Quasigroups and Related Systems **22** (2014), 247 − 254

# On strongly regular ordered Γ-semigroups

*Niovi Kehayopulu*

**Abstract.** We add here some further characterizations to the characterizations of strongly regular ordered Γ-semigroups already considered in Hacettepe J. Math. 42 (2013), 559–567. Our results generalize the characterizations of strongly regular ordered semigroups given in the Theorem in Math. Japon. 48 (1998), 213–215, in case of ordered Γ-semigroups. The aim of writing this paper is not just to add a publication in Γ-semigroups but, mainly, to publish a paper which serves as an example to show what a Γ-semigroup is and give the right information about this structure.

## 1. Introduction and prerequisites

We have already seen in [2, 3] the methodology we use to pass from ordered semigroups (semigroups) to ordered Γ-semigroups (Γ-semigroups). Some of the results of ordered semigroups can be transferred into ordered Γ-semigroups just putting a "Gamma" in the appropriate place, while there are results for which the transfer is not easy. But anyway, we never work directly on ordered Γ-semigroups. If we want to get a result on an ordered Γ-semigroup, then we have to prove it first in an ordered semigroup and then we have to be careful to define the analogous concepts in case of the ordered Γ-semigroup (if they do not defined directly) and put the "Γ" in the appropriate place. In that sense, although a result on ordered Γ-semigroup generalizes its corresponding one of ordered semigroup, we can never say "we obtain and establish some important results in ordered Γ-semigroups extending and generalizing those for semigroups" (as we have seen in the bibliography) and this is because we can do nothing on ordered Γ-semigroups if we do not examine it first for ordered semigroups. The same holds if we consider Γ-semigroups instead of ordered Γ-semigroups. In the present paper, the transfer was rather difficult, because it was not easy to give the definition of strongly regular ordered Γ-semigroups. An ordered Γ-semigroup $(M, \Gamma, \leqslant)$ is called *strongly regular* if for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma x\mu a$ and $a\gamma x = x\gamma a = x\mu a = a\mu x$. This concept has been first introduced in Hacettepe J. Math. [4], where this type of ordered Γ-semigroup has been characterized as an ordered Γ-semigroup $M$ which is both left and right regular and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$ for every $a \in M$. As a continuation of that result, we add here some further characterizations of this type of ordered

2010 Mathematics Subject Classification: 06F99, 06F05, 20M10
Keywords: ordered Γ-semigroup, strongly regular, filter, left ideal, semiprime.

$\Gamma$-semigroups, the main being the characterization of a strongly regular ordered $\Gamma$-semigroup as an ordered $\Gamma$-semigroup $M$ in which the $\mathcal{N}$-class $(a)_{\mathcal{N}}$ is a strongly regular subsemigroup of $M$ for every $a \in M$. So we prove that not only part of the Theorem in [5], but the whole Theorem can be transferred in case of ordered $\Gamma$-semigroups. The results of the present paper generalize the corresponding results on strongly regular ordered semigroups considered in [5].

A semigroup $S$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a = axa$, that is, if $a \in aSa$ for every $a \in S$ or $A \subseteq ASA$ for every $A \subseteq S$. A semigroup $S$ is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a = xa^2$, that is $a \in Sa^2$ for every $a \in S$ or $A \subseteq SA^2$ for every $A \subseteq S$. It is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a = a^2x$, that is $a \in a^2S$ for every $a \in S$ or $A \subseteq A^2S$ for every $A \subseteq S$. A semigroup $S$ is called *completely regular* if for every $a \in S$ there exists $x \in S$ such that $a = axa$ and $ax = xa$ [6]. It has been proved in [6] that a semigroup is completely regular if and only if it is at the same time regular, left regular and right regular.

When we pass from semigroups to ordered semigroups, to completely regular semigroup correspond two concepts: The completely regular and the strongly regular ordered semigroups and this is the difference between semigroups and ordered semigroups. For an ordered semigroup $S$ we denote by $(H]$ the subset of $S$ defined by $(H] := \{t \in S \mid t \leqslant h \text{ for some } h \in H\}$. An ordered semigroup $(S, ., \leqslant)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leqslant axa$, that is, if $a \in (aSa]$ for every $a \in S$ or $A \subseteq (ASA]$ for every $A \subseteq S$. It is called it left (resp. *right*) *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leqslant xa^2$ (resp. $a \leqslant a^2x$). An ordered semigroup $S$ is regular if and only if $a \in (aSa]$ for every $a \in S$ or $A \subseteq (ASA]$ for every $A \subseteq S$. It is left regular if and only if $a \in (Sa^2]$ for every $a \in S$ or $A \subseteq (SA^2]$ for every $A \subseteq S$. It is right regular if and only if $a \in (a^2S]$ for every $a \in S$ or $A \subseteq (A^2S]$ for every $A \subseteq S$. An ordered semigroup $S$ is called *completely regular* if it is regular, left regular and right regular. An ordered semigroup $S$ is called *strongly regular* if for every $a \in S$ there exists $x \in S$ such that $a \leqslant axa$ and $ax = xa$. The strongly regular ordered semigroups are clearly completely regular but the converse statement does not hold in general. Characterizations of completely regular semigroups have been given in [6], characterizations of strongly regular ordered semigroups have been given in [5].

For two nonempty sets $M$ and $\Gamma$, define $M\Gamma M$ as the set of all elements of the form $m_1\gamma m_2$, where $m_1, m_2 \in M$ and $\gamma \in \Gamma$.

$M$ is called a $\Gamma$-*semigroup* if the following assertions are satisfied:

(1) $M\Gamma M \subseteq M$.

(2) If $m_1, m_2, m_3, m_4 \in M$, $\gamma_1, \gamma_2 \in \Gamma$ such that $m_1 = m_3$, $\gamma_1 = \gamma_2$ and $m_2 = m_4$, then $m_1\gamma_1 m_2 = m_3\gamma_2 m_4$.

(3) $(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$ for all $m_1, m_2, m_3 \in M$ and all $\gamma_1, \gamma_2 \in \Gamma$.

This is the definition in [1] and it is a revised version of the definition of $\Gamma$-semigroups given by Sen and Saha in [7], which allows us in an expression of the form, say $A_1\Gamma A_2\Gamma\dots\Gamma A_n$ to put the parentheses anywhere beginning with some $A_i$ and ending in some $A_j$ $(i,j\in N=\{1,2,\dots,n\})$ or in an expression of the form $a_1\Gamma a_2\Gamma\dots\Gamma a_n$ or $a_1\gamma a_2\gamma\dots\gamma a_n$ to put the parentheses anywhere beginning with some $a_i$ and ending in some $a_j$ $(A_1,A_2,\dots,A_n$ being subsets and $a_1,a_2,\dots,a_n$ elements of $M)$. Unless the uniqueness condition (widely used and still in use by some authors) in an expression of the form, say $a\gamma b\mu c\xi d\rho e$ or $a\Gamma b\Gamma c\Gamma d\Gamma e$, it is not known where to put the parentheses.

Here is an example of a *po*-Γ-semigroup $M$ which is easy to check [3] and shows exactly what a Γ-semigroup is. Other examples in which $M$ has order 3, 5 or 6 and Γ order 2, one can find in [1–3]: Consider the two-elements set $M:=\{a,b\}$, and let $\Gamma=\{\gamma,\mu\}$ be the set of two binary operations on $M$ defined in the tables below:

| $\gamma$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $b$ | $a$ |

| $\mu$ | $a$ | $b$ |
|---|---|---|
| $a$ | $b$ | $a$ |
| $b$ | $a$ | $b$ |

One can check that $(x\rho y)\omega z=x\rho(y\omega z)$ for all $x,y,z\in M$ and all $\rho,\omega\in\Gamma$. So $M$ is a Γ-semigroup.

An *ordered* Γ-*semigroup* (shortly *po*-Γ-*semigroup*) $M$, denoted by $(M,\Gamma,\leqslant)$, is a Γ-semigroup $M$ endowed with an order relation "$\leqslant$" such that $a\leqslant b$ implies $a\gamma c\leqslant b\gamma c$ and $c\gamma a\leqslant c\gamma b$ for all $c\in M$ and all $\gamma\in\Gamma$ [8]. An equivalence relation $\sigma$ on $M$ is called *congruence* if $(a,b)\in\sigma$ implies $(a\gamma c,b\gamma c)\in\sigma$ and $(c\gamma a,c\gamma b)\in\sigma$ for every $c\in M$ and every $\gamma\in\Gamma$. A congruence $\sigma$ on $M$ is called *semilattice congruence* if (1) $(a\gamma a,b\gamma a)\in\sigma$ for all $a,b\in M$ and all $\gamma\in\Gamma$ and (2) $(a,a\gamma a)\in\sigma$ for all $a\in M$ and all $\gamma\in\Gamma$. A *po*-Γ-semigroup $M$ is called *regular* if $a\in(a\Gamma M\Gamma a]$ for every $a\in M$, equivalently if $A\subseteq(A\Gamma M\Gamma A]$ for every $A\subseteq M$. Keeping the already existing definition of left and right regular Γ (or ordered Γ)-semigroups in the bibliography, we will call an ordered Γ-semigroup $M$ *left* (resp. *right*) *regular* if $a\in(M\Gamma a\Gamma a]$ (resp. $a\in(a\Gamma a\Gamma M]$) for every $a\in M$. As in an ordered semigroup, we call a *po*-Γ-semigroup *completely regular* if it is at the same time regular, left regular and right regular. A nonempty subset $A$ of $M$ is called a *subsemigroup* of $M$ if $a,b\in A$ and $\gamma\in\Gamma$ implies $a\gamma b\in A$, that is if $A\Gamma A\subseteq A$. A nonempty subset $A$ of $(M,\Gamma,\leqslant)$ is called a *left* (resp. *right*) *ideal* of $M$ if (1) $M\Gamma A\subseteq A$ (resp. $A\Gamma M\subseteq A$) and (2) if $a\in A$ and $M\ni b\leqslant a$ implies $b\in A$. Clearly, the left ideals as well as the right ideals of $M$ are subsemigroups of $M$. A subsemigroup $F$ of $(M,\Gamma,\leqslant)$ is called a *filter* of $M$ if (1) $a,b\in M$ and $\gamma\in\Gamma$ such that $a\gamma b\in F$ implies $a,b\in F$ and (2) if $a\in F$ and $M\ni b\geq a$ implies $b\in F$. For an element $a$ of $M$ we denote by $N(a)$ the filter of $M$ generated by $a$ and by $\mathcal{N}$ the relation on $M$ defined by $\mathcal{N}:=\{(a,b)\mid N(a)=N(b)\}$. Exactly as in ordered semigroups one can prove that the relation $\mathcal{N}$ is a semilattice congruence on $M$. In this paper, for semiprime subsets we will also keep the existing definition in the bibliography. A subset $T$ of an ordered Γ-semigroup $M$ is called *semiprime* if $A\subseteq M$ such that $A\Gamma A\subseteq T$ implies $A\subseteq T$, equivalently if $a\in M$ such that $a\Gamma a\subseteq T$ implies $a\in T$. For a *po*-Γ-semigroup $M$, we clearly have $M=(M]$, and for any subsets $A$, $B$ of

$M$, we have $A \subseteq (A] = ((A]]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A]\Gamma(B] \subseteq (A\Gamma B]$ and $((A]\Gamma(B]] = ((A]\Gamma B] = (A\Gamma(B]] = (A\Gamma B]$.

## 2. Main results

**Definition.** (cf. [4]) A *po*-$\Gamma$-semigroup $(M, \Gamma, \leqslant)$ is called *strongly regular* if for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that

$$a \leqslant a\gamma x\mu a \quad \text{and} \quad a\gamma x = x\gamma a = x\mu a = a\mu x.$$

A subsemigroup $T$ of $(M, \Gamma, \leqslant)$ is called strongly regular if the set $T$ with the same $\Gamma$ and the order "$\leqslant$" of $M$ is strongly regular, that is, for every $a \in T$ there exist $y \in T$ and $\lambda, \rho \in \Gamma$ such that $a \leqslant a\lambda y\xi a$ and $a\lambda y = y\lambda a = y\xi a = a\xi y$. We write it also as $(T, \Gamma, \leqslant)$.

**Theorem.** *Let $M$ be an ordered $\Gamma$-semigroup. The following are equivalent:*

(1) *$M$ is strongly regular.*

(2) *For every $a \in M$, there exist $y \in M$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma y\mu a$, $y \leqslant y\mu a\gamma y$ and $a\gamma y = y\gamma a = y\mu a = a\mu y$.*

(3) *Every $\mathcal{N}$-class of $M$ is a strongly regular subsemigroup of $M$.*

(4) *The left and the right ideals of $M$ are semiprime and for every left ideal $L$ and every right ideal $R$ of $M$, the set $(L\Gamma R]$ is a strongly regular subsemigroup of $M$.*

(5) *$M$ is left regular, right regular, and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$ for every $a \in M$.*

(6) *For every $a \in M$ there exist $e_a, e_a^{'} \in M\Gamma a\Gamma a\Gamma M$ and $\rho, \mu \in \Gamma$ such that $e_a \leq e_a\rho e_a^{'}$, $a \leqslant e_a\mu a$, $a \leqslant a\rho e_a^{'}$, $(M\Gamma e_a\Gamma M] = (M\Gamma e_a^{'}\Gamma M] = (M\Gamma a\Gamma M]$, and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$.*

(7) *For every $a \in M$ there exist $e_a, e_a^{'} \in M$ and $\rho, \mu \in \Gamma$ such that $a \leq e_a\mu a$, $a \leqslant a\rho e_a^{'}$, and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$.*

(8) *For every $a \in M$, we have $a \in (M\Gamma a] \cap (a\Gamma M]$, and $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$.*

**Proof.** (1) $\Longrightarrow$ (2). For its proof we refer to [4]. For convenience, we sketch the proof: Let $a \in M$. Since $M$ is strongly regular, there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma x\mu a$ and $a\gamma x = x\gamma a = x\mu a = a\mu x$. Then we have

$$a \leqslant a\gamma x\mu a \leq (a\gamma x\mu a)\gamma x\mu a = a\gamma(x\mu a\gamma x)\mu a.$$

For the element $y := x\mu a\gamma x$ of $M$, we have $a \leqslant a\gamma y\mu a$, $y \leqslant y\mu a\gamma y$ and $a\gamma y = y\gamma a = y\mu a = a\mu y$.

(2) $\Longrightarrow$ (3). Let $b \in M$. The class $(b)_{\mathcal{N}}$ is a subsemigroup of $M$. Indeed: First of all, it is a nonempty subset of $M$. Let $x, y \in (b)_{\mathcal{N}}$ and $\gamma \in \Gamma$. Since $(x, b) \in \mathcal{N}$, $(b, y) \in \mathcal{N}$, and $\mathcal{N}$ is a semilattice congruence on $M$, we have $(x\gamma y, b\gamma y) \in \mathcal{N}$, $(b\gamma y, y\gamma y) \in \mathcal{N}$, $(y\gamma y, y) \in \mathcal{N}$, then $(x\gamma y, y) \in \mathcal{N}$, and $x\gamma y \in (y)_{\mathcal{N}} = (b)_{\mathcal{N}}$.

$(b)_{\mathcal{N}}$ is strongly regular. In fact: Let $a \in (b)_{\mathcal{N}}$. By (2), there exist $y \in M$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma y\mu a$, $y \leqslant y\mu a\gamma y$ and $a\gamma y = y\gamma a = y\mu a = a\mu y$. On the other hand, $y \in (b)_{\mathcal{N}}$. Indeed: Since $N(a) \ni a \leqslant a\gamma y\mu a$ and $N(a)$ is a filter of $M$, we have $a\gamma y\mu a \in N(a)$, $y \in N(a)$, $N(y) \subseteq N(a)$. Since $N(y) \ni y \leqslant y\mu a\gamma y$ and $N(y)$ is a filter, we have $y\mu a\gamma y \in N(y)$, $a \in N(y)$, $N(a) \subseteq N(y)$. Then $N(a) = N(y)$, $(a, y) \in \mathcal{N}$, and $y \in (a)_{\mathcal{N}} = (b)_{\mathcal{N}}$.

(3) $\Longrightarrow$ (4). Let $L$ be a left ideal of $M$ and $a \in M$ such that $a\Gamma a \subseteq L$. Then $a \in L$. In fact: Since $a \in (a)_{\mathcal{N}}$ and $(a)_{\mathcal{N}}$ is strongly regular, there exist $x \in (a)_{\mathcal{N}}$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma x\mu a$ and $a\gamma x = x\gamma a = x\mu a = a\mu x$. Then we have

$$a \leqslant (a\gamma x)\mu a = (x\gamma a)\mu a = x\gamma(a\mu a) \in M\Gamma(a\Gamma a) \subseteq M\Gamma L \subseteq L,$$

and $a \in L$. If $R$ is a right ideal of $M$, $a \in M$ and $a\Gamma a \subseteq R$, then

$$a \leqslant a\gamma(x\mu a) = a\gamma(a\mu x) = (a\gamma a)\mu x \in (a\Gamma a)\Gamma M \subseteq R\Gamma M \subseteq R,$$

so $a \in R$, and $R$ is also semiprime.

Let $L$ be a left ideal and $R$ a right ideal of $M$. The (nonempty) set $(L\Gamma R]$ is a subsemigroup of $M$. In fact: Let $a, b \in (L\Gamma R]$ and $\gamma \in \Gamma$. We have $a \leqslant y_1\gamma_1 x_1$ and $b \leqslant y_2\gamma_2 x_2$ for some $y_1, y_2 \in L$, $\gamma_1, \gamma_2 \in \Gamma$, $x_1, x_2 \in R$. Then $a\gamma b \leqslant (y_1\gamma_1 x_1)\gamma(y_2\gamma_2 x_2)$. Since $(y_1\gamma_1 x_1)\gamma y_2 \in (M\Gamma M)\Gamma L \subseteq M\Gamma L \subseteq L$, we have $(y_1\gamma_1 x_1\gamma y_2)\gamma_2 x_2 \in L\Gamma R$, so $a\gamma b \in (L\Gamma R]$.

Let now $a \in (L\Gamma R]$. Then there exist $x \in (L\Gamma R]$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma x\mu a$ and $a\gamma x = x\gamma a = x\mu a = a\mu x$. In fact: Since $a \in (a)_{\mathcal{N}}$ and $(a)_{\mathcal{N}}$ is strongly regular, there exist $t \in (a)_{\mathcal{N}}$ and $\gamma, \mu \in \Gamma$ such that $a \leqslant a\gamma t\mu a$ and $a\gamma t = t\gamma a = t\mu a = a\mu t$. Since $a \in (L\Gamma R]$, there exist $y \in L$, $\rho \in \Gamma$, $z \in R$ such that $a \leqslant y\rho z$. We have

$$a \leqslant a\gamma t\mu a \leqslant a\gamma t\mu(a\gamma t\mu a) = a\gamma(t\mu a\gamma t)\mu a.$$

For the element $x := t\mu a\gamma t$, we have

$$x = t\mu a\gamma t \leqslant t\mu(y\rho z)\gamma t = (t\mu y)\rho(z\gamma t).$$

Since $t\mu y \in M\Gamma L \subseteq L$ and $z\gamma t \in R\Gamma M \subseteq R$, we have $(t\mu y)\rho(z\gamma t) \in L\Gamma R$, then $x \in L\Gamma R$. Moreover, we have $a\gamma x = x\gamma a$, that is, $a\gamma(t\mu a\gamma t) = (t\mu a\gamma t)\gamma a$. Indeed,

$$\begin{aligned} a\gamma(t\mu a\gamma t) &= (a\gamma t)\mu(a\gamma t) = (t\mu a)\mu(t\gamma a) = t\mu(a\mu t)\gamma a \\ &= t\mu(a\gamma t)\gamma a = (t\mu a\gamma t)\gamma a. \end{aligned}$$

Also $x\mu a = a\mu x$, that is, $(t\mu a\gamma t)\mu a = a\mu(t\mu a\gamma t)$. Indeed,

$$\begin{aligned}(t\mu a\gamma t)\mu a &= (t\mu a)\gamma(t\mu a) = (a\mu t)\gamma(a\gamma t) = a\mu(t\gamma a)\gamma t\\ &= a\mu(t\mu a)\gamma t = a\mu(t\mu a\gamma t).\end{aligned}$$

And $x\gamma a = x\mu a$, that is, $(t\mu a\gamma t)\gamma a = (t\mu a\gamma t)\mu a$. Indeed,

$$(t\mu a\gamma t)\gamma a = (t\mu a)\gamma(t\gamma a) = (t\mu a)\gamma(t\mu a) = (t\mu a\gamma t)\mu a.$$

(4) $\Longrightarrow$ (5). Let $a \in M$. The set $(M\Gamma a\Gamma a]$ is a left ideal of $M$. This is because it is a nonempty subset of $M$ and we have

$$M\Gamma(M\Gamma a\Gamma a] = (M]\Gamma(M\Gamma a\Gamma a] \subseteq (M\Gamma M\Gamma a\Gamma a] = ((M\Gamma M)\Gamma a\Gamma a] \subseteq (M\Gamma a\Gamma a]$$

and $((M\Gamma a\Gamma a]] = (M\Gamma a\Gamma a]$. Since $(M\Gamma a\Gamma a]$ is a left ideal of $M$, by (4), it is semiprime.

Since $(a\Gamma a)\Gamma(a\Gamma a) \subseteq (M\Gamma a\Gamma a]$, we have $a\Gamma a \subseteq (M\Gamma a\Gamma a]$, and $a \in (M\Gamma a\Gamma a]$, thus $M$ is left regular. Similarly the set $(a\Gamma a\Gamma M]$ is a right ideal of $M$ and $M$ is right regular. For the rest of the proof, we prove that $(M\Gamma a\Gamma M] = ((M\Gamma a]\Gamma(a\Gamma M]]$. Then, since $(M\Gamma a]$ is a left ideal and $(a\Gamma M]$ a right ideal of $M$, by (4), the set $((M\Gamma a]\Gamma(a\Gamma M]]$ is a strongly regular subsemigroup of $M$, and so is $(M\Gamma a\Gamma M]$. We have

$$\begin{aligned}M\Gamma a\Gamma M &\subseteq M\Gamma(M\Gamma a\Gamma a]\Gamma M = (M]\Gamma(M\Gamma a\Gamma a]\Gamma(M]\\ &\subseteq ((M\Gamma M)\Gamma a\Gamma a\Gamma M] \subseteq (M\Gamma a\Gamma a\Gamma M] = ((M\Gamma a]\Gamma(a\Gamma M]],\end{aligned}$$

then

$$\begin{aligned}(M\Gamma a\Gamma M] &\subseteq (((M\Gamma a]\Gamma(a\Gamma M]]] = ((M\Gamma a]\Gamma(a\Gamma M]]\\ &= ((M\Gamma a)\Gamma(a\Gamma M)] \subseteq (M\Gamma a\Gamma M],\end{aligned}$$

and so $(M\Gamma a\Gamma M] = ((M\Gamma a]\Gamma(a\Gamma M]]$.

(5) $\Longrightarrow$ (6). Let $a \in M$. Since $M$ is left regular, we have $a \in (M\Gamma a\Gamma a]$, since $M$ is right regular, $a \in (a\Gamma a\Gamma M]$. Then there exist $x, y \in M$ and $\gamma, \mu, \rho, \xi \in \Gamma$ such that $a \leqslant x\gamma a\mu a$ and $a \leqslant a\rho a\xi y$. Let $e_a := x\gamma a\rho a\xi y$ and $e_a^{'} := x\gamma a\mu a\xi y$. Then $e_a, e_a^{'} \in M\Gamma a\Gamma a\Gamma M$ and we have

$$a \leqslant x\gamma a\mu a \leqslant x\gamma(a\rho a\xi y)\mu a = (x\gamma a\rho a\xi y)\mu a = e_a\mu a,$$

$$a \leqslant a\rho a\xi y \leqslant a\rho(x\gamma a\mu a)\xi y = a\rho(x\gamma a\mu a\xi y) = a\rho e_a^{'},$$

$$e_a = x\gamma a\rho a\xi y \leqslant x\gamma(a\rho a\xi y)\rho(x\gamma a\mu a)\xi y = e_a\rho e_a^{'}.$$

Moreover,

$$\begin{aligned}(M\Gamma e_a\Gamma M] &= (M\Gamma x\gamma a\rho a\xi y\Gamma M] \subseteq (M\Gamma M\Gamma M\Gamma a\Gamma M\Gamma M] \subseteq (M\Gamma a\Gamma M]\\ &\subseteq (M\Gamma(e_a\Gamma M]\Gamma M] = (M\Gamma e_a\Gamma M\Gamma M] \subseteq (M\Gamma e_a\Gamma M],\end{aligned}$$

so $(M\Gamma e_a\Gamma M] = (M\Gamma a\Gamma M]$, similarly $(M\Gamma e_a^{'}\Gamma M] = (M\Gamma a\Gamma M]$. In addition, by (5), $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$.

(6) $\Longrightarrow$ (7). This is obvious.

(7) $\Longrightarrow$ (8). Let $a \in M$. By hypothesis, there exist $e_a$, $e_a^{'} \in M$ and $\rho, \mu \in \Gamma$ such that $a \leqslant e_a\mu a$, $a \leqslant a\rho e_a^{'}$, and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$. Since $a \leqslant e_a\mu a$, we have $a \in (M\Gamma a]$. Since $a \leqslant a\rho e_a^{'} \in a\Gamma M$, we have $a \in (a\Gamma M]$. Then $a \in (M\Gamma a] \cap (a\Gamma M]$ and $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$, so condition (8) is satisfied.

(8) $\Longrightarrow$ (1). For its proof we refer to [4]. □

**Corollary.** *Let $M$ be an ordered $\Gamma$-semigroup. The following are equivalent:*

(1) *$M$ is strongly regular.*

(2) *If $a \in M$, then for the subset $e_a := M\Gamma a\Gamma a\Gamma M$ of $M$, we have*
$e_a \subseteq (e_a\Gamma e_a]$, $a \in (e_a\Gamma a]$, $a \in (a\Gamma e_a]$, $(M\Gamma e_a\Gamma M] = (M\Gamma a\Gamma M]$,
*and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$.*

(3) *If $a \in M$, then there exists a subset $e_a$ of $M$ such that*
$a \in (e_a\Gamma a]$, $a \in (a\Gamma e_a]$,
*and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$.*

**Proof.** (1) $\Longrightarrow$ (2). Let $a \in M$ and $e_a := M\Gamma a\Gamma a\Gamma M$. Since $M$ is strongly regular, by the Theorem (1) $\Rightarrow$ (5), $M$ is left regular, right regular, and the set $(M\Gamma a\Gamma M]$ is a strongly regular subsemigroup of $M$. Since $M$ is left regular and right regular, we have $a \in (M\Gamma a\Gamma a]$ and $a \in (a\Gamma a\Gamma M]$. Then we have

$$\begin{aligned} e_a &= M\Gamma a\Gamma a\Gamma M \subseteq M\Gamma(a\Gamma a\Gamma M]\Gamma(M\Gamma a\Gamma a]\Gamma M \\ &= (M]\Gamma(a\Gamma a\Gamma M]\Gamma(M\Gamma a\Gamma a]\Gamma(M] \\ &\subseteq ((M\Gamma a\Gamma a\Gamma M)\Gamma(M\Gamma a\Gamma a\Gamma M)] = (e_a\Gamma e_a], \end{aligned}$$

$$a \in (M\Gamma a\Gamma a] \subseteq (M\Gamma(a\Gamma a\Gamma M]\Gamma a] = ((M\Gamma(a\Gamma a\Gamma M)\Gamma a] = (e_a\Gamma a],$$

$$a \in (a\Gamma a\Gamma M] \subseteq (a\Gamma(M\Gamma a\Gamma a]\Gamma M] = (a\Gamma(M\Gamma a\Gamma a\Gamma M)] = (a\Gamma e_a],$$

$$\begin{aligned} (M\Gamma e_a\Gamma M] &= (M\Gamma(M\Gamma a\Gamma a\Gamma M)\Gamma M] \subseteq (M\Gamma a\Gamma M] \\ &\subseteq (M\Gamma(M\Gamma a\Gamma a]\Gamma M] = (M\Gamma(M\Gamma a\Gamma a)\Gamma M] \\ &\subseteq (M\Gamma(M\Gamma a\Gamma(a\Gamma a\Gamma M]\Gamma M] \\ &= (M\Gamma(M\Gamma a\Gamma a\Gamma a\Gamma M)\Gamma M] \\ &\subseteq (M\Gamma(M\Gamma a\Gamma a\Gamma M)\Gamma M] = (M\Gamma e_a\Gamma M], \end{aligned}$$

thus we have $(M\Gamma e_a\Gamma M] = (M\Gamma a\Gamma M]$.

(2) $\Longrightarrow$ (3). This is obvious.

(3) $\Longrightarrow$ (1). Let $a \in M$. By hypothesis, there exists a subset $e_a$ of $M$ such that $a \in (e_a \Gamma a]$, $a \in (a \Gamma e_a]$, and the set $(M \Gamma a \Gamma M]$ is a strongly regular subsemigroup of $M$. Since $a \in (e_a \Gamma a] \subseteq (M \Gamma a]$ and $a \in (a \Gamma e_a] \subseteq (a \Gamma M]$, we have $a \in (M \Gamma a] \cap (a \Gamma M]$. By the Theorem (8) $\Rightarrow$ (1), $M$ is strongly regular. $\square$

**Remark.** Here are some information we get about ordered semigroups: By the implication (1) $\Rightarrow$ (2) of the above Corollary (or by the implication (1) $\Rightarrow$ (2) of the Theorem in [5]), we have the following: If $S$ is a strongly regular ordered semigroup, $a \in S$ and $e_a := Sa^2S$, then we have $e_a \subseteq ({e_a}^2]$, $a \in (e_a a]$, $a \in (a e_a]$, $(S e_a S] = (SaS]$, and the set $(SaS]$ is a strongly regular subsemigroup of $S$. By the implication (3) $\Rightarrow$ (1) of the Corollary (or by the implication (8) $\Rightarrow$ (1) of the Theorem in [5]), we have the following: Let $S$ be an ordered semigroup. Suppose that for every $a \in S$ there exists a subset $e_a$ of $S$ such that $a \in (e_a a]$, $a \in (a e_a]$, and the set $(SaS]$ is a strongly regular subsemigroup of $S$. Then $S$ is strongly regular.

# References


[1] **N. Kehayopulu,** *On prime, weakly prime ideals in po-$\Gamma$-semigroups*, Lobachevskii J. Math. **30** (2009), 257 – 262.

[2] **N. Kehayopulu,** *On ordered $\Gamma$-semigroups*, Sci. Math. Jpn. **71** (2010), 179 – 185.

[3] **N. Kehayopulu,** *On regular duo po-$\Gamma$-semigroups*, Math. Slovaca **61** (2011), 871 – 884.

[4] **N. Kehayopulu,** *On bi-ideals of ordered $\Gamma$-semigroups –A Corrigendum*, Hacettepe J. Math. **42** (2013), 559 – 567.

[5] **N. Kehayopulu, M. Tsingelis,** *A characterization of strongly regular ordered semigroups*, Math. Japon. **48** (1998), 213 – 215.

[6] **M. Petrich,** *Introduction to Semigroups*, Charles E. Merrill Publishing Co., A Bell & Howell Company, Columbus, Ohio (1973).

[7] **M. K. Sen, N. K. Saha,** *On $\Gamma$-semigroup* **I**, Bull. Calcutta Math. Soc. **78** (1986), 180 – 186.

[8] **M. K. Sen, A. Seth,** *On po-$\Gamma$-semigroups*, Bull. Calcutta Math. Soc. **85** (1993), 445 – 450.


Received February 17, 2014
Revised June 26, 2014

University of Athens, Department of Mathematics, 15784 Panepistimiopolis, Athens, Greece
E-mail: nkehayop@math.uoa.gr