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Endotypes of equivalence relations

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Abstract. We classify all equivalence relations according to their type of an endomorphism (i.e., endotype).

1. Introduction

Semigroups of endomorphisms of relational systems have been studied by numerous authors. The main problems that have been investigated recently are the definability of relational systems by their semigroups of endomorphisms, the description of abstract characteristics and representations (see, [1], [2], [11]), the investigation of algebraic and combinatorial properties of monoids of endomorphisms [6], [7]. The basic concept which is studied in this paper is the notion of an endotype of a relational system. The notion of an endotype for a symmetric binary relation was introduced in [3] by means of interrelations between sets of six types of endomorphisms. Later, the notion of an endotype was defined for relations of any arity [8]. Using this concept all relations from a given class can be classified according to their type of an endomorphism. Thus, in [5] endotypes of generalized polygons were found and in [4] all values of an endotype for graphs of a complement of an arbitrary finite path were described. Endotypes of graphs of N-prisms have been calculated in [9]. Endomorphisms of relations of an equivalence have been described in [10]. Here we describe all endotypes of equivalence relations.

The paper is organized as follows. In Section 2, we give definitions of six types of endomorphisms of an arbitrary binary relation and their examples. In Section 3, we present all possible values of an endotype of equivalence relations.

2. Types of endomorphisms

Let X be an arbitrary set and $\Im(X)$ be the symmetric semigroup on X.

Definition 2.1. The transformation $f \in \Im(X)$ is called an *endomorphism* of a relation $\rho \subseteq X \times X$ if for all $a, b \in X$ the condition $(a; b) \in \rho$ implies $(af; bf) \in \rho$.

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The set of all endomorphisms of ρ is a semigroup with respect to the ordinary composition of transformations. This semigroup is called an *endomorphism semigroup* of a relation ρ and is denoted by $End(X,\rho)$.

Definition 2.2. The endomorphism $f \in End(X,\rho)$ is called a *half-strong endomorphism* if for all $a, b \in X$ the condition $(af; bf) \in \rho$ implies the existence of preimages $a' \in aff^{-1}$ and $b' \in bff^{-1}$ such that $(a'; b') \in \rho$.

The set of all half-strong endomorphisms of ρ we denote by $HEnd(X,\rho)$. It is known [3] that $HEnd(X,\rho)$ is not a semigroup in the general case.

Definition 2.3. The endomorphism $f \in End(X,\rho)$ is called a *locally strong endomorphism* if for all $a, b \in X$ the condition $(af; bf) \in \rho$ implies that for every $a' \in aff^{-1}$ there exists $b' \in bff^{-1}$ such that $(a'; b') \in \rho$ and analogously for all preimages of bf.

We denote the set of all locally strong endomorphisms of ρ by $LEnd(X,\rho)$. Observe that generally $LEnd(X,\rho)$ does not form a semigroup.

Definition 2.4. The endomorphism $f \in End(X,\rho)$ is called a *quasi-strong endomorphism* if for all $a, b \in X$ the condition $(af; bf) \in \rho$ implies that there exists $a' \in aff^{-1}$ such that for every $b' \in bff^{-1}$ we have $(a'; b') \in \rho$ and analogously for a suitable preimage of bf.

By $QEnd(X,\rho)$ we denote the set of all quasi-strong endomorphisms of ρ. Note that $QEnd(X,\rho)$ is not a semigroup in general.

Definition 2.5. The endomorphism $f \in End(X,\rho)$ is called a *strong endomorphism* if for all $a, b \in X$ the condition $(af; bf) \in \rho$ implies that $(a; b) \in \rho$.

We denote the set of all strong endomorphisms of ρ by $SEnd(X,\rho)$. It is not hard to check that $SEnd(X,\rho)$ is a subsemigroup of $End(X,\rho)$.

Definition 2.6. The endomorphism $f \in End(X,\rho)$ is called an *automorphism* if f is bijective and f^{-1} is an endomorphism of ρ.

The set of all automorphisms of ρ we denote by $Aut(X,\rho)$. Obviously, $Aut(X,\rho)$ is a subgroup of $End(X,\rho)$ for any relation $\rho \subseteq X \times X$.

Example 2.7. For the set $X = \{a, b, c, d, e\}$, let

$$\mu = \{(a;b), (b;a), (a;a), (b;b), (b;d), (d;e), (e;e)\} \quad \text{and}$$

$$\sigma = \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}, \{b,d\}, \{d\}\}.$$

Then

$$\begin{pmatrix} a & b & c & d & e \\ a & a & d & a & b \end{pmatrix} \in End(X,\mu) \backslash HEnd(X,\mu),$$

$$\begin{pmatrix} a & b & c & d & e \\ b & b & e & d & e \end{pmatrix} \in HEnd(X,\mu)\backslash LEnd(X,\mu),$$

$$\begin{pmatrix} a & b & c & d & e \\ a & a & c & a & a \end{pmatrix} \in LEnd(X,\mu)\backslash QEnd(X,\mu),$$

$$\begin{pmatrix} a & b & c & d & e \\ c & d & c & d & e \end{pmatrix} \in QEnd(X,\sigma)\backslash SEnd(X,\sigma),$$

$$\begin{pmatrix} a & b & c & d & e \\ c & b & c & d & e \end{pmatrix} \in SEnd(X,\sigma)\backslash Aut(X,\sigma),$$

$$\begin{pmatrix} a & b & c & d & e \\ c & b & a & d & e \end{pmatrix} \in Aut(X,\sigma). \qquad \square$$

3. Endotypes of equivalence relations

A relation $\rho \subseteq X \times X$ is called *trivial* if $\rho = i_X = \{(x;x)|x \in X\}$ or $\rho = \omega_X = X \times X$.

Let $Eq(X)$ be the set of all equivalence relations on X and $\alpha \in Eq(X)$. We denote the quotient set of X relative to α by X/α and the equivalence class which contains $x \in X$ by $\overline{x}$.

Lemma 3.1. *The endomorphism f of an equivalence relation $\alpha \in Eq(X)$ is a quasi-strong endomorphism if and only if $f \in SEnd(X,\alpha)$.*

Proof. Assume that $f \in QEnd(X,\alpha)$ and $(af;bf) \in \alpha$ for some $a, b \in X$. Since f is quasi-strong, then there exists $a' \in aff^{-1}$ which is adjacent to every element of bff^{-1} and analogously for a suitable preimage $b' \in bff^{-1}$. From here $bff^{-1} \subseteq \overline{a'}$ and $aff^{-1} \subseteq \overline{b'}$, in addition $(a';b') \in \alpha$. Therefore, $\overline{a} = \overline{b}$ and so $f \in SEnd(X,\alpha)$. The converse statement is obvious. $\square$

For an arbitrary binary relation ρ on a set X the following inclusions of sets of endomorphisms hold:

$$End(X,\rho) \supseteq HEnd(X,\rho) \supseteq LEnd(X,\rho) \supseteq QEnd(X,\rho) \supseteq SEnd(X,\rho) \supseteq Aut(X,\rho).$$

With this sequence we associate the sequence of number $(s_1, s_2, s_3, s_4, s_5)$, where $s_i \in \{0,1\}, i \in \{1,\dots,5\}$. Here 1 stands $\neq$ and 0 stands $=$ at the respective position in the above sequence for endomorphisms. For example, $s_2 = 0$ indicates that $HEnd(X,\rho) = LEnd(X,\rho)$, and $s_5 = 1$ means that $SEnd(X,\rho) \neq Aut(X,\rho)$.

The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called the *endotype* (or the *endomorphism type*) of ρ and is denoted by $Endotype(X,\rho)$.

The main result of the paper is the following theorem:

Theorem 3.2. *For any equivalence α on a set X, we have*

$$Endotype(X,\alpha) = \begin{cases} 0, & |X| = 1, \\ 4, & 2 \leq |X| < \infty,\ \alpha = i_X, \\ 16, & 2 \leqslant |X|,\ \alpha = \omega_X, \\ 20, & |X| = \infty,\ \alpha = i_X, \\ 23, & \alpha \neq i_X, \alpha \neq \omega_X. \end{cases}$$

Proof. 1) For a singleton set X, the set $Eq(X)$ consists of the trivial equivalence $\alpha = i_X = \omega_X$, in addition, obviously $End(X,\alpha) = Aut(X,\alpha)$ and therefore, $Endotype(X,\alpha) = \sum_{i=1}^{5} 0 \cdot 2^{i-1} = 0$.

2) Let X be a finite set such that $|X| \geqslant 2$ and $\alpha = i_X$. Assume that $\xi \in SEnd(X,\alpha)$ and $a\xi = b\xi$ for some $a, b \in X$. Then the condition $(a\xi; b\xi) \in \alpha$ implies $(a; b) \in \alpha$, so $a = b$. Since X is finite, then ξ is a bijection. Therefore, $SEnd(X,\alpha) = Aut(X,\alpha)$.

Define a transformation f of X as follows: $Xf = \{a\}$, $a \in X$. A direct verification shows that $f \in LEnd(X,\alpha)$. In addition, $f \notin QEnd(X,\alpha)$ since there is no $x \in X$ such that $(x; y) \in \alpha$ for all $y \in X$. Thus, $LEnd(X,\alpha) \neq QEnd(X,\alpha)$.

It is clear that $End(X,\alpha) = LEnd(X,\alpha)$ and so,

$$Endotype(X,\alpha) = 2^2 = 4.$$

3) Let $\alpha = \omega_X$, $|X| \geqslant 2$. In this case we have $End(X,\alpha) = SEnd(X,\alpha) = \Im(X)$ and $Aut(X,\alpha) = S(X)$, where $S(X)$ is the symmetric group on X. Thus, $SEnd(X,\alpha) \neq Aut(X,\alpha)$ and we obtain

$$Endotype(X,\alpha) = 2^4 = 16.$$

4) Let X be an infinite set and $\alpha = i_X$. In contrast to item 2) the equality $SEnd(X,\alpha) = Aut(X,\alpha)$ does not hold. An example of a suitable endomorphism from $SEnd(X,\alpha) \backslash Aut(X,\alpha)$ is any injection $g \in \Im(X)$ which is not a surjection. In this case, we have

$$End(X,\alpha) = HEnd(X,\alpha) = LEnd(X,\alpha) \supset QEnd(X,\alpha) = SEnd(X,\alpha) \supset Aut(X,\alpha),$$

therefore

$$Endotype(X,\alpha) = 2^2 + 2^4 = 20.$$

5) Let $\alpha \in Eq(X)$ be a nontrivial equivalence relation. Then $|X| \geqslant 3, |X/\alpha| \geqslant 2$ and there exists at least one class $A \in X/\alpha$ such that $|A| \geqslant 2$. Assume $\varphi \in \Im(X)$ such that $A\varphi = \{a\}, a \in A$ and for all $B \in X/\alpha, B \neq A$ the restriction $\varphi|_B$ is the identity transformation of B. Obviously, φ is not a bijection. Besides, for all $x, y \in X$ we obtain $(x; y) \in \alpha$ if and only if $x\varphi, y\varphi \in B$ for suitable $B \in X/\alpha$. This means that $\varphi \in SEnd(X,\alpha)$, but $\varphi \notin Aut(X,\alpha)$.

Further, take different $a, b \in X$ such that $(a; b) \in \alpha$ and define a transformation ψ of X as follows: $X\psi = \{a\}$. Similarly as in item 2) of this proof, it can be shown that $\psi \in LEnd(X, \alpha) \setminus QEnd(X, \alpha)$.

Now we consider a full transformation g of X such that

$$\overline{x}g = \begin{cases} \{a, b\}, & \text{if } \overline{x} = \overline{a}, \\ \{b\} & \text{otherwise} \end{cases}$$

for all $\overline{x} \in X/\alpha$. It is not hard to check that $g \in HEnd(X, \alpha)$. Since for every $b' \in bg^{-1} \setminus \overline{a}$ there is no $a' \in ag^{-1}$ such that $(a'; b') \in \alpha$, then $g \notin LEnd(X, \alpha)$. From here $HEnd(X, \alpha) \neq LEnd(X, \alpha)$.

Finally, define $h \in \Im(X)$ putting for all $x \in X$:

$$xh = \begin{cases} a, & \text{if } x \in \overline{a}, \\ b & \text{otherwise.} \end{cases}$$

It is clear that $h \in End(X, \alpha)$. Moreover, for all $x \in ah^{-1}, y \in bh^{-1}$ we have $(x; y) \notin \alpha$, therefore $h \notin HEnd(X, \alpha)$. From here $End(X, \alpha) \neq HEnd(X, \alpha)$ and then we obtain the following sequence of inclusions:

$$End(X, \alpha) \supset HEnd(X, \alpha) \supset LEnd(X, \alpha) \supset QEnd(X, \alpha) = SEnd(X, \alpha) \supset Aut(X, \alpha).$$

Thus, $Endotype(X, \alpha) = 1 + 2 + 4 + 16 = 23$. $\square$

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