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On right weakly regular ordered ternary semigroups

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Abstract. The concept of fuzzy subsemigroups and various types of fuzzy ideals of an ordered ternary semigroup is introduced and properties of such fuzzy sets are investigated. Characterizations of right weakly regular ordered ternary semigroups by fuzzy ideals and fuzzy quasi-ideals are presented.

1. Introduction

The concept of fuzzy subset was initiated by Zadeh [16]. After the introduction of fuzzy subsets, it has found many applications in the field of mathematics and related areas. The study of fuzzy algebraic strutures started with the introduction of the concepts of fuzzy subgroup in the pioneering Rosenfeld's paper [12]. Kuroki [7, 8] introduced and studied fuzzy (left, right) ideals in semigroups. Next Kehayopulu and Tsingelis [4] introduced and studied fuzzy bi-ideals in ordered semigroups. The same authors characterized in [5] intra-regular ordered semigroups in terms of fuzzy sets.

Investigations of ideals in ternary semigroups was initiated by Sioson [15]. He also introduced the notion of regular ternary semigroup and characterized them by using the notion of quasi-ideals. In [3] regular ternary semigroups (n-ary semigroups also) were characterized by principal ideals. Applications of such ideals to divisibility in n-ary semigroups one can find in [2], their connections with congruences on ternary semigroups are described in [14]. Orthodox ternary semigroups are characterized in [13].

Quasi-ideals and bi-ideals in ternary semigroups were discussed by Dixit and Devan in [1]. Rehman and Shabir [11] gave some characterizations of weakly regular ternary semigroups by $(\bar{\alpha}, \bar{\alpha} \vee \bar{q})$-fuzzy subsets.

In this paper, fuzzy left (right, lateral) ideals, fuzzy quasi-ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals in ordered ternary semigroups and several related properties are investigated. A weakly regular ordered ternary semigroups are characterized by the properties of these ideals.

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2. Preliminary notes

A *ternary semigroup* is an algebraic structure $(S, [\ ])$ such that S is a non-empty set and $[\ ] : S \times S \times S \to S$ a ternary operation satisfying associative law, that is,

$$[[a,b,c],d,e] = [a,[b,c,d],e] = [a,b,[c,d,e]]$$

for all $a,b,c,d,e \in S$. Further, for simplicity, we write abc instead of $[a,b,c]$.

An *ordered ternary semigroup* is an algebraic structure $(S, [\ ], \leqslant)$ such that $(S, [\ ])$ is a ternary semigroup and $(S, \leqslant)$ is a partially ordered set satisfying the following conditions:

$$a \leqslant b \Rightarrow acd \leqslant bcd \text{ and } cda \leqslant cdb \text{ and } cad \leqslant cbd$$

for all $a,b,c,d \in S$.

A non-empty subset A of an ordered ternary semigroup S is called a *left* (*right, lateral*) *ideal* of S if

(1) $SSA \subseteq A$ ($ASS \subseteq A$, $SAS \subseteq A$, respectively) and

(2) if $a \in A$ and $b \in S$ such that $b \leqslant a$, then $b \in A$.

If A is a left, right and lateral ideal of S, then it is called an *ideal* of S.

For a subset A of S, we denote by $(A]$ the subset of S defined by

$$(A] = \{t \in S \mid t \leqslant a \text{ for some } a \in A\}.$$

For subsets A, B, C of S, we have

(1) $A \subseteq (A]$,

(2) if $A \subseteq B$, then $(A] \subseteq (B]$,

(3) $(A](B](C] \subseteq (ABC]$,

(4) $((A]] = (A]$.

A non-empty subset A of S is called a *quasi-ideal* of S if

(1) $(ASS] \cap (SAS] \cap (SSA] \subseteq A$,

(2) $(ASS] \cap (SSASS] \cap (SSA] \subseteq A$,

(3) if $a \in A$ and $b \in S$ such that $b \leqslant a$, then $b \in A$.

A is called a *bi-ideal* of S if

(1) $AAA \subseteq A$,

(2) $ASASA \subseteq A$,

(3) if $a \in A$ and $b \in S$ such that $b \leqslant a$, then $b \in A$.

A non-empty subset A of S is called a *generalized bi-ideal* of S if

(1) $ASASA \subseteq A$,

(2) if $a \in A$ and $b \in S$ such that $b \leqslant a$, then $b \in A$.

The intersection of all the left ideals of S containing $a \in S$ is the smallest left ideal of S containing a. It is denoted by $L(a)$ and called the *left ideal generated by* a. Clearly $L(a) = (a \cup SSa]$.

Similarly, $R(a) = (a \cup aSS]$, $I(a) = (a \cup aSS \cup SSa \cup SSaSS,]$ and $B(a) = (a \cup SaSaS]$ are the right ideal, ideal and bi-ideal of S generated by a, respectively.

A fuzzy subset f of an ordered ternary semigroup S, i.e., the mapping from S to the interval $[0,1]$, is called *fuzzy left* (resp. *right, lateral*) *ideal* of S if

(1) $f(abc) \geqslant f(c)$ (resp. $f(abc) \geqslant f(a)$, $f(abc) \geqslant f(b)$) for all $a, b, c \in S$ and

(2) if $a \leqslant b$, then $f(a) \geqslant f(b)$ for all $a, b \in S$.

Let f and g be two fuzzy subsets of S, we define the relation $\subseteq$ between f and g, the union and the intersection of f and g, respectively, as

$$f \subseteq g \text{ if } f(x) \leqslant g(x),$$

$$(f \cup g)(x) = \max\{f(x), g(x)\},$$

$$(f \cap g)(x) = \min\{f(x), g(x)\},$$

for all $x \in S$.

Let a be an element of an ordered ternary semigroup S. Then we define the new set

$$A_a := \{(x, y, z) \in S \times S \times S \mid a \leqslant xyz\}.$$

The *product* of three fuzzy subsets f, g, h of a ternary semigroup S is defined in the following way:

$$(f \circ g \circ h)(a) = \begin{cases} \sup\limits_{(u,v,w) \in A_a} \{\min\{f(u), g(v), h(w)\}\} & \text{if } A_a \neq \emptyset, \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

3. Weakly regular ordered ternary semigroups

In this section we characterize right weakly regular ordered ternary semigroups in terms of fuzzy subsets.

Definition 3.1. An ordered ternary semigroup S is said to be *right* (*left*) *weakly regular*, if for each $x \in S$ we have $x \in ((xSS)^3]$ (resp. $x \in ((SSx)^3]$).

Every regular ordered ternary semigroup induced by ordered semigroup is right (left) weakly regular but the converse is not true.

As a simple consequence of the transfer principle for fuzzy sets (see [6]) we obtain

Lemma 3.2. *Let S be an ordered ternary semigroup and $\emptyset \neq R \subset S$. Then R is a right ideal* (*resp., left ideal, lateral ideal, two-sided ideal, ideal*) *of S if and only if its characteristic function f_R is a fuzzy right ideal* (*resp., fuzzy left ideal, fuzzy lateral ideal, fuzzy two-sided ideal, fuzzy ideal*) *of S.* □

Lemma 3.3. *For an ordered ternary semigroup S, the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *$R_1 \cap R_2 \cap R_3 \subseteq (R_1R_2R_3]$ for every right ideals of S.*

Proof. (1) $\Rightarrow$ (2). Let R_1, R_2 and R_3 be right ideals of S and $a \in R_1 \cap R_2 \cap R_3$. Since S is right weakly regular, $a \leqslant (as_1t_1)(as_2t_2)(as_3t_3) \in (R_1SS)(R_2SS)(R_3SS) \subseteq R_1R_2R_3$. That is $a \in (R_1R_2R_3]$.

(2) $\Rightarrow$ (1). Let $a \in S$. Since $a \in R(a) = (a \cup aSS]$ and $R(a)$ is a right ideal of S, by assumption we have $a \in (R(a)R(a)R(a)] \subseteq (aaa \cup aaaSS \cup aaSSa \cup aaSSaSS \cup aSSaa \cup aSSaaSS \cup aSSaSSa \cup aSSaSSaSS]$. All that cases we have $a \in ((aSS)^3]$, so S is right weakly regular. □

Lemma 3.4. *Let S be an ordered ternary semigroup and f a fuzzy subset of S. Let R_1, R_2 and R_3 be right ideals of S. Then the following conditions are equivalent:*

(1) *$f_{R_1} \cap f_{R_2} \cap f_{R_3} \subseteq f_{R_1} \circ f_{R_2} \circ f_{R_3}$,*

(2) *$R_1 \cap R_2 \cap R_3 \subseteq (R_1R_2R_3]$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R_1 \cap R_2 \cap R_3$. By assumption, $1 = (f_{R_1} \cap f_{R_2} \cap f_{R_3})(a) \subseteq (f_{R_1} \circ f_{R_2} \circ f_{R_3})(a)$. That is $1 = (f_{R_1} \circ f_{R_2} \circ f_{R_3})(a)$. It follows that $a \leqslant u_1u_2u_3$ where $u_i \in R_i$ and $i \in \{1, 2, 3\}$. Hence $a \in (R_1R_2R_3]$.

(2) $\Rightarrow$ (1). Let $a \in S$. If $a \in R_1 \cap R_2 \cap R_3$. Then $(f_{R_1} \cap f_{R_2} \cap f_{R_3})(a) = 1$ and by assumption we have $a \leqslant r_1r_2r_3$ for some $r_i \in R_i$ where $i \in \{1, 2, 3\}$. That is $A_a \neq \emptyset$. Hence

$$\begin{aligned}(f_{R_1} \circ f_{R_2} \circ f_{R_3})(a) &= \sup_{(u,v,w)\in A_a} \{\min\{f_{R_1}(u), f_{R_2}(v), f_{R_3}(w)\}\} \\ &\geqslant \min\{f_{R_1}(r_1), f_{R_2}(r_2), f_{R_3}(r_3)\} = 1.\end{aligned}$$

Thus $(f_{R_1} \circ f_{R_2} \circ f_{R_3})(a) = 1$. If there is some $i \in \{1, 2, 3\}$ such that $a \notin R_i$, then $(f_{R_1} \cap f_{R_2} \cap f_{R_3})(a) = 0$. Therefore we have $f_{R_1} \cap f_{R_2} \cap f_{R_3} \subseteq f_{R_1} \circ f_{R_2} \circ f_{R_3}$. □

Theorem 3.5. *For an ordered ternary semigroup S, the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *$f \cap g \cap h \subseteq f \circ g \circ h$ for all fuzzy right ideals of S.*

Proof. (1) $\Rightarrow$ (2). Let f, g and h be fuzzy right ideals of S and $a \in S$. Since S is right weakly regular, $a \leqslant (ax_1t_1)(ax_2t_2)(ax_3t_3)$ for some $x_i, t_i \in S$ where $i \in \{1, 2, 3\}$. This shows $A_a \neq \emptyset$, so we have

$$\begin{aligned}(f \circ g \circ h)(a) &= \sup_{(u,v,w)\in A_a} \{\min\{f(u), g(v), h(w)\}\} \\ &\geq \min\{f(ax_1t_1), g(ax_2t_2), h(ax_3t_3)\} \\ &\geq \min\{f(a), g(a), h(a)\} \\ &= (f \cap g \cap h)(a).\end{aligned}$$

Thus $f \cap g \cap h \subseteq f \circ g \circ h$.

(2) $\Rightarrow$ (1). Let $a \in S$ and R_1, R_2 and R_3 be right ideals of S. Then by Lemma 3.2, f_{R_1}, f_{R_2} and f_{R_3} are fuzzy right ideals of S. Thus by hypothesis we have $f_{R_1} \cap f_{R_2} \cap f_{R_3} \subseteq f_{R_1} \circ f_{R_2} \circ f_{R_3}$. By Lemma 3.4 and Lemma 3.3, S is a right weakly regular. □

Corollary 3.6. *For an ordered ternary semigroup S, the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *$(f \cap g \cap h) \subseteq (f \circ g \circ h) \cap (g \circ h \circ f) \cap (h \circ f \circ g)$ for fuzzy right ideals of S.*

Proof. (1) $\Rightarrow$ (2). Let f, g and h be fuzzy right ideals of S and $a \in S$. Since S is right weakly regular, $a \leqslant (as_1t_1)(as_2t_2)(as_3t_3)$ for some u_i and t_i where $i \in \{1, 2, 3\}$. That is $A_a \neq \emptyset$. Hence

$$\begin{aligned}(f \circ g \circ h)(a) &= \sup_{(r,s,t) \in A_a} \{\min\{f(r), g(s), h(t)\}\} \\ &\geq \min\{f(as_1t_1), g(as_2t_2), h(as_3t_3)\} \\ &\geq \min\{f(a), g(a), h(a)\} \\ &= (f \cap g \cap h)(a).\end{aligned}$$

So, $f \cap g \cap h \subseteq f \circ g \circ h$. Similarly, we have $g \cap h \cap f \subseteq f \circ g \circ h$ and $h \cap f \cap g \subseteq f \circ g \circ h$. Thus $f \cap g \cap h \subseteq (f \circ g \circ h) \cap (g \circ h \circ f) \cap (h \circ f \circ g)$.

(2) $\Rightarrow$ (1). Let $a \in S$. Since $(f \circ g \circ h) \cap (g \circ h \circ f) \cap (h \circ f \circ g) \subseteq f \circ g \circ h$, by assumption we have $f \cap g \cap h \subseteq (f \circ g \circ h)$. Thus by Theorem 3.5, S is right weakly regular. □

Lemma 3.7. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *$R \cap I = (RII]$ for every right ideal R and every two-sided ideal I of S.*

Proof. (1)$\Rightarrow$(2). Let R be a right ideal and I a two-sided ideal of S. By Lemma 3.3, we have $R \cap I \subseteq (RII]$. Since $(RII] \subseteq (R] = R$ and $(RII] \subseteq (I] = I$. Then $(RII] \subseteq R \cap I$. Thus $R \cap I = (RII]$.

(2)$\Rightarrow$(1). It is a consequence of Lemma 3.3. □

Lemma 3.8. *Let S be an ordered ternary semigroup and f be a fuzzy subset of S. Let R and I be a right ideal and a two-sided ideal of S, respectively. Then the following conditions are equivalent:*

(1) *$f_R \cap f_I = f_R \circ f_I \circ f_I$,*

(2) *$R \cap I = (RII]$.*

Proof. (1) $\Rightarrow$ (2). By Lemma 3.4 we have $R \cap I \subseteq (RII]$. Let $a \in (RII]$. Then $a \leqslant u_1u_2u_3$ for some $u_1 \in R$ and $u_2, u_3 \in I$. It follows that $(f_R \circ f_I \circ f_I)(a) = 1 = (f_R \cap f_I)(a)$. That is $a \in R \cap I$.

(2) $\Rightarrow$ (1). Let $a \in S$. If $a \in R \cap I$, then $(f_I \cap f_R)(a) = 1$. And since $R \cap I = (RII]$, $a \leqslant u_1u_2u_3$ for some $u_1 \in R$ and $u_2, u_3 \in I$. Thus $A_a \neq \emptyset$. Hence

$$\begin{aligned}(f_R \circ f_I \circ f_I)(a) &= \sup_{(r,s,t)\in A_a} \{\min\{f_I(r), f_R(s), f_I(t)\}\} \\ &\geqslant \min\{f_I(u_1), f_R(u_2), f_I(u_3)\} = 1.\end{aligned}$$

Thus $(f_R \circ f_I \circ f_I)(a) = 1$. If $a \notin R$ or $a \notin I$, then $(f_I \cap f_R)(a) = 0$. And also, $A_a = \emptyset$. Thus $(f_R \circ f_I \circ f_I)(a) = 0$. Therefore $f_R \cap f_I = f_R \circ f_I \circ f_I$. □

Theorem 3.9. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *$f \cap g = f \circ g \circ g$ for every fuzzy right ideal f and every fuzzy ideal g of S.*

Proof. (1) $\Rightarrow$ (2). Assume that S is right weakly regular. Let f and g be a fuzzy right ideal and a fuzzy ideal of S, respectively. Let $a \in S$. Then, since S is right weakly regular, $a \leqslant (as_1t_1)(as_2t_2)(as_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$, that is, $A_a \neq \emptyset$, and then

$$\begin{aligned}(f \circ g \circ g)(a) &= \sup_{(p,q,r)\in A_a} \{\min\{f(p), g(q), g(r)\}\} \\ &\leqslant \sup_{(p,q,r)\in A_a} \{\min\{f(pqr), g(pqr), g(pqr)\}\} \\ &\leqslant \sup_{(p,q,r)\in A_a} \{\min\{f(a), g(a), g(a)\}\} \\ &= \min\{f(a), g(a)\} = (f \cap g)(a).\end{aligned}$$

Therefore $f \circ g \circ g \subseteq f \cap g$. On the other hand, we have

$$\begin{aligned}(f \circ g \circ g)(a) &= \sup_{(x,y,z)\in A_a} \{\min\{f(x), g(y), g(z)\}\} \\ &\geqslant \min\{f(as_1t_1), g(as_2t_2), g(as_3t_3)\} \\ &\geqslant \min\{f(a), g(a), g(a)\} \\ &= \min\{f(a), g(a)\} = (f \cap g)(a).\end{aligned}$$

Thus $f \cap g \subseteq f \circ g \circ g$. Therefore $f \cap g = f \circ g \circ g$.

(2) $\Rightarrow$ (1). Let f be fuzzy subset of S and R and I is a right ideal and an ideal of S, respectively. We show that S is right weakly regular. By Lemma 3.2, we see that f_R and f_I are a fuzzy right ideal and a fuzzy ideal of S, by hypothesis, $f_R \cap f_I = f_R \circ f_I \circ f_I$. By Lemma 3.8, we have $R \cap I = (RII]$. Therefore by Lemma 3.7, S is right weakly regular. □

We recall that a fuzzy subset f of an ordered ternary semigroup S is called *idempotent* if $f = f \circ f \circ f$.

Lemma 3.10. *Let S be an ordered ternary semigroup. Then we have $f \circ 1 \circ 1 \subseteq f$, $1 \circ g \circ 1 \subseteq g$ and $1 \circ 1 \circ h \subseteq h$ for each fuzzy right ideal f of S, fuzzy lateral ideal g of S and fuzzy left ideal h of S.* □

Lemma 3.11. *Let S be an ordered ternary semigroup and f a fuzzy subset of S. Let A be a non-empty subset of S. Then the following conditions are equivalent:*

(1) $f_A = f_A \circ f_A \circ f_A$,

(2) $A = (A^3]$.

Proof. (1) $\Rightarrow$ (2). Let $x \in A$. By assumption, $f_A(x) = 1 = (f_A \circ f_A \circ f_A)(x)$. It follows that $x \leqslant u_1u_2u_3$ for some $u_1, u_2, u_3 \in A$. This shows $A \subseteq (A^3]$. Next, let $y \in (A^3]$. Then $y \leqslant v_1v_2v_3$ for some $v_1, v_2, v_3 \in A$. Hence

$$\begin{aligned}(f_A \circ f_A \circ f_A)(y) &= \sup_{(r,s,t)\in A_y} \{\min\{f_A(r), f_A(s), f_A(t)\}\} \\ &\geqslant \min\{f_A(v_1), f_A(v_2), f_A(v_3)\} = 1.\end{aligned}$$

That is $1 = (f_A \circ f_A \circ f_A)(y) = f_A(y)$. Thus $y \in A$. Therefore $A = (A^3]$.

(2) $\Rightarrow$ (1). Let $x \in S$. If $x \notin A$, then there is no $u_1, u_2, u_3 \in A$ such that $x \leqslant u_1u_2u_3$. So $(f_A \circ f_A \circ f_A)(x) = 0 = f_A(x)$. If $x \in A$, then $x \leqslant u_1u_2u_3$ for some $u_1, u_2, u_3 \in A$. Hence

$$\begin{aligned}(f_A \circ f_A \circ f_A)(x) &= \sup_{(r,s,t)\in A_y} \{\min\{f_A(r), f_A(s), f_A(t)\}\} \\ &\geq \min\{f_A(u_1), f_A(u_2), f_A(u_3)\} = 1.\end{aligned}$$

Thus $(f_A \circ f_A \circ f_A)(x) = 1 = f_A(x)$, so $f_A \circ f_A \circ f_A = 1 = f_A$. □

Theorem 3.12. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *every fuzzy right ideal of S is idempotent.*

Proof. (1)$\Rightarrow$(2). Assume that S is right weakly regular. Let f be a fuzzy right ideal of S. We show that $f = f \circ f \circ f$. Let $a \in S$. It is clear from Lemma 3.10 that $f \circ f \circ f \subseteq f$. On other hand, since S is right weakly regular, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a \leqslant (as_1t_1)(as_2t_2)(as_3t_3)$, that is, $A_a \neq \emptyset$, and then

$$\begin{aligned}(f \circ f \circ f)(a) &= \sup_{a\leqslant pqr} \{\min\{f(p), f(q), f(r)\}\} \\ &\geq \min\{f(as_1t_1), f(as_2t_2), f(as_3t_3)\} \\ &\geq \min\{f(a), f(a), f(a)\} = f(a).\end{aligned}$$

This implies that $f \subseteq f \circ f \circ f$. Hence $f = f \circ f \circ f$.

(2) $\Rightarrow$ (1). Assume that f is an idempotent fuzzy right ideal of S. Let $a \in S$. We show that $a \in ((aSS)^3]$. Let $A = (a \cup aSS]$. We see that A is a right ideal of S. By Lemma 3.2, f_A is a fuzzy right ideal of S. Thus by hypothesis, $f_A = f_A \circ f_A \circ f_A$. By Lemma 3.11, $A = (A^3]$. Since $a \in A$, it follows that $a \in (A^3]$. This implies that

$$\begin{aligned} a \in\ & ((a \cup aSS)(a \cup aSS)(a \cup aSS)] \\ &= (aaa \cup aaSSa \cup aSSaa \cup aSSaSSa \cup aaaSS \cup aaSSaSS \cup aSSaaSS \\ &\quad \cup aSSaSSaSS]. \end{aligned}$$

For any case, we have $a \in ((aSS)^3]$. Therefore S is right weakly regular. □

Theorem 3.13. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *$f \cap g \cap h \subseteq f \circ g \circ h$ for every fuzzy generalized bi-ideal f, every fuzzy two-sided ideal g and every fuzzy right ideal h of S,*

(3) *$f \cap g \cap h \subseteq f \circ g \circ h$ for every fuzzy bi-ideal f, every fuzzy two-sided ideal g and every fuzzy right ideal h of S.*

(4) *$f \cap g \cap h \subseteq f \circ g \circ h$ for every fuzzy quasi-ideal f, every fuzzy two-sided ideal g and every fuzzy right ideal h of S.*

Proof. (1) $\Rightarrow$ (2). Assume that S is a right weakly regular. Let f, g and h be a fuzzy generalized bi-ideal, a fuzzy two-sided ideal and a fuzzy right ideal of S, respectively. Let $a \in S$. Then, since S is right weakly regular, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a \leqslant (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$, that is, $A_a \neq \emptyset$, and then

$$\begin{aligned} (f \circ g \circ h)(a) &= \sup_{a \leqslant pqr} \{\min\{f(p), g(q), h(r)\}\} \\ &\geq \min\{f(a), g(s_1t_1as_2t_2), f(as_3t_3)\}\} \\ &\geq \min\{f(a), g(a), h(a)\} \\ &= (f \cap g \cap h)(a). \end{aligned}$$

Therefore $f \cap g \cap h \subseteq f \circ g \circ h$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). It is clear since every fuzzy bi-ideal of S is fuzzy generalized bi-ideal of S and every fuzzy quasi-ideal of S is fuzzy bi-ideal of S.

(4) $\Rightarrow$ (1). Let f be a fuzzy right ideal of S and g a fuzzy ideal of S. Take $h = g$ since every fuzzy right ideal is also fuzzy quasi-ideal. Thus by hypothesis, $f \cap g \cap g \subseteq f \circ g \circ g$. This implies that $f \cap g \subseteq f \circ g \circ g$. But $f \circ g \circ g \subseteq f \cap g$. Thus $f \cap g = f \circ g \circ g$. Therefore, by Theorem 3.9, S is right weakly regular. □

Corollary 3.14. *Let S be an ordered ternary semigroup. Then the following conditions are equivalent:*

(1) *S is right weakly regular,*

(2) *$f \cap g \subseteq f \circ g \circ g$ for every fuzzy generalized bi-ideal f and every fuzzy two-sided ideal g of S,*

(3) *$f \cap g \subseteq f \circ g \circ g$ for every fuzzy bi-ideal f and every fuzzy two-sided ideal g of S,*

(4) *$f \cap g \subseteq f \circ g \circ g$ for every fuzzy quasi-ideal f and every fuzzy two-sided ideal g of S.* □

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