x = RS$, $y = R$ and $t = ST$. As a corollary of the proof it is shown that a similar theorem for the triangle group $\Delta(2, 6, 6)$ is given by the presentation $\langle x, y : x^2 = y^6 = (xy)^6 = 1\rangle$. In [7] it was shown by Q. Mushtaq, M. Ashiq that same result is true for the group $G^{5,5,24}$. It was also explained by Conder [3] that "we lose no generality in assuming that $l \leqslant m \leqslant n$ because $G^{l,m,n}$ is isomorphic to $G^{p,q,r}$ for any rearrangement (p, q, r) of (l, m, n). The

2010 Mathematics Subject Classification: Primary 20F05, Secondary 20G40
Keywords and phrases: Coxeter groups, symmetric and alternating groups, coset diagrams, permutation action, imprimitivity.

group $H = \langle x, y : x^2 = y^l = (xy)^m = 1\rangle$ is of index 1 or 2 in $G^{l,m,n} = \langle x, y, t : x^2 = y^l = t^2 = (xt)^2 = (yt)^2 = (xy)^m = (xyt)^n = 1\rangle$ and is isomorphic to $\Delta(2, l, m; q) = \langle x, y, : x^2 = y^l = (xy)^m = (x^{-1}y^{-1}xy)^q = 1\rangle$ where $q = n$ if n is odd and $q = \frac{p}{2}$ if n is even. It was the question asked by Higman that how small the integers l, m, n can be made while maintaining the property that all but finitely many A_n and S_n are factor groups of $G^{l,m,n}$. In many cases $G^{l,m,n}$ is isomorphic to $PSL(2, q)$ or $PGL(2, q)$ for some prime power q, when l, m, n are small. For all values of n, Coxeter [4] has mentioned that: $G^{5,5,m}$ is trivial when $m = 1$ or 2. $G^{5,5,3}$ is homomorphic to $PSL(2, 5)$.

In this paper, we use pictorial argument to show that alternating groups A_n and symmetric groups S_n of degree n can be obtained as quotients of the group $G^{5,5,12} = \langle x, y, t : x^2 = y^5 = t^2 = (xt)^2 = (yt)^2 = (xy)^5 = (xyt)^{12} = 1\rangle$ for all but finitely many positive integers n.

2. Diagrams for $G^{5,5,12}$

To prove our result for the group $G^{5,5,12}$ we will use coset diagrams as used in [3], [6] and [7]. We also need a method for combination of smaller diagrams in order to make large diagrams of desired type. A coset diagram for $G^{5,5,12}$ with n vertices is the action of its generators on the cosets of some particular subgroup in the usual right representation. Generators x, y, t are used to draw the coset diagrams. The coset diagrams, accredited to Higman show an action of $G^{5,5,12} = \langle x, y, t : x^2 = y^5 = t^2 = (xt)^2 = (yt)^2 = (xy)^5 = (xyt)^{12} = 1\rangle$ on a finite set and defined as follows: Pentagons represents cycles of y and vertices of pentagons permuted anti-clockwise by y. An edge denotes those vertices of x which are interchanged by involution while reflection about vertical line of axis represents action of t. A method is also required to connect smaller diagrams to obtain a larger diagram of same condition.

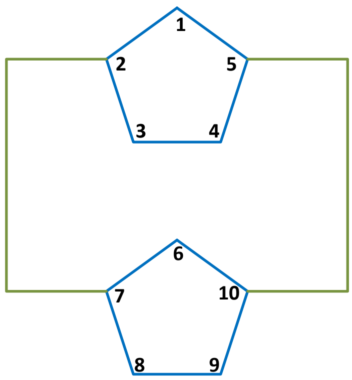

Figure 1: Basic Example

For example, Figure 1 is transitive depiction of $G = \langle x, y, t : x^2 = y^5 = t^2 = (xt)^2 = (yt)^2 = (xy)^5 = 1\rangle$ of degree 10. In this diagram, x act as: (5 6) (2 8) (1) (3) (4) (7) (9) (10); y act as: (1 2 3 4 5) (6 7 8 9 10); t act as: (2 5) (3 4) (6 8) (9 10) (1) (7).

For proof of we will need basic diagrams and portion of a coset diagram for connecting them in different ways and in different numbers. This fragment is known as handle and is denoted as $[A, B]$ as shown in Figure 2 which means a coset diagram containing vertices A and B fixed by x while vertex A is mapped onto B by both y and t and A is fixed by xyt.

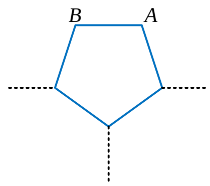

Figure 2: 1-handle

We can combine two coset diagrams namely R and S by placing them one above the other on common vertical axis of symmetry and then by joining them as shown in Figure 3:

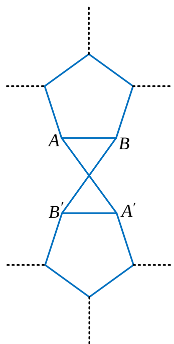

Figure 3: 2-Handle

The diagram thus we get is again a coset diagram for $G^{5,5,12}$ because it satisfies the relation $x^2 = y^5 = t^2 = (xt)^2 = (yt)^2 = 1$. Also, if (A, B, b_1, b_2, b_3) and (A, B, b'_1, b'_2, b'_3) are the suitable 5-cycles of xy then since we have
$(B, B')(A, A')(A, B, b_1, b_2, b_3)(A, B, b'_1, b'_2, b'_3) = (A, B', b_1, b_2, b_3)(A', B, b'_1, b'_2, b'_3)$
the two afterward 5-cycles will be cycles of xy while other cycles remains unaffected in the new diagram. So in resulting diagram xy is of order 5. The cycles ending in A and A' will be juxtaposed to form a single cycle if B and B' will be joined in same way.

3. Jordan's Theorem

Let p be a prime number and G is a primitive group of degree $n = p + k$, $k = 3$. If G contains an element of degree and order p, then G is either alternating or symmetric (Theorem 3.9 [8]).

4. Basic Diagrams

To prove our result, we will need three basic diagrams and fragment of a coset diagram in order to connect them in different ways and in different numbers. Specification is given to each diagram consisting of the degree of the corresponding permutation representation of the group $G^{5,5,12}$, number of handles to be used, parity of action of t and the cycle organization of xyt and xy^2t. Let we have a coset diagram of n vertices denoted by $D(n)$ and here we need copies of three diagrams $D(20), D(21)$ and $D(10)$ for the construction of required diagram of n vertices.

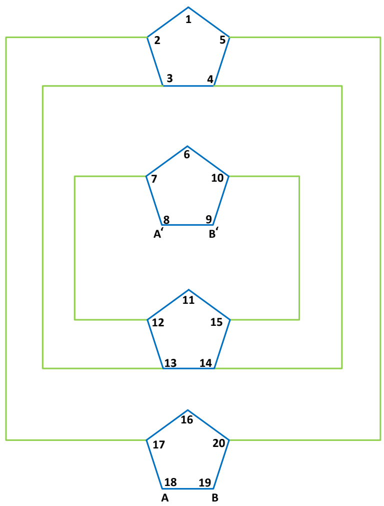

Figure 4: D(20)

Specification of $D(20)$ as in Figure 4: 2(1)-handles: x even, y even, t even. $xyt : AA'(B3B'3)2.4^2$, $xy^2t : (A3A'3)(B1B'1)2^2.4$, which means that on 20 points of the diagram, t is even and xyt has 5 cycles namely two of length 1 corresponding to A and A', one containing both the points B and B' is of length 8, one of length 2 and two of length 4. Similarly xy^2t has five cycles namely one containing points A and A' is of length 8, one of length 2 corresponding to point B, one of length 2 corresponding to point B', two of lengths 2 and one of length 4.

Specification of D(21) as in Figure 5: 1(1)-handle: x even, y even, t even. xyt : $A(B7)2.6.4$, xy^2t : $(A6)(B1)2.5$. which means that on 21 points of the diagram, t is even and xyt has 5 cycles namely one corresponding to A is of length 1, one corresponding to B is having length 8, one of length 2, one of length 4 and one of length 6.

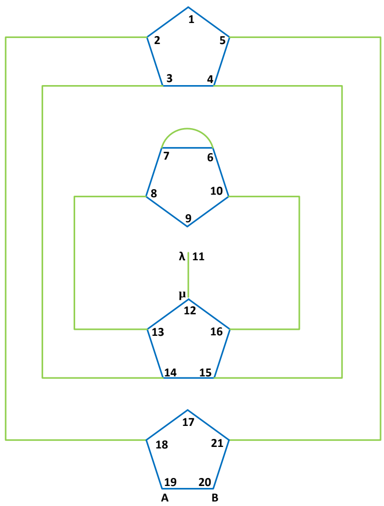

Figure 5: D(21)

Similarly xy^2t has 4 cycles namely one of length 7 containing point A, one of length 2 containing points B two of lengths 5 and one of length 2.

Specification of D(10) as in Figure 6: 2(1)-handles: x even, y even, t even. $xyt : AA'(B1B'1)4$, $xy^2t : (B1)(A1A'1)(B'1).2$.

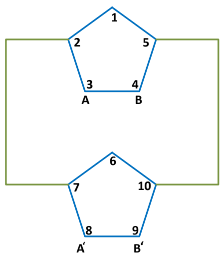

Figure 6: D(10)

Theorem. *For all but finitely many positive integers n, the alternating group A_n and the symmetric group S_n can be obtained as a quotient of the group $G^{5,5,12}$.*

Proof. Take P copies of $D(20)$, Q copies of $D(10)$ and R copies of $D(21)$ and then connect P copies of $D(20)$ with Q copies of $D(10)$ where P, Q are positive integers. Now join R copies of $D(21)$ with $D(10)$ where $R = 1$ or 2. However we cannot connect any copy of $D(21)$ with $D(20)$. The diagram thus obtained will have n vertices and is a coset diagram of group $G^{5,5,12}$. Depending upon the values of P, Q, R reflection t will act as even or odd permutation. The diagram $D(n)$ gives a permutation depiction of group $G^{5,5,12}$ because every cycle of xyt divide 12. Also, we noticed that xy^2t has cycle lengths $2, 4, 5, 7, 8$ with the exception 5, all lengths of xy^2t are divisors of 56, thus the element $(xy^2t)^{56}$ produces exponent of the cycle, fixing the remaining $n-5$ vertices.Now, we have to show that the representation $G^{5,5,12}$ is primitive on the vertices of $D(n)$. By contradiction, suppose that $G^{5,5,12}$ is not primitive then since $(xy^2t)^{12}$ fixes $n-5$ vertices, then 5 vertices of the cycle should lie down in the same area, say Z, of imprimitivety. Amongst the vertices in this cycle are the ones labeled λ and μ . But, x takes λ to μ and μ lies on the vertical line of axis. Thus, Z is conserved by all generators x, y and t. By transitivity it implies that Z has n number of vertices, which is a contradiction to assumption of imprimitivety. Hence, the depiction is primitive. Hence by Jordan's Theorem (Theorem 3.9, [8]), permutation representation is alternating or symmetric of degree n. Thus either A_n or S_n is homomorphic image of $G^{5,5,12}$. Since y and xy is odd order, they yield even permutations and as a result, x will also be even. Depending upon the values of P, Q and R, t is even or odd and in the case t produces an even permutation, we get A_n and if t produces odd permutation, we will get S_n. In either case, A_n is the group $\Delta(2,5,5;6)$ and is of index 2 in the group $G^{5,5,12}$. Hence for all but finitely many positive integers n the alternating group A_n and the symmetric group S_n can be obtained as a quotient of the group $G^{5,5,12}$. □

Corollary. *For all but finitely many positive integers n, the groups A_n and S_n have the presentation $\langle x, y : x^2 = y^5 = (x^{-1}y^{-1}xy)^6 = 1\rangle$.*

References

[1] **W. Burnside**, *Theory of groups of finite order*, Dover publications Inc. (1995).

[2] **A. Cayley**, *The theory of groups: graphical representation*, Amer. J. Math. **1** (1878), 174 − 176.

[3] **M.D.E. Conder**, *On the group $G^{6,6,6}$*, Qurat. J. Math. Oxford **39** (1988), 175 − 183.

[4] **H.S.M. Coxeter**, *The abstract groups $G^{m,n,p}$*, Trans. Amer. Math. Soc. **45** (1939), 73 − 150.

[5] **H.S.M. Coxeter and W.O.J Moser**, *Generators and relations for discrete groups*, Springer-Verlag, Berlin (1980).

[6] **Q. Mushtaq, M. Ashiq and T. Maqsood**, *Factor groups of $G^{5,5,12}$*, Commun. Algebra **12** (1992), 3759 − 3767.

[7] **Q. Mushtaq, M. Ashiq, T. Maqsood and M. Aslam**, *The Coxeter group $G^{5,5,24}$*, Commun. Algebra **30** (2002), 2175 − 2182.

[8] **W.W. Stothers**, *Subgroup of the $(2,3,7)$ triangle group*, Manuscripta Math. **20** (1997), 323 − 334.

[9] **H. Wielandt**, *Finite permutation groups*, Academic Press, London (1964).

Received March 15, 2014

M. Ashiq
Department of Mathematics, National University of Sciences and Technology (NUST), MCS Campus, Rawalpindi, Pakistan
E-mail: ashiqjaved@yahoo.co.uk

T. Imran
Riphah International University, Islamabad, Pakistan
E-mail: tahirimran_78@yahoo.com