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Fundamental residuated lattices

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Abstract. For any countable set X, we construct a residuated lattice and a weak hyper residuated lattice on X. Next, using the notion of (strongly) regular relation we construct the corresponding quotient weak hyper residuated lattice. Finally, by considering the fundamental relation we define the fundamental residuated lattice and we show that any residuated lattice is a fundamental residuated lattice.

1. Introduction

The study of residuated lattices originated in the context of the theory of ring ideals in 1930. The collection of all two-sided ideals of a ring forms a lattice upon which one can impose a natural monoid structure making this object into a residuated lattice. Such ideas were investigated by Ward and Dilworth in a series of papers such as [11]. Since that time, there has been substantial research regarding some specific classes of residuated structures, see for example [4] and [7]. The study of hyperstructures started in 1934 by Marty at 8th Congress of Scandinavian Mathematicians [10], which introduce the concept of hypergroup. Since then other classic hyperstructures have also been studied and many researches have been worked on this new field of modern algebra and developed it [3]. Recently, S. Ghorbani et al. [6], applied the hyperstructures to MV-algebras and introduced the concept of hyper MV-algebra, which is a generalization of MV-algebra. In the follow we constructed weak hyper residuated lattices [2] as a generalization of the concept of residuated lattices that contain of the classes of MV-algebras, BL-algebras, and Heyting algebras, subclasses of the class of residuated lattices. Now we prove that any residuated lattice is a fundamental residuated lattice.

2. Preliminaries

Recall that a hypergroupoid $(H, *, 1)$ is called a *commutative semihypergroup* with 1 as the identity, if it satisfies the following axioms:

(i) $x * (y * z) = (x * y) * z$,
(ii) $x * y = y * x$,
(iii) $x \in 1 * x$.

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By a *residuated lattice*, we mean an algebraic structure $(L, \vee, \wedge, *, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following conditions:
(RL1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
(RL2) $(L, *, 1)$ is a commutative monoid,
(RL3) the pair $(*, \to)$ is an adjoint pair, i.e., $x * y \leqslant z$ if and only if $x \leqslant y \to z$ for $x, y, z \in L$.

Proposition 2.1. [2] *Let A, B and C be nonempty subsets of a weak hyper residuated lattice $(L, \vee, \wedge, \odot, \to, 0, 1)$. Then for all $x, y, z \in L$ we have:*

(*i*) $1 \ll A \Leftrightarrow 1 \in A$, *and* $A \ll 0 \Leftrightarrow 0 \in A$,

(*ii*) $x \leqslant y \Rightarrow 1 \in x \to y$, *and* $A \ll B \Rightarrow 1 \in A \to B$,

(*iii*) *if* 1 *is a scalar, then* $1 \in x \to y \Rightarrow x \leqslant y$, *and* $1 \in A \to B \Rightarrow A \ll B$,

(*iv*) $1 \in (x \to x) \cap (x \to 1) \cap (0 \to x)$,

(*v*) *if* 1 *is a scalar, then* $x \in 1 \to x$,

(*vi*) $A \ll B \to C \Leftrightarrow A \odot B \ll C \Leftrightarrow B \ll A \to C$,

(*vii*) $x \odot y \ll x, y$, *and* $A \odot B \ll A, B$.

Definition 2.2. [5] Let R be an equivalence relation on a hypergroupoid (H, o). If A and B are two nonempty subsets of H, then

(*i*) $A\bar{R}B$ means that for each $a \in A$ there exists $b \in B$ such that aRb and for all $b \in B$ there exists $a \in A$ such that bRa,

(*ii*) $A\bar{\bar{R}}B$ means that aRb for all $a \in A$ and $b \in B$,

(*iii*) R is called *regular on the right* (*on the left*) if aRb implies $(aox)\bar{R}(box)$ (resp., $(xoa)\bar{R}(xob)$) for all $a, b, x \in H$. A relation regular on the right and left is called *regular*,

(iv) R is *strongly regular on the right* (*on the left*) if aRb implies $(aox)\bar{\bar{R}}(box)$ (resp., $(xoa)\bar{\bar{R}}(xob)$), for all $a, b, x \in H$. If R is regular on the right and left, then it is called *strongly regular*.

From now by $\mathcal{L}$ will be denoted a weak hyper residuated lattice $(L, \vee, \wedge, \odot, \to, 0, 1)$.

3. Countable (weak hyper) residuated lattices

Let $\mathbb{W}_k$ be a finite set and let $\mathbb{Q}_n$ be the set of all rational numbers in the interval $[1, n]$, $n > 1$. Below using these sets we construct finite and infinite residuated lattices and show that (weak hyper) residuated lattices with the same cardinality are isomorphic.

Theorem 3.1. *Let* $\mathbb{W}_k = \{0, 1, \ldots, k-1\}$. *Then* $(\mathbb{W}_k, \vee, \wedge, \odot, \rightarrow, 0, k-1)$, *where*

$$\begin{aligned} x \wedge y &= \min\{x, y\}, & x \odot y &= \begin{cases} 0, & x+y \leqslant k-1 \\ (x+y)-(k-1), & x+y > k-1 \end{cases} \\ x \vee y &= \max\{x, y\}, & x \rightarrow y &= \begin{cases} k-1, & x \leqslant y \\ (k-1)-x+y, & x > y \end{cases} \end{aligned}$$

is a residuated lattice.

Proof. It is easy to see that $(\mathbb{W}_k, \vee, \wedge, 0, k-1)$ is a bounded lattice. Since, $+$ is commutative and associative, then $\odot$ is commutative and associative. Also, for any $x \neq 0$, $x + (k-1) > k-1$ implies that $x \odot (k-1) = x + (k-1) - (k-1) = x$, i.e., $(k-1)$ is the identity and so $(\mathbb{W}_k, \odot, (k-1))$ is a commutative monoid. Hence, it is enough to show that $(\odot, \rightarrow)$ are adjoint pair. Let $x, y, z \in \mathbb{W}_k$ and $x \odot y \leqslant z$. If $y \leqslant z$, then $x \leqslant k-1 = y \rightarrow z$. Now, let $y > z$. If $x + y \leqslant k-1$, then we get $x \leqslant (k-1) - y \leqslant (k-1) - y + z = y \rightarrow z$. If $x + y > k-1$, then $x \odot y = (x+y) - (k-1) \leqslant z$ and so $x \leqslant (k-1) - y + z = y \rightarrow z$.

Conversely, let $x \leqslant y \rightarrow z$. If $x + y \leqslant k-1$, then $x \odot y = 0 \leqslant z$. Now, let $x + y > k-1$. If $y \leqslant z$, then $x + y \leqslant x + z$ and so we get $k-1 < x+y \leqslant x+z \leqslant (k-1)+z$. Thus $x + y \leqslant (k-1) + z$. Hence $x \odot y = x + y - (k-1) \leqslant z$. If $y > z$, then $x \leqslant y \rightarrow z = (k-1) - y + z$ implies that $x + y - (k-1) \leqslant z$. Hence $x \odot y \leqslant z$. □

Proposition 3.2. $(\mathbb{Q}_n, \vee, \wedge, \odot, \rightarrow, 1, n)$ *is a residuated lattice with respect to the operations:*

$$\begin{aligned} x \wedge y &= \min\{x, y\}, & x \odot y &= \begin{cases} 1, & xy \leqslant n, \\ \frac{xy}{n}, & xy > n \end{cases} \\ x \vee y &= \max\{x, y\}, & x \rightarrow y &= \begin{cases} n, & x \leqslant y, \\ \frac{n}{x}y, & x > y \end{cases} \end{aligned}$$

Proof. It is easy to check that $(\mathbb{Q}_n, \vee, \wedge, 1, n)$ is a bounded lattice and $(\mathbb{Q}_n, \odot, n)$ is a commutative monoid. Hence, we will prove the condition (RL3). Let $x, y, z \in \mathbb{Q}_n$ and $x \odot y \leqslant z$. If $y \leqslant z$ then $x \leqslant n = y \rightarrow z$. Now, let $y > z$. If $xy \leqslant n$ then $x \leqslant \frac{n}{y} \leqslant \frac{n}{y}z = y \rightarrow z$. If $xy > n$ then $x \odot y = \frac{xy}{n} \leqslant z$ and so $x \leqslant \frac{n}{y}z = y \rightarrow z$.

Conversely, let $x \leqslant y \rightarrow z$. If $xy \leqslant n$, then $x \odot y = 1 \leqslant z$. Now, let $xy > n$. If $y \leqslant z$, then $n < xy \leqslant xz$. Since $\frac{x}{n} \leqslant 1$, then $1 < \frac{xy}{n} \leqslant \frac{x}{n}z \leqslant z$. Thus $x \odot y \leqslant z$. Now, if $y > z$ then $x \leqslant y \rightarrow z = \frac{n}{y}z$ and so $\frac{xy}{n} \leqslant z$. Hence, $x \odot y \leqslant z$. □

Theorem 3.3. *For any infinite countable set* X, *there exist binary operations* $\odot, \rightarrow, \vee$ *and* $\wedge$ *on* X *and constants* $0, 1 \in X$ *such that* $(X, \vee, \wedge, \odot, \rightarrow 0, 1)$ *is a residuated lattice.*

Proof. Let X be an infinite countable set. Since $\mathbb{Q}_n$ in Proposition 3.2 is an infinite countable set, then $|\mathbb{Q}_n| = |X|$. Thus there exists a bijection $f : \mathbb{Q}_n \rightarrow X$. Hence, for any $x, y \in X$, there exist $t, s \in \mathbb{Q}_n$ such that $f(t) = x$ and $f(s) = y$. Now, we define the order $\preceq$ on X as: $x \preceq y$ iff $t \leqslant s$. It is easy to check that $(X, \preceq)$ is a lattice with $0_X = f(1)$ as the least and $1_X = f(n)$ as the greatest element of it. Hence, there exist operations $\vee$ and $\wedge$ on X such that $(X, \vee, \wedge, 0_X, 1_X)$ is a bounded lattice. Now, we define the operations $\odot$ and $\rightarrow$ on X as follows:

$$x \odot y = \begin{cases} f(1), & ts \leqslant n, \\ f(\frac{ts}{n}), & ts > n \end{cases} \qquad x \rightarrow y \quad = \quad \begin{cases} f(n), & t \leqslant s, \\ f(\frac{ns}{t}), & t > s \end{cases}$$

Since f is a bijection, then operations $\odot$ and $\rightarrow$ are well-defined. Moreover, $\odot$ is commutative and associative. Since, for any $x \in X$ there exists $t \in \mathbb{Q}_n$ such that $x = f(t)$. Hence, $x \odot 1_X = f(t) \odot f(n) = f(\frac{tn}{n}) = f(t) = x$. So 1_X is the identity and $(X, \odot, 1_X)$ is a commutative monoid. Now, we claim that $(\odot, \rightarrow)$ are adjoint pair. Let $x, y, z \in X$, then there exist $t, s, u \in \mathbb{Q}_n$ such that $x = f(t)$, $y = f(s)$ and $z = f(u)$. Let $x \odot y \preceq z$. If $s \leqslant u$, then $x \preceq f(n) = y \rightarrow z$. Now, let $s > u$. If $ts \leqslant n$, then $t \leqslant \frac{n}{s} \leqslant \frac{n}{s}u$. So, we get $x = f(t) \preceq f(\frac{n}{s}u) = y \rightarrow z$. If $ts > n$, since $x \odot y = f(\frac{ts}{n}) \preceq f(u) = z$, then $\frac{ts}{n} \leqslant u$. So $t \leqslant n\frac{u}{s}$. Hence, $x = f(t) \preceq f(n\frac{u}{s}) = y \rightarrow z$. Conversely, let $x \preceq y \rightarrow z$. If $ts \leqslant n$, then $x \odot y = f(1) \preceq z$. Now, let $ts > n$. If $s \leqslant u$, then we get $n < ts \leqslant tu$ and so $1 < \frac{t}{n}s \leqslant \frac{t}{n}u \leqslant u$. Hence, $x \odot y = f(\frac{ts}{n}) \preceq f(u) = z$. If $s > u$, since $x = f(t) \preceq y \rightarrow z = f(n\frac{u}{s})$, then $t \leqslant n\frac{u}{s}$. So, $\frac{ts}{n} \leqslant u$. Hence, $x \odot y = f(\frac{ts}{n}) \preceq f(u) = z$. Therefore, $(X, \vee, \wedge, \odot, \rightarrow, 0_X, 1_X)$ is an infinite countable residuated lattice. □

Theorem 3.4. *All (weak hyper) residuated lattices with the same cardinality are isomorphic.*

Proof. Let $(X, \vee_X, \wedge_X, \odot_X, \rightarrow_X, 0_X, 1_X)$ be a (weak hyper) residuated lattice and let Y be the set with the same cardinality as X. Then there exists a bijection $f : X \rightarrow Y$. Put $0_Y = f(0_X)$ and $1_Y = f(1_X)$. Since for any $y_1, y_2 \in Y$ there exist $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ on Y we can define the following hyperoperations:

$$y_1 \vee_Y y_2 = f(x_1 \vee_X x_2), \qquad y_1 \wedge_Y y_2 = f(x_1 \wedge_X x_2),$$

$$y_1 \odot_Y y_2 = f(x_1 \odot_X x_2), \qquad y_1 \rightarrow_Y y_2 = f(x_1 \rightarrow_X x_2).$$

It is not difficult to check that $(Y, \vee_Y, \wedge_Y, \odot_Y, \rightarrow_Y, 0_Y, 1_Y)$ is a (weak hyper) residuated lattice. The map $\varphi : X \rightarrow Y$ defined by $\varphi(x) = f(x)$ is an isomorphosm isomorphism between these two (weak hyper) residuated lattices. □

Corollary 3.5. Let X be a nonempty countable set. Then we can construct a residuated lattice on X such that $X \cong \mathbb{W}_k$ or $X \cong \mathbb{Q}_n$.

Proof. Since X is a finite or an infinite countable set, then $|X| = |\mathbb{W}_k|$, for $k \in \mathbb{N}$ or $|X| = |\mathbb{Q}_n|$ for $n \in \mathbb{N}$. By Theorem 3.1 and Proposition 3.2, $\mathbb{W}_k$ and $\mathbb{Q}_n$ are residuated lattices and so by Theorem 3.4, we can construct a residuated lattice on X such that $X \cong \mathbb{W}_k$ or $X \cong \mathbb{Q}_n$. □

Theorem 3.6. *Consider two residuated lattices* $\mathcal{L}_1 = (L_1, \vee_1, \wedge_1, \rightarrow_1, \odot_1, 0_1, 1_1)$ *and* $\mathcal{L}_2 = (L_2, \vee_2, \wedge_2, \rightarrow_2, \odot_2, 0_2, 1_2)$*. Then* $(L_1 \times L_2, \vee, \wedge, \rightarrow, \odot, (0_1, 0_2), (1_1, 1_2))$ *denoted by* $\mathcal{L}_1 \times \mathcal{L}_2$ *with the operations:*

$$\begin{aligned}
(x_1, y_1) \vee (x_2, y_2) &= (x_1 \vee_1 x_2,\ y_1 \vee_2 y_2),\\
(x_1, y_1) \wedge (x_2, y_2) &= (x_1 \wedge_1 x_2,\ y_1 \wedge_2 y_2),\\
(x_1, y_1) \rightarrow (x_2, y_2) &= \{(x_1 \rightarrow_1 x_2, y_2),\ (x_1 \rightarrow_1 x_2, y_1 \rightarrow_2 y_2)\},\\
(x_1, y_1) \odot (x_2, y_2) &= \{(x_1 \odot_1 x_2, y_1),\ (x_1 \odot_1 x_2, y_2),\ (x_1 \odot_1 x_2, y_1 \odot_2 y_2)\}
\end{aligned}$$

is a weak hyper residuated lattice (*with the natural ordering*).

Proof. It is obvious that $(L_1 \times L_2, \vee, \wedge, (0_1, 0_2), (1_1, 1_2))$ is a bounded lattice and the hyperoperation " $\odot$ " is associative and commutative. Since,

$$\begin{aligned}
(1, 1) \odot (x, y) &= \{(1 \odot_1 x, 1),\ (1 \odot_1 x, y),\ (1 \odot_1 x, 1 \odot_2 y)\}\\
&= \{(x, 1), (x, y)\} = (x, y) \odot (1, 1),
\end{aligned}$$

$(L_1 \times L_2, \odot, (1_1, 1_2))$ is a commutative semihypergroup with $(1, 1)$ as the identity.

Now, we will show that it satisfies (WHRL3), that is:

$$(x_1, y_1) \odot (x_2, y_2) \ll (z_1, z_2) \Leftrightarrow (x_1, y_1) \ll (x_2, y_2) \rightarrow (z_1, z_2).$$

Let $(x_1, y_1), (x_2, y_2), (z_1, z_2) \in L_1 \times L_2$. If $(x_1, y_1) \odot (x_2, y_2) \ll (z_1, z_2)$, then

$$\{(x_1 \odot_1 x_2, y_1),\ (x_1 \odot_1 x_2, y_2),\ (x_1 \odot_1 x_2, y_1 \odot_2 y_2)\} \ll (z_1, z_2).$$

Now,

(i) if $(x_1 \odot_1 x_2, y_1) \leqslant (z_1, z_2)$, then $x_1 \odot_1 x_2 \ll z_1$ and $y_1 \leqslant z_2$. By (WHRL3), $x_1 \ll x_2 \rightarrow_1 z_1$ and so $(x_1, y_1) \leqslant (x_2 \rightarrow_1 z_1, z_2) \in (x_2, y_2) \rightarrow (z_1, z_2)$.

(ii) if $(x_1 \odot_1 x_2, y_2) \leqslant (z_1, z_2)$, then $(x_1 \odot_1 x_2) \ll z_1$ and $y_2 \leqslant z_2$. By (WHRL3), $x_1 \ll x_2 \rightarrow_1 z_1$ and by Proposition 2.6 (ii), $1 \in y_2 \rightarrow_2 z_2$. So,

$$(x_1, y_1) \leqslant (x_2 \rightarrow_1 z_1, 1) \subseteq (x_2 \rightarrow_1 z_1, y_2 \rightarrow_2 z_2) \in (x_2, y_2) \rightarrow (z_1, z_2).$$

(iii) if $(x_1 \odot_1 x_2, y_1 \odot_2 y_2) \leqslant (z_1, z_2)$, then $(x_1 \odot_1 x_2 \ll z_1$ and $y_1 \odot_2 y_2 \ll z_2$. So, by (WHRL3), we have $x_1 \ll x_2 \rightarrow_1 z_1$ and $y_1 \ll y_2 \rightarrow_2 z_2$. Thus

$$(x_1, y_1) \leqslant (x_2 \rightarrow_1 z_1, y_2 \rightarrow_2 z_2) \in (x_2, y_2) \rightarrow (z_1, z_2).$$

Conversely, if $(x_1, y_1) \ll (x_2, y_2) \rightarrow (z_1, z_2)$, then

$$(x_1, y_1) \ll \{(x_2 \rightarrow_1 z_1, z_2), (x_2 \rightarrow_1 z_1, y_2 \rightarrow_2 z_2)\}.$$

Now,

(i) if $(x_1, y_1) \leqslant (x_2 \to_1 z_1, z_2)$, then $x_1 \ll x_2 \to_1 z_1$ and $y_1 \leqslant z_2$. So, by (WHRL3), we have $x_1 \odot_1 x_2 \ll z_1$. Therefore, $(x_1 \odot_1 x_2, y_1) \leqslant (z_1, z_2)$.

(ii) if $(x_1, y_1) \leqslant (x_2 \to_1 z_1, y_2 \to_2 z_2)$, then $x_1 \ll x_2 \to_1 z_1$ and $y_1 \leqslant y_2 \to_2 z_2$. So, by (WHRL3), we have $x_1 \odot_1 x_2 \ll z_1$ and $y_1 \odot_2 y_2 \ll z_2$. Therefore, $(x_1 \odot_1 x_2, y_1 \odot_2 y_2) \leqslant (z_1, z_2)$. □

Corollary 3.7. *For any nonempty countable set X we can construct a weak hyper residuated lattice on X.*

Proof. By Corollary 3.5, we can construct a residuated lattice on X and by Theorem 3.6, we can construct a weak hyper residuated lattice on $X \times \mathcal{L}$, for any residuated lattice $\mathcal{L}$. □

Theorem 3.8. *Let $(L, \vee, \wedge)$ be a lattice and two elements $e_0, e_1 \notin L$. If $\bar{L} = L \cup \{e_0, e_1\}$, then $(\bar{L}, \bar{\vee}, \bar{\wedge}, \odot, \to, e_0, e_1)$ is a weak hyper residuated lattice, where*

$$x \bar{\wedge} y = \begin{cases} x \wedge y, & x, y \in L, \\ e_0, & x = e_0 \text{ or } y = e_0, \\ x, & x \in \bar{L} \text{ and } y = e_1, \\ y, & x = e_1 \text{ and } y \in \bar{L}. \end{cases} \qquad x \odot y = \{e_0, x \bar{\wedge} y\},$$

$$x \bar{\vee} y = \begin{cases} x \vee y & x, y \in L, \\ e_1 & x = e_1 \text{ or } y = e_1, \\ x, & x \in \bar{L} \text{ and } y = e_0, \\ y & x = e_0 \text{ and } y \in \bar{L}, \end{cases} \qquad x \to y = \begin{cases} \{e_0, x, e_1\}, & x = y, \\ \{x, e_1\}, & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that $(\bar{L}, \bar{\vee}, \bar{\wedge}, e_0, e_1)$ is a bounded lattice. Since $\wedge$ is commutative, associative and $x \in \{e_0, x\} = \{e_0, x \bar{\wedge} e_1\} = x \odot e_1$, then $(\bar{L}, \odot, e_1)$ is a commutative semihypergroup with e_1 as the identity. Moreover, since, $e_0 \in x \odot y$ and $e_1 \in y \to z$, then we get $x \odot y \ll z$ if and only if $x \ll y \to z$. Hence, the proof is completed and $(\bar{L}, \bar{\vee}, \bar{\wedge}, \odot, \to, e_0, e_1)$ is a weak hyper residuated lattice. □

4. Quotient weak hyper residuated lattice

Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be a weak hyper residuated lattice. For any subset $A \subseteq L$, we denote by $\mathcal{L}(A)$, the set of all finite combinations of elements A with $*$, where $* \in (\vee, \wedge, \odot, \to)$.

Let $\beta_1 = \{(x, x) \mid x \in L\}$. For any integer $n > 1$, the relation β_n is defined as follows:

$$x \beta_n y \;\Leftrightarrow\; \exists (a_1, a_2, \ldots, a_n) \in L^n,\ \exists u \in \mathcal{L}(a_1, a_2, \ldots, a_n) \text{ such that } \{x, y\} \subseteq u.$$

It is clear that β_1 is reflexive and β_n is symmetric for any $n > 1$. Then the relation $\beta = \bigcup_{n \geq 1} \beta_n$ is reflexive and symmetric. Now, let $\beta^\star$ be the transitive closure of β.

Proposition 4.1. *$\beta^\star$ is a strongly regular relation with respect to $\to$ and $\odot$, and is compatible with $\vee$ and $\wedge$.*

Proof. For any $x, y \in L$, if $x\beta^\star y$, then there exist $a_0, a_1, \ldots, a_t \in L$ such that $a_0 = x$, $a_t = y$ and there exist $\{q_1, \ldots, q_t\} \subseteq \mathbb{N}$ such that

$$x = a_0\ \beta_{q_1}\ a_1\ \beta_{q_2}\ a_2\ \ldots\ a_{t-2}\ \beta_{q_{t-1}}\ a_{t-1}\ \beta_{q_t}\ a_t = y.$$

Since, for all $i \in \{1, \ldots, t\}$, we have $a_{i-1}\ \beta_{q_i}\ a_i$, then there exist $z_1, \ldots, z_{q_i} \in L$ such that $\{a_{i-1}, a_i\} \subseteq u \in \mathcal{L}(z_1, \ldots, z_{q_i})$. Now, for all $i \in \{1, \ldots, t\}$, $j \in \{1, \ldots, q_i\}$ and $s \in L$, we have

$$a_{i-1} \odot s \subseteq u \odot s \in \mathcal{L}(z^i_j \odot s), \qquad a_i \odot s \subseteq u \odot s \in \mathcal{L}(z^i_j \odot s).$$

So, for all $v \in a_{i-1} \odot s$, and $\acute{v} \in a_i \odot s$ we have $v\ \beta_{q_{i+1}}\ \acute{v}$. Thus, by definition of $\beta^\star$, we get

$$\forall z \in x \odot s = a_0 \odot s \ \text{ and } \ \forall w \in y \odot s = a_t \odot s \text{ we have } z\ \beta^\star\ w.$$

Therefore, $\beta^\star$ is a right (and similarly a left) strongly regular relation with respect to $\odot$.

Now, by replacing $\to$ with $\odot$ in the above, it is easy to see that $\beta^\star$ is a strongly regular relation on L with respect to $\to$. Moreover,

$$a_{i-1} \wedge s \in u \wedge s \in \mathcal{L}(z^i_j \wedge s), \qquad a_i \wedge s \in u \wedge s \in \mathcal{L}(z^i_j) \wedge s.$$

So, $\{a_{i-1} \wedge s, a_i \wedge s\} \subseteq u \in \mathcal{L}(z^i_j \wedge s)$. Hence, $(a_{i-1} \wedge s, a_i \wedge s) \in \beta_{q_{i+1}}$ and we get $x \wedge s = a_0 \wedge s\ \beta^\star\ a_t \wedge s = y \wedge s$. Similarly, we get that $x \vee s\ \beta^\star\ y \vee s$. Therefore $\beta^\star$ is compatible with respect to $\wedge$ and $\vee$. □

Note: From now on, when we say R is a regular-compatible relation on $\mathcal{L}$, we means that R is regular with respect to $\odot$ and $\to$, and it is compatible with respect to $\vee$ and $\wedge$.

Definition 4.2. Let R be a regular-compatibe relation on $\mathcal{L}$. We denote the set of all equivalence classes of R by $\frac{L}{R}$ and for any $R(x), R(y) \in \frac{L}{R}$ we define the operations

$$R(x)\bar{\vee}R(y) = R(x \vee y), \qquad R(x)\bar{\wedge}R(y) = R(x \wedge y),$$

$$R(x)\bar{\odot}R(y) = \bigcup_{t \in x \odot y} R(t), \qquad R(x)\bar{\to}R(y) = \bigcup_{t \in x \to y} R(t).$$

Also, $R(x) \preceq R(y)$ if and only if $x \leqslant y$.

Theorem 4.3. *Let R be an equivalence relation on $\mathcal{L}$. Then R is a regular-compatible relation on $\mathcal{L}$ if and only if $(\frac{L}{R}, \bar{\vee}, \bar{\wedge}, \bar{\odot}, \bar{\to}, R(0), R(1))$ is a weak hyper residuated lattice.*

Proof. ($\Rightarrow$) Let R be a regular relation on $\mathcal{L}$. First, we claim that $\bar{\vee}$, $\bar{\wedge}$, $\bar{\odot}$ and $\bar{\rightarrow}$ are well-defined. Let $R(x_1) = R(x_2)$ and $R(y_1) = R(y_2)$. Then x_1Rx_2 and y_1Ry_2. Since R is compatible with respect to $\vee$ and $\wedge$, then $(x_1 \vee y_1)R(x_2 \vee y_2)$ and $(x_1 \wedge y_1)R(x_2 \wedge y_2)$ i.e., $R(x_1)\bar{\vee}R(y_1) = R(x_2)\bar{\vee}R(y_2)$ and $R(x_1)\bar{\wedge}R(y_1) = R(x_2)\bar{\wedge}R(y_2)$ and so $\bar{\vee}$ and $\bar{\wedge}$ are well-defined.

Since R is regular relation, then x_1Rx_2 implies $(x_1 \rightarrow y_1)\bar{R}(x_2 \rightarrow y_1)$ and y_1Ry_2 implies $(x_2 \rightarrow y_1)\bar{R}(x_2 \rightarrow y_2)$. By definition of $\bar{R}$, we get $(x_1 \rightarrow y_1)\bar{R}(x_2 \rightarrow y_2)$. Now, we show that $R(x_1)\bar{\rightarrow}R(y_1) = R(x_2)\bar{\rightarrow}R(y_2)$: If $R(s) \in R(x_1)\bar{\rightarrow}R(y_1)$, then $R(s) = R(t)$ for some $t \in x_1 \rightarrow y_1$. Since $(x_1 \rightarrow y_1)\bar{R}(x_2 \rightarrow y_2)$, then there exists $t' \in x_2 \rightarrow y_2$ such that tRt'. Thus $R(t) = R(t')$ and so $R(s) = R(t') \in R(x_2)\bar{\rightarrow}R(y_2)$. Therefore, $R(x_1)\bar{\rightarrow}R(y_1) \subseteq R(x_2)\bar{\rightarrow}R(y_2)$. By the similar way, we can prove that $R(x_2)\bar{\rightarrow}R(y_2) \subseteq R(x_1)\bar{\rightarrow}R(y_1)$. So, $\bar{\rightarrow}$ is well-defined. By replacing $\odot$ with $\rightarrow$, we conclude that $\bar{\odot}$ is well-defined, too.

Now, we prove that $(\frac{L}{R}, \bar{\vee}, \bar{\wedge}, R(0), R(1))$ is a bounded lattice. Since for all $x \in L$, $0 \leqslant x \leqslant 1$, then we get $R(0) \preceq R(x) \preceq R(1)$. Also, since R is compatible with respect to $\vee, \wedge$, then it is clear that $(\frac{L}{R}, \bar{\vee}, \bar{\wedge})$ is a bounded lattice with $R(0)$ as least element and $R(1)$ as greatest one.

Moreover, we show that $(\frac{L}{R}, \bar{\odot}, R(1))$ is a commutative semihypergroup with 1 as the identity. Let $R(x), R(y), R(z) \in \frac{L}{R}$. Then

$$R(x)\bar{\odot}R(y) = \bigcup_{t \in x \odot y} R(t) = \bigcup_{t \in y \odot x} R(t) = R(y)\bar{\odot}R(x).$$

Hence, $(\frac{L}{R}, \bar{\odot})$ is commutative. Since

$$\begin{aligned} R(x)\bar{\odot}(R(y)\bar{\odot}R(z)) &= R(x)\bar{\odot} \bigcup_{t \in y \odot z} R(t) = \bigcup_{\acute{t} \in x \odot (y \odot z)} R(\acute{t}) \\ &= \bigcup_{s \in (x \odot y) \odot z} R(s) = (\bigcup_{\acute{s} \in x \odot y} R(\acute{s}))\bar{\odot}R(z) \\ &= (R(x)\bar{\odot}(R(y))\bar{\odot}R(z), \end{aligned}$$

we see that $\bar{\odot}$ is associative. Also $R(x) \in R(x)\bar{\odot}R(1)$, since $x \in x \odot 1$ and $R(x)\bar{\odot}R(1) = \bigcup_{t \in x \odot 1} R(t)$. Thus $R(1)$ is an identity.

Finally, we verify the condition (WHRL3). For this, let $R(x)\bar{\odot}R(y) \preceq R(z)$. Then $\bigcup_{t \in x \odot y} R(t) \preceq R(z)$ and so there exists $t \in x \odot y$ such that $R(t) \preceq R(z)$ i.e., $t \leqslant z$ (by definition of $\preceq$). Thus $x \odot y \ll z$ and so, by (WHRL3), we get $x \ll y \rightarrow z$. Therefore, there exists $\acute{t} \in y \rightarrow z$ such that $x \leqslant \acute{t}$ and so

$$R(x) \preceq R(\acute{t}) \in \bigcup_{\acute{t} \in y \rightarrow z} R(\acute{t}) = R(y)\bar{\rightarrow}R(z).$$

Hence, we get $R(x) \preceq R(y)\bar{\rightarrow}R(z)$. The converse can be proved, by the similar way.

($\Leftarrow$) Let $(\frac{L}{R}, \bar{\vee}, \bar{\wedge}, \bar{\odot}, \bar{\rightarrow}, R(0), R(1))$ be a weak hyper residuated lattice and let

$a, b, c \in L$ be such that aRb. Then $R(a) = R(b)$ and so $R(a) \bar{\rightarrow} R(c) = R(b) \bar{\rightarrow} R(c)$. Hence, $\bigcup_{t \in a \to c} R(t) = \bigcup_{\acute{t} \in b \to c} R(\acute{t})$. So, for any $t \in a \to c$ there exists $\acute{t} \in b \to c$ such that $R(t) = R(\acute{t})$ i.e. $t\ R\acute{t}$. Thus $a \to c\ \bar{R}\ b \to c$. Similarly, $c \to a\ \bar{R}\ c \to b$. By the similar way, we can prove that $a \odot c\bar{R}b \odot c$ and $c \odot a\bar{R}c \odot b$. Hence, R is regular respect to $\to$ and $\odot$. In the follow, it is easy to prove that R is compatible respect to $\vee$ and $\wedge$. Therefore R is a regular-compatible relation on $\mathcal{L}$. □

Theorem 4.4. *Let $f : \mathcal{L} \to \mathcal{L}'$ be a homomorphism between two weak hyper residuated lattices. Then there exists a regular-compatible relation R_f on $\mathcal{L}$, such that $f(\mathcal{L}) \cong \frac{\mathcal{L}}{R_f}$. Specially, if f is onto, then $\mathcal{L}' \cong \frac{\mathcal{L}}{R_f}$.*

Proof. Let $R_f = \{(x, y) \in L^2 | f(x) = f(y)\})$. It is obvious that R_f is an equivalence relation on $\mathcal{L}$. Let $x, y, a \in L$ be such that $x\ R_f\ y$. Then $f(x) = f(y)$. Since f is a homomorphism, then it is easy to see that R_f is compatible with $\vee$ and $\wedge$. Also, $f(x \to a) = f(x) \to' f(a) = f(y) \to' f(a) = f(y \to a)$, it follows that for all $u \in x \to a$ there exists $v \in y \to a$ such that $f(u) = f(v)$ i.e., uR_fv. Similarly, regularity to the left can be shown. Also, by replacing $\odot$ with $\to$, we get R_f is regular with respect to $\odot$. Hence, R_f is a regular-compatible relation on $\mathcal{L}$ and so by Theorem 4.3, $\frac{L}{R_f}$ is a weak hyper residuated lattice. Now, let $g : f(L) \to \frac{L}{R_f}$ is defined by $g(f(x)) = R_f(x)$. Since

$$f(x) = f(y) \Leftrightarrow R_f(x) = R_f(y) \Leftrightarrow g(f(x)) = g(f(y)),$$

then g is well defined and one-to-one. Moreover, by Theorem 4.3, g is a homomorphism and it is easy to see that g is onto. Therefore, g is an isomorphism and so $f(\mathcal{L}) \cong \frac{\mathcal{L}}{R_f}$. Now, if f is onto, then $\mathcal{L} \cong \frac{\mathcal{L}}{R_f}$. □

Theorem 4.5. *If R is a strongly regular-compatible relation on $\mathcal{L}$, then $\frac{\mathcal{L}}{R}$ is a residuated lattice.*

Proof. Let R be a strongly regular-compatible relation on $\mathcal{L}$. By Theorem 4.3, $\frac{\mathcal{L}}{R}$ is a weak hyper residuated lattice. Now, it is enough to prove that for any $x, y \in L$, $|R(x) \bar{\rightarrow} R(y)| = 1$ and $|R(x) \bar{\odot} R(y)| = 1$. If $R(t), R(\acute{t}) \in R(x) \bar{\rightarrow} R(y)$, then $t, \acute{t} \in x \to y$. Since R is strongly regular and xRx, then $x \to y\ \bar{\bar{R}}\ x \to y$ and so $t\ R\ \acute{t}$. Hence, $R(t) = R(\acute{t})$ and this implies that $|R(x) \bar{\rightarrow} R(y)| = 1$. By the similar way, we can prove that $|R(x) \bar{\odot} R(y)| = 1$. Therefore, $\frac{\mathcal{L}}{R}$ is a residuated lattice on $\mathcal{L}$. □

Proposition 4.6. *Let R be a strongly regular-compatible relation on $\mathcal{L}$. If $\mathcal{L}'$ is a residuated lattice and $f : \mathcal{L} \to \mathcal{L}'$ is a homomorphism, then the equivalence relation R_f is a strongly regular-compatible relation.*

Proof. Let R be a strongly regular-compatible relation on $\mathcal{L}$, xR_fy and $a \in L$. Since $f(x) = f(y)$ we get

$$f(x \wedge a) = f(x) \wedge f(a) = f(y) \wedge f(a) = f(y \wedge a)$$

and so $(x\wedge a)R_f(y\wedge a)$. Similarly, $(x\vee a)R_f(y\vee a)$. Thus R_f is compatible respect to $\wedge$ and $\vee$. Also, for any $u\in x\to a$ and $v\in y\to a$, we get

$$f(u)\in f(x\to a)=f(x)\to f(a)=f(y)\to f(a)=f(y\to a)\ni f(v).$$

Since $\mathcal{L}'$ is a residuated lattice, then $|f(x\to a)|=1=|f(y\to a)|$ and so $f(u)=f(v)$ i.e., uRv. Thus $x\to a\ \bar{\bar{R}}_f\ y\to a$. Hence R_f is a strongly regular on the right and similarly on the left respect to $\to$. By the similar way, we can prove that R_f is strongly regular respect to $\odot$. Therefore, R_f is a strongly regular-compatible relation. □

Theorem 4.7. *The relation $\beta^\star$ is the smallest strongly regular-compatible relation on $\mathcal{L}$ such that the quotient $\frac{\mathcal{L}}{\beta^\star}$ is a residuated lattice.*

Proof. By Proposition 4.1, $\beta^\star$ is a strongly regular-compatible relation on $\mathcal{L}$ and so by Theorem 4.5, $\frac{\mathcal{L}}{\beta^\star}$ is a residuated lattice. Let ρ be an equivalence relation on $\mathcal{L}$ such that $\frac{L}{\rho}$ is a residuated lattice. Now, if $\pi: L\to\frac{L}{\rho}$ be the natural projection $\pi(x)=\rho(x)$ and $x\beta^\star y$, then there exist $z_1,\ldots,z_n\in L^n$ such that $\{x,y\}\subseteq u\in\mathcal{L}(z_1,\ldots,\ z_n)$. Since $\{\pi(x),\pi(y)\}\subseteq\pi(u)$ and $|\pi(u)|=1$, then $\pi(x)=\pi(y)$ and so $\rho(x)=\rho(y)$ i.e., $x\rho y$. Thus $\beta^\star\subseteq\rho$. □

Proposition 4.8. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two weak hyper residuated lattices, then the cartesian product $\mathcal{L}_1\times\mathcal{L}_2$ is a weak hyper residuated lattice with the operations*

$$\begin{aligned}(x_1,y_1)\vee(x_2,y_2)&=(x_1\vee x_2,y_1\vee y_2),\\(x_1,y_1)\wedge(x_2,y_2)&=(x_1\wedge x_2,y_1\wedge y_2),\\(x_1,y_1)\odot(x_2,y_2)&=\{(a,b)|a\in x_1\odot x_2,b\in y_1\odot y_2\},\\(x_1,y_1)\to(x_2,y_2)&=\{(a',b')|a'\in x_1\to x_2,b'\in y_1\to y_2\}.\end{aligned}$$

Proof. The proof is straightforward. □

Lemma 4.9. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two weak hyper residuated lattices. Then for $a,c\in\mathcal{L}_1$, $b,d\in\mathcal{L}_2$ we have $(a,b)\beta^\star_{\mathcal{L}_1\times\mathcal{L}_2}(c,d)$ if and only if $a\beta^\star_{\mathcal{L}_1}c$ and $b\beta^\star_{\mathcal{L}_2}d$.*

Proof. Since $u\in\mathcal{L}(L_1\times L_2)$ if and only if there exist $u_1\in\mathcal{L}(L_1)$ and $u_2\in\mathcal{L}(L_2)$ such that $u=u_1\times u_2$, then $(a,b)\beta^\star_{\mathcal{L}_1\times\mathcal{L}_2}(c,d)$ if and only if there exist $u\in\mathcal{L}_1$ and $v\in\mathcal{L}_2$ such that $\{(a,b),(c,d)\}\subseteq u\times v$ if and only if $\{a,c\}\subseteq u$ and $\{b,d\}\subseteq v$ if and only if $a\beta^\star_{\mathcal{L}_1}c$ and $b\beta^\star_{\mathcal{L}_2}d$. □

Theorem 4.10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two weak hyper residuated lattices. Then $\frac{\mathcal{L}_1\times\mathcal{L}_2}{\beta^\star_{\mathcal{L}_1\times\mathcal{L}_2}}\cong\frac{\mathcal{L}_1}{\beta^\star_{\mathcal{L}_1}}\times\frac{\mathcal{L}_2}{\beta^\star_{\mathcal{L}_2}}$.*

Proof. Let $\varphi:\frac{\mathcal{L}_1\times\mathcal{L}_2}{\beta^\star_{\mathcal{L}_1\times\mathcal{L}_2}}\to\frac{\mathcal{L}_1}{\beta^\star_{\mathcal{L}_1}}\times\frac{\mathcal{L}_2}{\beta^\star_{\mathcal{L}_2}}$ be defined by $\varphi(\beta^\star(x,y))=(\beta^\star(x),\beta^\star(y))$. First, by Lemma 4.9, we have $\beta^\star(x_1,y_1)=\beta^\star(x_2,y_2)$ if and only if $\beta^\star(x_1)=$

$\beta^\star(x_2)$ and $\beta^\star(y_1) = \beta^\star(y_2)$ if and only if $\varphi(\beta^\star(x_1, y_1)) = \varphi(\beta^\star(x_2, y_2))$, for any $(x_1, y_1), (x_2, y_2) \in \mathcal{L}_1 \times \mathcal{L}_2$. So, φ is well defined and one to one. Also, we have

$$\begin{aligned}\varphi(\beta^\star(x_1, y_1)\bar{\vee}\beta^\star(x_2, y_2)) &= \varphi(\beta^\star(x_1 \vee x_2, y_1 \vee y_2)) \\ &= (\beta^\star(x_1 \vee x_2),\ \beta^\star(y_1 \vee y_2) \\ &= (\beta^\star(x_1)\bar{\vee}\beta^\star(x_2),\ \beta^\star(y_1)\bar{\vee}\beta^\star(y_2)) \\ &= (\beta^\star(x_1),\ \beta^\star(y_1)) \vee\ (\beta^\star(x_2), \beta^\star(y_2)) \\ &= \varphi(\beta^\star(x_1, y_1)) \vee\ \varphi(\beta^\star(x_2, y_2))\end{aligned}$$

and similarly, $\varphi(\beta^\star(x_1, y_1)\bar{\wedge}\beta^\star(x_2, y_2)) = \varphi(\beta^\star(x_1, y_1)) \wedge \varphi(\beta^\star(x_2, y_2))$. Moreover,

$$\begin{aligned}\varphi(\beta^\star(x_1, y_1)\bar{\rightarrow}\beta^\star(x_2, y_2)) &= \varphi(\bigcup_{a\in x_1\to x_2, b\in y_1\to y_2} \beta^\star(a, b)) \\ &= \bigcup_{a\in x_1\to x_2, b\in y_1\to y_2} \varphi(\beta^\star(a, b)) \\ &= \bigcup_{a\in x_1\to x_2, b\in y_1\to y_2} (\beta^\star(a), \beta^\star(b)) \\ &= (\bigcup_{a\in x_1\to x_2} \beta^\star(a), \bigcup_{b\in y_1\to y_2} \beta^\star(b)) \\ &= (\beta^\star(x_1)\bar{\rightarrow}\beta^\star(x_2), \beta^\star(y_1)\bar{\rightarrow}\beta^\star(y_2)) \\ &= (\beta^\star(x_1), \beta^\star(y_1)) \to (\beta^\star(x_2), \beta^\star(y_2)) \\ &= \varphi(\beta^\star(x_1, y_1)) \to \varphi(\beta^\star(x_2, y_2)).\end{aligned}$$

By the similar way, $\varphi(\beta^\star(x_1, y_1)\bar{\odot}\beta^\star(x_2, y_2)) = \varphi(\beta^\star(x_1, y_1)) \odot \varphi(\beta^\star(x_2, y_2))$. Hence, φ is an isomorphism. □

Corollary 4.11. *Let all $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be a weak hyper residuated lattice. Then*

$$\frac{\mathcal{L}_1 \times \mathcal{L}_2 \times ... \times \mathcal{L}_n}{\beta^\star_{\mathcal{L}_1 \times \mathcal{L}_2 \times ... \times \mathcal{L}_n}} \cong \frac{\mathcal{L}_1}{\beta^\star_1} \times \frac{\mathcal{L}_2}{\beta^\star_2} \times ... \times \frac{\mathcal{L}_n}{\beta^\star_n}.$$

Proof. The proof is similar to the proof of Theorem 4.10. □

Theorem 4.12. *Let X and Y be two sets such that $|X| = |Y|$. If X is a weak hyper residuated lattice, then on Y we can construct a weak hyper residuated lattice such that $\frac{X}{\beta^\star_X} \cong \frac{Y}{\beta^\star_Y}$.*

Proof. By Theorem 3.4, on Y we can construct a weak hyper residuated lattice and an isomorphism $\varphi : X \to Y$. Now, we define $\psi : \frac{X}{\beta^\star_X} \to \frac{Y}{\beta^\star_Y}$ as $\psi(\beta^\star(x)) = \beta^\star(\varphi(x))$.

Let $y_1, y_2 \in Y$ be arbitrary. Since φ is an isomorphism, then ψ is onto and there exist $x_1, x_2 \in X$ such that $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$ and by

$$\begin{aligned}
\beta^\star_X(x_1) = \beta^\star_X(x_2) &\Leftrightarrow x_1 \beta^\star x_2 \\
&\Leftrightarrow \exists u \in \mathcal{L}(X) \text{ such that } \{x_1, x_2\} \subseteq u \\
&\Leftrightarrow \varphi(u) \in \mathcal{L}(Y) \text{ such that } \{\varphi(x_1), \varphi(x_2)\} \subseteq \varphi(u) \\
&\Leftrightarrow \beta^\star_Y(y_1) = \beta^\star_Y(y_2).
\end{aligned}$$

we see that ψ is well-defined and one-to-one.

Now, we show $\psi(\beta^\star_X(x_1)\bar{\odot}\beta^\star_X(x_2)) = \psi(\beta^\star_X(x_1)) \odot \psi(\beta^\star_X(x_2))$. Indeed,

$$\begin{aligned}
\psi(\beta^\star_X(x_1)\bar{\odot}\beta^\star_X(x_2)) &= \psi(\bigcup_{t \in x_1 \odot x_2} \beta^\star_X(t)) = \bigcup_{t \in x_1 \odot x_2} \psi(\beta^\star_X(t)) \\
&= \bigcup_{t \in x_1 \odot x_2} \beta^\star_Y(\varphi(t)) = \bigcup_{\acute{t} \in \varphi(x_1 \odot x_2)} \beta^\star_Y(\acute{t}) \\
&= \bigcup_{\acute{t} \in \varphi(x_1) \odot \varphi(x_2)} \beta^\star_Y(\acute{t}) = \beta^\star_Y(\varphi(x_1))\bar{\odot}\beta^\star_Y \varphi(x_2) \\
&= \psi(\beta^\star_X(x_1)) \odot \psi(\beta^\star_X(x_2)).
\end{aligned}$$

Similarly for $\to$, $\wedge$ and $\vee$. Hence, ψ is a homomorphism and so it is an isomorphism. Therefore, $\frac{X}{\beta^\star_X} \cong \frac{Y}{\beta^\star_Y}$. $\square$

5. Fundamental residuated lattice

In this section, we define the fundamental rasiduated lattice and show that every residuated lattice is fundamental.

Theorem 5.1. *Let $\mathcal{L}$ and $\acute{\mathcal{L}}$ be two residuated lattices and $\mathcal{L} \times \mathcal{L}'$ be weak hyper residuated lattice as in Theorem 3.6. Then $\frac{\mathcal{L} \times \mathcal{L}'}{\beta^\star_{L \times L'}} \cong \mathcal{L}$.*

Proof. Let $\mathcal{L} \times \mathcal{L}'$ be a weak hyper residuated lattice as in Theorem 3.6 and $u \in \mathcal{L}((x_1, y_1), \ldots, (x_n, y_n)) \subseteq \mathcal{L}(L \times L')$, where $(x_i, y_i) \in L \times L'$. Then

$$u \in (\mathcal{L}(x_1, \ldots, x_n), \mathcal{L}(y_1, \ldots, y_n)) \bigcup (\mathcal{L}(x_1, \ldots, x_n), y_i), \qquad 1 \leqslant i \leqslant n.$$

So, there exist $x_0 \in \mathcal{L}(x_1, \ldots, x_n) \subseteq L$ and $y_0 \in \mathcal{L}(y_1, \ldots, y_n) \subseteq L'$ such that $u \in \{(x_0, y_0), (x_0, y_i)_{i=1}^n\}$. Hence, for any finite combination $u \in \mathcal{L}(L \times L')$, we have u is the form of $u = \{(x_0, t) \,|\, x_0 \text{ is fixed in L}, \ t \in L' \text{is variable}\}$. Thus, for all $(a, c) \in L^2$, $(b, d) \in L'^2$, we get $(a, b)\beta^\star(c, d)$ if and only if there exists $u \in \mathcal{L}(L \times L')$ such that $\{(a, b), (c, d)\} \subseteq u$ if and only if there exist $u_1 \in \mathcal{L}(L)$ and $u_2 \in \mathcal{L}(L')$ such that $\{a, c\} \subseteq u_1$ and $\{b, d\} \subseteq u_2$ if and only if $a = c$. Now, define

$\psi : \frac{\mathcal{L}\times\mathcal{L}'}{\beta^\star_{L\times L'}} \to \mathcal{L}$, by $\psi(\beta^\star(a,b)) = a$. Since $\beta^\star(a,b) = \beta^\star(c,d)$ if and only if $a = c$ if and only if $\psi(\beta^\star(a,b)) = \psi(\beta^\star(c,d))$, then ψ is well-defined and one to one. Also,

$$\begin{aligned}\psi(\beta^\star(a,b) \Rightarrow \beta^\star(c,d)) &= \psi(\bigcup_{(t,t')\in(a,b)\to(c,d)} (\{\beta^\star(t,t'))\\ &= \{\psi(\beta^\star(t,t'))|(t,t') \in \{(a \to_L c, d), (a \to_L c, b \to_{L'} d)\}\}\\ &= \{t|t \in a \to_L c\} = a \to_L c\\ &= \psi(\beta^\star(a,b)) \to_L \psi(\beta^\star(c,d)).\end{aligned}$$

Similarly for other operations. Hence ψ is a homomorphism. It is clear that ψ is onto. Thus ψ is an isomorphism and $\frac{\mathcal{L}\times\mathcal{L}'}{\beta^\star_{L\times L'}} \cong \mathcal{L}$. $\square$

Definition 5.2. A residuated lattice $\mathcal{L}$ is called a *fundamental residuated lattice*, if there exists a nontrivial weak hyper residuated lattice $\mathcal{X}$ such that $\frac{\mathcal{X}}{\beta^\star} \cong \mathcal{L}$.

Proposition 5.3. *Any residuated lattice $\mathcal{L}$ is a fundamental residuated lattice.*

Proof. By Theorem 3.6, for any residuated lattice $\mathcal{L}$, we can construct a weak hyper residuated lattice $\mathcal{L} \times \mathcal{L}'$ that $\mathcal{L}'$ is a nontrivial residuated lattice. Now, by Theorem 5.1, $\frac{\mathcal{L}\times\mathcal{L}'}{\beta^\star_{L\times L'}} \cong \mathcal{L}$ and so $\mathcal{L}$ is fundamental. $\square$

Corollary 5.4. *Any countable set can be considered as a fundamental residuated lattice.*

Proof. This follows by Corollary 3.7 and Proposition 5.3. $\square$

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