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Int-soft interior ideals of semigroups

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Abstract. The soft version of interior ideals in semigroups is considered. The notion of an int-soft interior ideal is introduced, and related properties are investigated. A relation between an int-soft ideal and an int-soft interior ideal is provided, and conditions for an int-soft interior ideal to be an int-soft two-sided ideal are given. Characterizations of int-soft interior ideals are discussed. Using the notion of int-soft left (right) ideals, characterizations of a left (right) simple semigroup are provided. The int-soft interior ideal generated by a soft set is established.

1. Introduction

As a new mathematical tool for dealing with uncertainties, Molodtsov [11] introduced the notion of soft sets, and pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [10] described the application of soft set theory to a decision making problem. Maji et al. [9] also studied several operations on the theory of soft sets. Çağman et al. [5] introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory, and provided an example which shows that the method can be successfully applied to the problems that contain uncertainties. Feng [6] considered the application of soft rough approximations in multicriteria group decision making problems. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. In the aspect of algebraic strudtures, the soft set theory has been applied to rings, fields and modules (see [1, 3]), groups (see [2]), semirings (see [7]), (ordered) semigroups (see [8, 13]). Song et al. [13] introduced the notion of int-soft semigroups and int-soft left (resp. right) ideals, and investigated several properties.

In this paper, we consider the soft version of interior ideals in semigroups. We introduce the notion of an int-soft interior ideal, and investigate related properties. We give a relation between an int-soft ideal and an int-soft interior ideal. We provide conditions for an int-soft interior ideal to be an int-soft two-sided ideal. We consider characterizations of int-soft interior ideals. Using the notion of int-soft left (right) ideals, we discuss characterizations of a left (right) simple semigroup. We establish the int-soft interior ideal generated by a soft set.

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2. Preliminaries

Recall that a semigroup S is said to be

- *regular* if for every $x \in S$ there exists $a \in S$ such that $xax = x$,
- *left* (*right*) *simple* if it contains no proper left (right) ideal,
- *simple* if it contains no proper two-sided ideal.

A nonempty subset A of S is called

- a *left* (resp., *right*) *ideal* of S if $SA \subseteq A$ (resp., $AS \subseteq A$),
- a *two-sided ideal* of S if it is both a left and a right ideal of S.
- an *interior ideal* of S if $SAS \subseteq A$.

Let U be an initial universe set, E be a set of parameters, $\mathcal{P}(U)$ be the power set of U and $A, B, C, \ldots \subseteq E$.

Definition 2.1 (cf. [4, 11]). A *soft set* (α, A) over U is defined to be the set of ordered pairs

$$(\alpha, A) := \{(x, \alpha(x)) \,|\, x \in E,\, \alpha(x) \in \mathcal{P}(U)\},$$

where $\alpha : E \to \mathcal{P}(U)$ such that $\alpha(x) = \emptyset$ if $x \notin A$.

The function α is called *approximate function* of the soft set (α, A). The subscript A in the notation α indicates that α is the approximate function of (α, A).

For a soft set (α, A) over U and a subset γ of U, the γ-*inclusive set* of (α, A), denoted by $i_A(\alpha; \gamma)$, is defined to be the set

$$i_A(\alpha; \gamma) := \{x \in A \,|\, \gamma \subseteq \alpha(x)\}.$$

For any soft sets (α, S) and (β, S) over U, we define

$$(\alpha, S) \,\tilde{\subseteq}\, (\beta, S) \text{ if } \alpha(x) \subseteq \beta(x) \;\text{ for all } x \in S.$$

The *soft union* of (α, S) and (β, S) is defined to be the soft set $(\alpha \,\tilde{\cup}\, \beta, S)$ over U in which $\alpha \,\tilde{\cup}\, \beta$ is defined by

$$(\alpha \,\tilde{\cup}\, \beta)(x) = \alpha(x) \cup \beta(x) \;\text{ for all } x \in S.$$

The *soft intersection* of (α, S) and (β, S) is defined to be the soft set $(\alpha \,\tilde{\cap}\, \beta, S)$ over U in which $\alpha \,\tilde{\cap}\, \beta$ is defined by

$$(\alpha \,\tilde{\cap}\, \beta)(x) = \alpha(x) \cap \beta(x) \;\text{ for all } x \in S.$$

The *int-soft product* of (α, S) and (β, S) is defined to be the soft set $(\alpha \,\tilde{\circ}\, \beta, S)$ over U in which $\alpha \,\tilde{\circ}\, \beta$ is a mapping from S to $\mathcal{P}(U)$ given by

$$(\alpha \,\tilde{\circ}\, \beta)(x) = \begin{cases} \bigcup\limits_{x=yz} \{\alpha(y) \cap \beta(z)\} & \text{if } \exists\, y, z \in S \text{ such that } x = yz \\ \emptyset & \text{otherwise.} \end{cases}$$

Let (α, S) be a soft set over U. For a subset γ of U with $i_S(\alpha;\gamma) \neq \emptyset$, define a soft set (α^*, S) over U by

$$\alpha^* : S \to \mathcal{P}(U), \quad x \mapsto \begin{cases} \alpha(x) & \text{if } x \in i_S(\alpha;\gamma), \\ \delta & \text{otherwise,} \end{cases}$$

where δ is a subset of U with $\delta \subsetneq \alpha(x)$.

Definition 2.2 (cf. [13])**.** A soft set (α, S) over U is called an *int-soft semigroup* over U if it satisfies:

$$(\forall x, y \in S)\,(\alpha(x) \cap \alpha(y) \subseteq \alpha(xy))\,. \tag{1}$$

Definition 2.3 (cf. [13])**.** A soft set (α, S) over U is called an *int-soft left* (resp., *right*) *ideal* over U if it satisfies:

$$(\forall x, y \in S)\,(\alpha(xy) \supseteq \alpha(y) \text{ (resp., } \alpha(xy) \supseteq \alpha(x)))\,. \tag{2}$$

If a soft set (α, S) over U is both an int-soft left ideal and an int-soft right ideal over U, we say that (α, S) is an *int-soft two-sided ideal* over U.

Obviously, every int-soft (resp., right) ideal over U is an int-soft semigroup over U. But the converse is not true in general (see [13]).

3. Int-soft interior ideals

In what follows, we take $E = S$, as a set of parameters, which is a semigroup unless otherwise specified.

Definition 3.1. A soft set (α, S) over U is called an *int-soft interior ideal* over U if

$$(\forall a, x, y \in S)\;\;(\alpha(xay) \supseteq \alpha(a))\,. \tag{3}$$

Example 3.2. Consider the semigroup $S = \{a, b, c, d\}$ with the multiplication: $cc = dc = dd = b$ and $xy = a$ in other cases.

Let (α, S) be a soft set over $U = \mathbf{Z}$ defined as follows:

$$\alpha : S \to \mathcal{P}(U), \;\; x \mapsto \begin{cases} 2\mathbb{Z} & \text{if } x = a, \\ 4\mathbb{N} & \text{if } x \in \{b, d\}, \\ 2\mathbb{N} & \text{if } x = c. \end{cases}$$

Then $\alpha(xyz) = \alpha(a) = 2\mathbb{Z} \supseteq \alpha(y)$ for every $x, y, z \in S$. Therefore (α, S) is an int-soft interior ideal over U. □

As is easy easily seen, every int-soft two-sided ideal is an int-soft interior ideal, but the converse is not true in general. In fact, the int-soft interior ideal (α, S) over $U = \mathbb{Z}$ in Example 3.2 is not an int-soft left ideal over $U = \mathbb{Z}$ since $\alpha(dc) = \alpha(b) = 4\mathbb{N} \subsetneq 2\mathbb{N} = \alpha(c)$, and so it is not an int-soft two-sided ideal over $U = \mathbb{Z}$.

We now provide conditions for an int-soft interior ideal to be an int-soft two-sided ideal.

Theorem 3.3. *In a regular semigroup S, every int-soft interior ideal over U is an int-soft two-sided ideal over U.*

Proof. Let (α, S) be an int-soft interior ideal over U and let $a, b \in S$. Since S is regular, there exist $x, y \in S$ such that $a = axa$ and $b = byb$. Hence

$$\alpha(ab) = \alpha((axa)b) = \alpha((ax)ab) \supseteq \alpha(a)$$

and

$$\alpha(ab) = \alpha(a(byb)) = \alpha(ab(yb)) \supseteq \alpha(b).$$

Therefore (α, S) is an int-soft two-sided ideal over U. □

Theorem 3.4. *If S is a monoid, then every int-soft interior ideal over U is an int-soft two-sided ideal over U.*

Proof. Let (α, S) be an int-soft interior ideal over U. Then $\alpha(xy) = \alpha(xye) \supseteq \alpha(y)$ and $\alpha(xy) = \alpha(exy) \supseteq \alpha(x)$ where e is the identity of S. Hence (α, S) is an int-soft two-sided ideal over U. □

For a nonempty subset A of S and $\Phi, \Psi \in \mathcal{P}(U)$ with $\Phi \supsetneqq \Psi$, define a map $\chi_A^{(\Phi,\Psi)}$ as follows:

$$\chi_A^{(\Phi,\Psi)} : S \to \mathcal{P}(U), \quad x \mapsto \begin{cases} \Phi & \text{if } x \in A, \\ \Psi & \text{otherwise.} \end{cases}$$

Then $(\chi_A^{(\Phi,\Psi)}, S)$ is a soft set over U, which is called the (Φ, Ψ)-*characteristic soft set*. The soft set $(\chi_S^{(\Phi,\Psi)}, S)$ is called the (Φ, Ψ)-*identity soft set* over U. The (Φ, Ψ)-characteristic soft set with $\Phi = U$ and $\Psi = \emptyset$ is called the *characteristic soft set*, and is denoted by (χ_A, S). The (Φ, Ψ)-identity soft set with $\Phi = U$ and $\Psi = \emptyset$ is called the *identity soft set*, and is denoted by (χ_S, S).

Theorem 3.5. *A nonempty subset A of a semigroup S is its an interior ideal if and only if the (Φ, Ψ)-characteristic soft set $(\chi_A^{(\Phi,\Psi)}, S)$ over U is an int-soft interior ideal over U for any $\Phi, \Psi \in \mathcal{P}(U)$ with $\Phi \supsetneqq \Psi$.*

Proof. Assume that A is an interior ideal of S. Let $\Phi, \Psi \in \mathcal{P}(U)$ with $\Phi \supsetneqq \Psi$ and $x, y, z \in S$. If $y \in A$, then $\chi_A^{(\Phi,\Psi)}(y) = \Phi$ and $xyz \in SAS \subseteq A$. Hence

$$\chi_A^{(\Phi,\Psi)}(xyz) = \Phi = \chi_A^{(\Phi,\Psi)}(y).$$

If $y \notin A$, then $\chi_A^{(\Phi,\Psi)}(y) = \Psi$ and so

$$\chi_A^{(\Phi,\Psi)}(xyz) \supseteq \Psi = \chi_A^{(\Phi,\Psi)}(y).$$

Therefore $(\chi_A^{(\Phi,\Psi)}, S)$ is an int-soft interior ideal over U for any $\Phi, \Psi \in \mathcal{P}(U)$ with $\Phi \supsetneqq \Psi$.

Conversely, suppose that the (Φ, Ψ)-characteristic soft set $(\chi_A^{(\Phi,\Psi)}, S)$ over U is an int-soft interior ideal over U for any $\Phi, \Psi \in \mathcal{P}(U)$ with $\Phi \supsetneq \Psi$. Let a be any element of SAS. Then $a = xyz$ for some $x, z \in S$ and $y \in A$. Then

$$\chi_A^{(\Phi,\Psi)}(a) = \chi_A^{(\Phi,\Psi)}(xyz) \supseteq \chi_A^{(\Phi,\Psi)}(y) = \Phi,$$

and so $\chi_A^{(\Phi,\Psi)}(a) = \Phi$. Thus $a \in A$, which shows that $SAS \subseteq A$. Therefore A is an interior ideal of S. □

Theorem 3.6. *For the identity soft set (χ_S, S) and a soft set (α, S) over U, the following are equivalent:*

(1) *(α, S) is an int-soft interior ideal over U,*

(2) *$(\chi_S \,\tilde{\circ}\, \alpha \,\tilde{\circ}\, \chi_S, S) \,\tilde{\subseteq}\, (\alpha, S)$.*

Proof. Assume that (α, S) is an int-soft interior ideal over U. Let x be any element of S. If there exist $y, z, u, v \in S$ such that $x = yz$ and $y = uv$, then

$$\begin{aligned}
(\chi_S \,\tilde{\circ}\, \alpha \,\tilde{\circ}\, \chi_S)(x) &= \bigcup_{x=yz} \{(\chi_S \,\tilde{\circ}\, \alpha)(y) \cap \chi_S(z)\} \\
&= \bigcup_{x=yz} \left\{ \left(\bigcup_{y=uv} \{\chi_S(u) \cap \alpha(v)\} \right) \cap \chi_S(z) \right\} \\
&\subseteq \bigcup_{x=yz} \left\{ \left(\bigcup_{y=uv} \{U \cap \alpha(v)\} \right) \cap U \right\} \\
&\subseteq \alpha(x)
\end{aligned}$$

since $\alpha(uvz) \supseteq \alpha(v)$. In other case, we have $(\chi_S \,\tilde{\circ}\, \alpha \,\tilde{\circ}\, \chi_S)(x) = \emptyset \subseteq \alpha(x)$. Therefore $(\chi_S \,\tilde{\circ}\, \alpha \,\tilde{\circ}\, \chi_S, S) \,\tilde{\subseteq}\, (\alpha, S)$.

Conversely, suppose that $(\chi_S \,\tilde{\circ}\, \alpha \,\tilde{\circ}\, \chi_S, S) \,\tilde{\subseteq}\, (\alpha, S)$. For any $a, x, y \in S$, we have

$$\begin{aligned}
\alpha(xay) &\supseteq (\chi_S \,\tilde{\circ}\, \alpha \,\tilde{\circ}\, \chi_S, S)(xay) \\
&= \bigcup_{xay=uv} \{(\chi_S \,\tilde{\circ}\, \alpha)(u) \cap \chi_S(v)\} \\
&\supseteq (\chi_S \,\tilde{\circ}\, \alpha)(xa) \cap \chi_S(y) \\
&= (\chi_S \,\tilde{\circ}\, \alpha)(xa) \cap U \\
&= \bigcup_{xa=uv} \{\chi_S(u) \cap \alpha(v)\} \\
&\supseteq \chi_S(x) \cap \alpha(a) \\
&= \alpha(a).
\end{aligned}$$

Therefore (α, S) is an int-soft interior ideal over U. □

Lemma 3.7 (cf. [13]). *A nonempty subset A of a semigroup S is its a left (right) ideal if and only if the characteristic soft set (χ_A, S) over U is an int-soft left (right) ideal over U.*

Theorem 3.8. *A semigroup S is a left (right) simple if and only if every its int-soft left (right) ideal over U is a constant function.*

Proof. Assume that S is a left simple semigroup and let (α, S) be an int-soft left ideal over U. Note that for any $a, b \in S$, there exist $x, y \in S$ such that $b = xa$ and $a = yb$. Hence

$$\alpha(a) = \alpha(yb) \supseteq \alpha(b) = \alpha(xa) \supseteq \alpha(a),$$

and so $\alpha(a) = \alpha(b)$. This implies that $\alpha : S \to \mathcal{P}(U)$ is constant since a and b are arbitrarily in S. Similarly, if S is a right simple semigroup, then every int-soft right ideal over U is a constant function.

Conversely, suppose that in a semigroup S every its int-soft left (right) ideal over U is a constant function. Let A be a left ideal of S. Then the characteristic soft set (χ_A, S) over U is an int-soft left ideal over U by Lemma 3.7, and so it is constant by assumption. For any $x \in S$, we have $\chi_A(x) = U$ since A is nonempty, and thus $x \in A$. This shows that $A = S$. Therefore S is left simple. In the case that S is right simple, the proof follows similarly. $\square$

Theorem 3.9. *In a simple semigroup S, every int-soft interior ideal over U is constant.*

Proof. Let (α, S) be an int-soft interior ideal over U. For any $a, b \in S$, there exist $x, y \in S$ such that $a = xby$ (see [12, Lemma I.3.9]). Then

$$\alpha(a) = \alpha(xby) \supseteq \alpha(b).$$

Similarly, we have $\alpha(a) \subseteq \alpha(b)$, and so $\alpha(a) = \alpha(b)$. This shows that $\alpha : S \to \mathcal{P}(U)$ is constant since a and b are arbitrarily in S. $\square$

Theorem 3.10. *A soft set (α, S) over U is an int-soft interior ideal over U if and only if the nonempty γ-inclusive set of (α, S) is an interior ideal of S for all $\gamma \subseteq U$.*

Proof. Assume that (α, S) is an int-soft interior ideal over U. Let $\gamma \subseteq U$ be such that $i_S(\alpha; \gamma) \neq \emptyset$. Let $x, y \in S$ and $a \in i_S(\alpha; \gamma)$. Then $\alpha(a) \supseteq \gamma$. It follows from (3) that

$$\alpha(xay) \supseteq \alpha(a) \supseteq \gamma$$

and that $xay \in i_S(\alpha; \gamma)$. Thus $i_S(\alpha; \gamma)$ is an interior ideal of S.

Conversely, suppose that the nonempty γ-inclusive set of (α, S) is an interior ideal of S for all $\gamma \subseteq U$. Let $x, y, z \in S$ be such that $\alpha(y) = \gamma$. Then $y \in i_S(\alpha; \gamma)$, and so $xyz \in i_S(\alpha; \gamma)$. Hence

$$\alpha(xyz) \supseteq \gamma = \alpha(y)$$

which shows that (α, S) is an int-soft interior ideal over U. $\square$

Theorem 3.11. *If (α, S) is an int-soft interior ideal over U, then so is (α^*, S).*

Proof. Let $x, y, z \in S$. If $y \notin i_S(\alpha; \gamma)$, then it is clear that $\alpha(xyz) \supseteq \alpha(y)$. Assume that $y \in i_S(\alpha; \gamma)$. Then $xyz \in i_S(\alpha; \gamma)$ since $i_S(\alpha; \gamma)$ is an interior ideal of S by Theorem 3.10. Hence $\alpha^*(xyz) = \alpha(xyz) \supseteq \alpha(y) = \alpha^*(y)$. Therefore (α^*, S) is an int-soft interior ideal over U. □

For any soft set (α, S) over U, the smallest int-soft interior ideal over U containing (α, S) is called the *int-soft interior ideal over U generated by* (α, S), and is denoted by $(\alpha, S)_I$.

Theorem 3.12. *Let S be a semigroup with identity e. Then $(\alpha, S)_I = (\beta, S)$, where*

$$\beta(a) = \bigcup \{\alpha(y) \mid a = xyz,\ x, y, z \in S\}$$

for all $a \in S$.

Proof. Let $a \in S$. Since $a = eae$, we have

$$\beta(a) = \bigcup \{\alpha(y) \mid a = xyz,\ x, y, z \in S\} \supseteq \alpha(a),$$

and so $(\alpha, S) \tilde{\subseteq} (\beta, S)$. For all $x, a, y \in S$, we have

$$\begin{aligned}
\beta(xay) &= \bigcup \{\alpha(x_2) \mid xay = x_1x_2x_3,\ x_1, x_2, x_3 \in S\} \\
&\supseteq \bigcup \{\alpha(z_2) \mid xay = (xz_1)z_2(z_3y),\ a = z_1z_2z_3,\ z_1, z_2, z_3 \in S\} \\
&= \beta(a).
\end{aligned}$$

Thus (β, S) is an int-soft interior ideal over U. Now let (γ, S) be an int-soft interior ideal over U such that $(\alpha, S) \tilde{\subseteq} (\gamma, S)$. Then $\alpha(a) \subseteq \gamma(a)$ for all $a \in S$ and

$$\begin{aligned}
\beta(a) &= \bigcup \{\alpha(x_2) \mid a = x_1x_2x_3,\ x_1, x_2, x_3 \in S\} \\
&\subseteq \bigcup \{\gamma(x_2) \mid a = x_1x_2x_3,\ x_1, x_2, x_3 \in S\} \\
&\subseteq \bigcup \{\gamma(x_1x_2x_3) \mid a = x_1x_2x_3,\ x_1, x_2, x_3 \in S\} \\
&= \gamma(a).
\end{aligned}$$

which implies that $(\beta, S) \tilde{\subseteq} (\gamma, S)$. Therefore $(\alpha, S)_l = (\beta, S)$. □

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