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Structure and representations of finite dimensional Malcev algebras

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Abstract. The paper [6] is devoted to the study of the basic structure theory of finite dimensional Malcev algebras. Similarly to the structure of finite dimensional Lie algebras, this theory has attracted a lot of attention and stimulated further research in this area. However, for the sake of brevity, detailed proofs of some results were omitted. Some authors experienced some difficulty owing to the lack of detailed proofs (see, for example [14]). The present work mostly follows the outline of [6] and fills the gaps in the literature.

Editors' Preface

This is an English translation of *Structure and representations of finite dimensional Malcev algebras* by E. N. Kuzmin, originally published in *Akademiya Nauk SSSR, Sibirskoe Otdelenie, Trudy Instituta Matematiki (Novosibirsk), Issledovaniya po Teorii Kolets i Algebr* 16 (1989), 75–101. The translation by Marina Tvalavadze was edited by Murray Bremner and Sara Madariaga. A brief survey of recent developments is included at the end of the paper.

1. Representations of nilpotent Malcev algebras. Cartan subalgebras

1.1. Malcev algebras were first introduced in [10] as Moufang-Lie algebras. They are defined by the identities,

$$x^2 = 0, \tag{1}$$

$$J(x, y, xz) = J(x, y, z)x, \tag{2}$$

where $J(x, y, z)$ is the so-called *Jacobian* of x, y, z:

$$J(x, y, z) = (xy)z + (yz)x + (zx)y.$$

In any anticommutative algebra, the Jacobian $J(x, y, z)$ is a skew-symmetric function of its arguments. Expanding the Jacobian, the Malcev identity (2) can be

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rewritten as follows,

$$xyzx + yzxx + zxxy = (xy)(xz), \tag{3}$$

where, for convenience, we omit parentheses in left-normed products: $((xy)z)x$.

Following [11] let us establish some basic identities which hold in all Malcev algebras. If A is a Malcev algebra, $x \in A$ and $R_x\colon a \mapsto ax$ is the operator of right multiplication by the element x, then the associative algebra A^* generated by $\{R_x \mid x \in A\}$ is called the *multiplication algebra* of A. Identity (3) implies that the following identities hold in A^*:

$$R_y R_x^2 = R_x^2 R_y + R_{yx} R_x + R_x R_{yx}, \tag{4}$$

$$R_x R_z R_x = R_z R_x^2 - R_x R_{zx} - R_{zxx}. \tag{5}$$

Linearizing identity (2) in x we obtain

$$J(x, y, uz) + J(u, y, xz) = J(x, y, z)u + J(u, y, z)x. \tag{6}$$

Hence the following identity holds in A^*:

$$R_{xyz} + R_{yzx} = R_y R_z R_x - R_z R_x R_y - R_{yz} R_x - R_{zx} R_y + R_y R_{zx} + R_z R_{xy}. \tag{7}$$

Adding the three identities obtained by cyclic permutations of the variables in (7), and assuming we work in characteristic different from 2, we obtain

$$R_{J(x,y,z)} = [R_y, R_{zx}] + [R_z, R_{xy}] + [R_x, R_{yz}], \tag{8}$$

where the square brackets denote the commutator of two operators:

$$[X, Y] = XY - YX.$$

Subtracting (7) from (8) we obtain

$$R_{zxy} = R_z R_x R_y - R_y R_z R_x + R_x R_{yz} - R_{xy} R_z,$$

or equivalently,

$$R_{xyz} = R_x R_y R_z - R_z R_x R_y + R_y R_{zx} - R_{yz} R_y. \tag{9}$$

Identity (9) implies that the following identity holds in A:

$$xyzt + yztx + ztxy + txyz = (xz)(yt); \tag{10}$$

this becomes identity (3) when setting $t = x$. If the characteristic of the field is other than 2 then identity (10) (also known as the Sagle identity) is equivalent to identity (3). Identity (10) presents some advantages: it is linear in each variable and invariant under cyclic permutations of the variables. Therefore, it is reasonable to use it (together with the anticommutative identity $x^2 = 0$) as the definition of the class of Malcev algebras in characteristic 2.

It is easy to check that Lie algebras in particular satisfy identity (10). On the other hand, it is easy to show that any Malcev algebra is binary-Lie: if u, v, w are arbitrary nonassociative words in two variables then, using induction on the length of u, v, w and identities (1), (2) and (6), it can be shown that $J(u, v, w) = 0$ when substituting for the variables any two elements of A. Therefore, the class of Malcev algebras can be regarded as an intermediate class between Lie algebras and binary-Lie algebras.

If we set $\Delta(x, y) = [R_x, R_y] - R_{xy}$ then $z\Delta(x, y) = J(z, x, y)$. From (8) we obtain

$$\begin{aligned}\Delta(y, zx) + \Delta(z, xy) + \Delta(x, yz) &= [R_y, R_{zx}] + [R_z, R_{xy}] + [R_x, R_{yz}] + R_{J(x,y,z)} \\ &= 2R_{J(x,y,z)},\end{aligned}$$

which can be written in the form

$$2wJ(x, y, z) = J(w, y, zx) + J(w, z, xy) + J(w, x, yz). \tag{11}$$

Define $D(x, y) = R_{xy} + [R_x, R_y]$. Since (9) is symmetric in x and y we obtain

$$2R_{xyz} = [[R_x, R_y], R_z] + [R_y, R_{zx}] + [R_x, R_{yz}]. \tag{12}$$

Subtracting (12) from (8) we obtain

$$R_{yzx+zxy-xyz} = R_{zD(x,y)} = [R_z, R_{xy}] + [R_z, [R_x, R_y]] = [R_z, D(x, y)],$$

which can be written in the form

$$(tz)D(x, y) = (tD(x, y))\, z + t\,(zD(x, y))\,. \tag{13}$$

This means that $D(x, y)$ is a derivation of A. If we set $R(x, y) = 2R_{xy} + [R_x, R_y]$ then it follows from (12) that

$$[R(x, y), R_z] = 2[R_{xy}, R_z] + 2R_{xyz} - [R_y, R_{zx}] - [R_x, R_{yz}]. \tag{14}$$

Adding (14) to (12) multiplied by 2 we obtain

$$[R(x, y), R_z] + 2R_{yzx} + 2R_{zxy} = [R_y, R_{zx}] + [R_x, R_{yz}],$$

which can be written in the form

$$[R(x, y), R_z] = R(xz, y) + R(x, yz). \tag{15}$$

Note that the identity

$$R_xR_yR_x = R_x^2R_y + R_{yx}R_x - R_{yxx}, \tag{16}$$

is a consequence of (4) and (5). A more general identity follows from (16) using induction on n:

$$R_x^n R_y R_x = R_x^{n+1} R_y + R_x^n R_{yx} - R_x R_{yx^n} - R_{yx^{n+1}} + R_{yx^n} R_x. \qquad (17)$$

To perform the inductive step it suffices to multiply both sides of (17) by R_x on the left and then substitute the term $R_x R_{yx^n} R_x$ using (16).

1.2. Let A be a Malcev algebra over a field F. According to [1], by a representation of A on a vector space V over F we understand a linear map $\rho\colon A \to \mathrm{End}(V)$ which provides the direct sum of vector spaces $V \oplus A$ with the structure of a Malcev algebra by setting

$$(v_1 + x)(v_2 + y) = v_1\rho(y) - v_2\rho(x) + xy \qquad (v_1, v_2 \in V,\ x, y \in A).$$

The algebra defined this way is called the *semidirect* or *split extension* of A by V, in which V (resp. A) appears as an abelian ideal (resp. Malcev subalgebra) of $V \oplus A$. The identities satisfied by ρ are similar to (9):

$$\rho(xyz) = \rho(x)\rho(y)\rho(z) - \rho(z)\rho(x)\rho(y) + \rho(y)\rho(zx) - \rho(yz)\rho(x).$$

A vector space V on which a representation is defined is called *Malcev A-module*. There is a special representation of the form $x \mapsto R_x$ (the regular representation). We will denote an arbitrary representation by R_x instead of $\rho(x)$. This will not lead to any confusion because it should be clear from the context which representation we mean†. It is easy to check that identities (12), (15)–(17) hold not only for the regular representation but also for arbitrary representations of a Malcev algebra A.

If a linear representation $\rho\colon A \to \mathrm{End}(V)$ satisfies

$$R_{xy} = [R_x, R_y], \qquad (18)$$

for any $x, y \in A$ then (9) follows from (18). Therefore, ρ is a representation of A (a homomorphism from A to the Lie algebra of endomorphisms of V). Representations of this type play a special role in the theory of Lie algebras. However, in the theory of Malcev algebras they are not very significant.

Generally speaking, the kernel $\mathrm{Ker}\rho$ of a representation ρ of a Malcev algebra A is not necessarily an ideal of A. Obviously, there exists a maximal ideal of A contained in $\mathrm{Ker}\rho$: the sum of all ideals of A contained in $\mathrm{Ker}\rho$. This ideal will be called the *quasi-kernel* of the representation ρ and it will be denoted by $\widetilde{\mathrm{Ker}\rho} = I$. For every representation ρ of a Malcev algebra A with quasi-kernel $\widetilde{\mathrm{Ker}\rho} = I$ there exists an induced nearly faithful representation of the quotient algebra A/I in the same vector space. Sometimes it can be useful to consider an arbitrary associative

† Translator's note: The author denotes the representation map by ρ and the image of an element x of the Malcev algebra under ρ by R_x.

algebra with identity E instead of $\mathrm{End}(V)$, where the right regular representation of E is isomorphic to[†] the algebra of endomorphisms of E.

Furthermore, we will restrict our attention to finite dimensional Malcev algebras, so we also assume that their representations are finite dimensional. We will denote by A_ρ^* the associative enveloping algebra of the representation ρ, i.e., the associative algebra generated by $\{R_x \mid x \in A\}$.

1.3. It is well-known that Engel's theorem plays an important role in the theory of finite dimensional Lie algebras. An analogue of this theorem holds for binary-Lie algebras [4]. The following theorems for the regular representation are found in [16].

Theorem 1.1. *Let ρ be a representation of a Malcev algebra A by nilpotent operators. Then A_ρ^* is nilpotent, and if ρ is a nearly faithful representation then A is also nilpotent.*

Proof. Let us first prove that A_ρ^* is nilpotent. To a subalgebra $B \subseteq A$ we assign the subalgebra $B^* \subseteq A_\rho^*$ generated by $\{R_x \mid x \in B\}$. Let B be a maximal subalgebra of A for which B^* is nilpotent and assume that $B \neq A$. Let $x \notin B$. Then for some natural number n we have $x_n = xb_1b_2\cdots b_n \in B$ for any $b_i \in B$. Indeed, using (9), R_{x_n} can be written as a linear combination of "R-words" from A_ρ^*, each of them having sufficiently many operators R_b ($b \in B$) if n is large enough. By our assumption, B^* is nilpotent, hence for some n we have $R_{x_n} = 0$ and $R_{x_n b} = 0$ ($b \in B$). If now $x_n \notin B$ then B is a proper subalgebra of B_1 generated by $\{x_n, B\}$ and $B_1^* = B^*$. Therefore, B_1^* is nilpotent, which contradicts the maximality of B. Hence we can choose u from the sequence $\{x_k \mid k \geq 0\}$ such that $u \notin B$, $uB \subseteq B$. We write $C = (u) + B$ and show that C is nilpotent, which contradicts the maximality of B. For this we consider "long" R-words depending on R_u, R_{b_i} ($b_i \in B$). It follows from the nilpotency of R_u that such words are either trivially equal to 0 in A_ρ^* or have many operators R_b ($b \in B$). For definiteness we assume that $R_u^m = 0$, $(B^*)^n = 0$. Then, nontrivial words of R-length $N \geq mt$ contain at least t operators R_{b_i}. We apply the following transformations to these words:

(a) Transformations of subwords of the form $R_{b_i}R_u^2$, $R_uR_{b_i}R_u$ using identities (4) and (16) in which $x = u$ and $y = b_i$. Operators R_u either shift to the left or disappear and the total number t of operators R_{b_i} remains invariant.

(b) If we run out of transformations of the first type then we consider the rightmost operator R_u and assume that R_{b_1}, R_{b_2} precede it. By setting $x = b_1$, $y = b_2$, $z = u$ in (9) we transform $R_{b_1}R_{b_2}R_u$. Then the operator R_u either shifts to the left or disappears and the total number of operators R_{b_i} decreases by 1 only in the term $R_{b_1b_2u}$. At the same time, the rightmost operator R_u disappears.

[†] Translator's note: In other words, the right regular representation $x \mapsto R_x$ is an isomorphism between E and $\mathrm{End}(E)$.

If we write $N \geq 2mn$ then $t \geq 2n$. Using transformations of the first and second types (note that if both transformations are possible then transformations of the first type will be applied) we obtain a linear combination of words and at the right side of each of them will be at least n operators R_{b_i}. However, such words are equal to 0 because the nilpotency index of B^* is n. Hence nilpotency of C is proved and therefore A^*_ρ is nilpotent. Let $(A^*_\rho)^n = 0$. In the same way as when we chose the element u, we have to ensure that $A^N \subseteq \mathrm{Ker}\rho$ when $N \geq 2n$. However, A^N is an ideal of A so if the representation R is nearly faithful then $A^N = 0$, i.e., A is nilpotent. □

The following is a useful generalization of Theorem 1.1.

Theorem 1.2. *Let B be an ideal of a Malcev algebra A, let ρ be a nearly faithful representation of A, and for every $x \in B$ assume that the operator R_x is nilpotent. Then the ideal B is nilpotent and the algebra B^* generated by $\{R_x \mid x \in B\}$ lies in the radical of A^*_ρ.*

Proof. By Theorem 1.1, B^* is nilpotent of index n. To every R-word from A^*_ρ that has at least $2n$ operators R_{b_i} ($b_i \in B$) we can apply a transformation similar to transformations (a) and (b) from Theorem 1.1. Using (9) we change subwords $R_b R_{a_1} R_{a_2}$ and $R_{a_1} R_b R_{a_2}$ ($b \in B$) shifting R_b to the right and keeping the total number of them unchanged in each term. If we run out of transformations of this type then we consider the rightmost operator R_a where $a \notin B$. Let R_{b_1} and R_{b_2} precede it. Using (9) we transform $R_{b_1} R_{b_2} R_a$ so that R_{b_i} shifts to the right and the total number of them remains invariant. In the term with a factor $R_{b_1 b_2 a}$ the total number of operators R_{b_i} decreases by 1 and a factor R_a disappears. As a result all terms can be reduced to 0 and B^* generates a nilpotent ideal of A^*_ρ with nilpotency index less than or equal to $2n$, i.e., $B^* \subseteq \mathrm{Rad}\, A^*_\rho$. The subalgebra B^{4n} generates the ideal B_0 of A, whose elements are represented by zero operators. However, $\mathrm{Ker}\rho$ contains nonzero ideals $B^{4n} = 0$, so the theorem is proved. □

It follows from the proof above that the sum of nilpotent ideals in an arbitrary (not necessarily finite dimensional) Malcev algebra is a nilpotent ideal and any finite dimensional Malcev algebra contains the largest nilpotent ideal $N(A)$ which is called the *nil-radical* of the algebra A [16].

1.4. In the general case, operators of a representation of a Malcev algebra are not necessarily nilpotent. The theory of such representations is based on lemmas which are analogues to some well-known lemmas from the theory of Lie algebras [3].

Lemma 1.3. *Let ρ be a representation of a Malcev algebra A in the vector space V, let $x, y \in A$ and let $yx^m = 0$ for some $m > 0$. Then the Fitting components V_0 and V_1 of V with respect to R_x are invariant with respect to R_y.*

Proof. First note that V_0 and V_1 coincide with the kernel and image of R_x^n respectively, for sufficiently large n, for example $n \geq \dim V$. For convenience, we use induction on m. If $m = 1$ then the lemma follows from identity (4). If $m > 1$ then we can use identity (4) and we also note that the operators R_x and R_{yx} leave V_0 and V_1 invariant. To the decomposition of the characteristic polynomial $f(\lambda)$ of the operator R_x into irreducible factors $\pi(\lambda)$ corresponds a decomposition of V into the direct sum of primary components V_π annihilated by certain powers of the operator $\pi(R_x)$. □

Lemma 1.4. *Under the hypotheses of Lemma 1.3, let V be decomposed into its primary components V_π with respect to the operator R_x. Then the subspaces V_π are invariant with respect to R_y.*

Proof. By Lemma 1.3 it suffices to consider subspaces V_π on which R_x acts as a non-singular transformation. Let us again use induction on m. If $m = 1$ then for any polynomial $P(\lambda)$ we use identity (16),

$$V_\pi R_y P(R_x) = V_\pi R_x R_y P(R_x) = V_\pi R_x P(R_x) R_y = V_\pi P(R_x) R_y,$$

which proves our lemma. If $m > 1$ then using (16) we note that the operators R_x, R_{yx} and R_{yxx} leave the subspace V_π invariant. □

Proposition 1.5. *Let A be a nilpotent Malcev algebra, let ρ be a representation of A in a vector space V, let V_0^x and V_1^x be the Fitting components of V with respect to R_x, and let $x \in A$. Then $V = V_0 + V_1$, where*

$$V_0 = \bigcap_{x \in A} V_0^x, \qquad V_1 = \sum_{x \in A} V_1^x = \sum_{k=1}^{\infty} V(A_\rho^*)^k.$$

Proof. The proof is standard: we only need to note that if $V = V_0^x$ for all $x \in A$, i.e., ρ is a representation of A by nilpotent transformations of V, then A_R^* is nilpotent, so $(A_R^*)^k = 0$ for some $k > 0$ (by Theorem 1.1 this fact is even true without the assumption that A is nilpotent). If $V_1^x \neq 0$ for some $x \in A$ then V can be decomposed into the direct sum of A-submodules $V_0^x + V_1^x$. Moreover, $V_1^x \subseteq V_1$, $\dim V_0^x < \dim V$, and then we use induction on the dimension of V_0^x. □

Proposition 1.6. *Under the same hypotheses of Proposition 1.5, V can be decomposed into a direct sum of A-submodules V_i. Moreover, the minimal polynomial of a transformation induced by any operator R_x on V_i is some power of an irreducible polynomial.*

Proof. The proof trivially follows from Lemma 1.4 and it uses induction on the dimension of V. Here we remark that every subspace V_i can be constructed as an intersection of primary components for a finite number of operators R_x $(x \in A)$ and the decomposition of Proposition 1.6 is uniquely determined. □

A representation ρ is called *split* if the characteristic roots of each operator R_x belong to the base field F. The next theorem follows from Proposition 1.6.

Theorem 1.7. *Let ρ be a split representation of a nilpotent Malcev algebra A. Then the representation space V can be decomposed into the direct sum of subspaces V_α characterized by the following conditions:*

1) V_α is invariant with respect to A_ρ^* (it is an A-submodule of V).

2) Each operator R_x has a unique characteristic root $\alpha(x)$.

3) If $\alpha \neq \beta$ then there exists an element $x \in A$ such that $\alpha(x) \neq \beta(x)$.

As we did in the proof of Proposition 1.6, we remark that each subspace V_α coincides with the intersection of root subspaces of V with respect to operators R_x for some finite number of elements $x \in A$. A map $\alpha\colon A \to F$ is called a *weight* of the algebra A with respect to the given representation ρ, and the corresponding subspaces V_α are called *weight spaces.*

In the case where H is a nilpotent subalgebra of a Malcev algebra A, and the given representation of H in A is split and induced by the regular representation of A, the subspaces A_α are said to be *root spaces* and the map $\alpha\colon H \to F$ is called a *root* of H in A. Below we will see that split representations of nilpotent Malcev algebras over fields of characteristic 0 can be completely described. In particular the weights are linear maps.

1.5. Let H be a nilpotent subalgebra of a Malcev algebra A whose regular representation on A is split. We now want to study relations between the root spaces A_α. The technique used to obtain these relations is similar to that used in [11] where they were derived for the case $\dim H = 1$. Using the results of the previous section we offer a simpler proof for a more general case. We will identify operators of scalar multiplication with elements of the base field F of arbitrary characteristic.

Let h be an arbitrary nonzero element of H and let $A = A_0^h + A_\alpha^h + \cdots$ be the decomposition of A into root spaces with respect to the operator R_h. Then Lemma 1.4 implies that $A_0^h A_\alpha^h \subseteq A_\alpha^h$. In particular A_0^h is a subalgebra of A. By setting $x = h$, $y = x_\alpha \in A_\alpha^h$ in (2) we obtain

$$J(h, x_\alpha, hx_\beta) = J(h, x_\alpha, x_\beta)h, \text{ or } J(h, x_\alpha, x_\beta(\beta - R_h)) = J(h, x_\alpha, x_\beta)(\beta + R_h).$$

By induction,

$$J(h, x_\alpha, x_\beta(\beta - R_h)^n) = J(h, x_\alpha, x_\beta)(\beta + R_h)^n,$$

and so $J(h, x_\alpha, x_\beta) \in A_{-\beta}^h$. Similarly, $J(h, x_\alpha, x_\beta) \in A_{-\alpha}^h$. Thus

$$J(h, x_\alpha, x_\beta) = 0 \quad (\alpha \neq \beta). \tag{19}$$

Substituting $u = x_0 \in A_0^h$ in (6) we obtain

$$J(x_0, x_\alpha, hx_\beta) + J(h, x_\alpha, x_0 x_\beta) = J(x_0, x_\alpha, x_\beta)h + J(h, x_\alpha, x_\beta)x_0,$$
$$J(x_0, x_\alpha, hx_\beta) = J(x_0, x_\alpha, x_\beta)h.$$

Therefore, similarly to (19) we get

$$J(A_0^h, A_\alpha^h, A_\beta^h) = 0 \ \ (\alpha \neq \beta). \tag{20}$$

If α, β are different roots of H in A, and A_α, A_β are the corresponding root spaces, then there exists an element $h \in H$ such that $\alpha(h) \neq \beta(h)$. Then

$$A_\alpha \subseteq A_{\alpha(h)}^h, \qquad A_\beta \subseteq A_{\beta(h)}^h, \qquad A_0 \subseteq A_0^h,$$

and it follows from (20) that

$$J(A_0, A_\alpha, A_\beta) = 0 \ (\alpha \neq \beta). \tag{21}$$

In particular, $J(H, A_\alpha, A_\beta) = 0$ $(\alpha \neq \beta)$ or

$$(x_\alpha x_\beta)h = (x_\alpha h)x_\beta + x_\alpha(x_\beta h) \ \ (\alpha \neq \beta), \tag{22}$$

for any $x_\alpha \in A_\alpha$, $x_\beta \in A_\beta$, $h \in H$. Identity (22) shows that each operator R_h $(h \in H)$ is a derivation of the linear space $A_\alpha A_\beta$. Thus

$$A_\alpha A_\beta \subseteq A_{\alpha+\beta} \ (\alpha \neq \beta). \tag{23}$$

Here $\alpha + \beta$ is not necessarily a root. If $\gamma\colon H \to F$ is not a root of H in A then we can assume that $A_\gamma = 0$. Reasoning in the same way when $\alpha = \beta$ we obtain

$$J(h, A_\alpha^h, A_\alpha^h) \subseteq A_{-\alpha}^h, \qquad J(A_0^h, A_\alpha^h, A_\alpha^h) \subseteq A_{-\alpha}^h,$$

and in particular $J(A_0, A_\alpha, A_\alpha) \subseteq A_{-\alpha(h)}^h$. Any vector from $J(A_0, A_\alpha, A_\alpha)$ appears as a root vector for the operator R_h $(h \in H)$ with eigenvalue $-\alpha(h)$. Therefore,

$$J(A_0, A_\alpha, A_\alpha) \subseteq A_{-\alpha}. \tag{24}$$

In particular, for any $h \in H$ the following identity holds:

$$(x_\alpha y_\alpha)h = (x_\alpha h)y_\alpha + x_\alpha(y_\alpha h) + z_{-\alpha}. \tag{25}$$

Decomposing $x_\alpha y_\alpha$ into a sum of components from different root spaces of A gives

$$x_\alpha y_\alpha = u_{2\alpha} + u_\beta + \cdots . \tag{26}$$

Then for any $\beta \neq 2\alpha$ there exists an element $h \in H$ such that $\beta(h) \neq 2\alpha(h)$. If we apply the operator $(R_h - 2\alpha(h))^n$, where n is sufficiently large, to (26) then we obtain on one side an element of $A_{-\alpha}$ by (25) and on the other side an element $u_\beta(R_h - 2\alpha(h))^n + \cdots$. Moreover, the component $u'_\beta = u_\beta(R_h - 2\alpha(h))^n \in A_\beta$ is nonzero if $u_\beta \neq 0$, and the restriction of $(R_h - 2\alpha(h))$ to A_β has as its only characteristic root $\beta(h) - 2\alpha(h) \neq 0$, and it acts on A_β as a nonsingular map. Therefore the only nonzero component u_β in (26) except for $u_{2\alpha}$ is $u_{-\alpha}$:

$$x_\alpha y_\alpha \in A_{2\alpha} + A_{-\alpha}. \tag{27}$$

In particular, $A_0^2 \subseteq A_0$, which is clear since A_0 is the intersection of subspaces A_0^h, each of which is a subalgebra of A, and an intersection of subalgebras is itself a subalgebra.

Let α, β, γ be pairwise distinct weights of H in A. We show that $J(A_\alpha, A_\beta, A_\gamma) = 0$. If one of the weights α, β, γ is 0 then this follows from (21). Thus it is enough to consider the case $\alpha\beta\gamma \neq 0$. We first assume that $\alpha + \beta \neq \gamma$, $\alpha + \gamma \neq \beta$. Then it follows from (6), (21) and (23) that $J(x_\alpha, x_\beta, hx_\gamma) = J(x_\alpha, x_\beta, x_\gamma)h$, which implies $J(x_\alpha, x_\beta, x_\gamma) \in A_{-\gamma}$. Interchanging β and γ we obtain $J(x_\alpha, x_\beta, x_\gamma) \in A_{-\beta}$, which implies $J(x_\alpha, x_\beta, x_\gamma) = 0$. Further let $\alpha + \beta = \gamma$ and $\mathrm{char} F \neq 2$. Then $\gamma + \alpha \neq \beta$, $\gamma + \beta \neq \alpha$, and we go back to the previous case if we interchange α and γ. Finally let $\alpha + \beta = \gamma$ and $\mathrm{char} F = 2$. Then $\beta + \gamma = \alpha$ is symmetric in α, β and γ. It follows from (6), (21), (23) and (24) that

$$J(x_\alpha, x_\beta, hx_\gamma) = J(x_\alpha, x_\beta, x_\gamma)h + z_\beta, \tag{28}$$

where $z_\beta = J(h, x_\beta, x_\alpha x_\gamma) \in A_\beta$. Similar to (27), (28) implies that $J(x_\alpha, x_\beta, x_\gamma) \in A_\gamma + A_\beta$. By symmetry,

$$J(x_\alpha, x_\beta, x_\gamma) \in A_\alpha + A_\beta, \quad J(x_\alpha, x_\beta, x_\gamma) \in A_\alpha + A_\gamma,$$

which implies that $J(x_\alpha, x_\beta, x_\gamma) = 0$. Thus the proof is complete.

We now consider the Jacobian $J(x_\alpha, y_\alpha, x_\beta)$ where $\alpha, \beta \neq 0$. We first assume that $\beta \neq \alpha, -\alpha, 2\alpha$. Using (6) repeatedly and also (21), (23) and (27) we obtain $J(x_\alpha, x_\beta, hy_\alpha) = J(x_\alpha, x_\beta, y_\alpha)h$. Therefore, $J(x_\alpha, y_\alpha, x_\beta) \in A_{-\alpha}$. On the other hand, $J(x_\alpha, y_\alpha, hx_\beta) = J(x_\alpha, y_\alpha, x_\beta)h$. Therefore, $J(x_\alpha, y_\alpha, x_\beta) \in A_{-\beta}$. Therefore, $J(x_\alpha, y_\alpha, x_\beta) \in A_{-\beta}$ and thus $J(x_\alpha, y_\alpha, x_\beta) = 0$. Let $\beta = 2\alpha$ and $2\alpha \neq 0, \alpha, -\alpha$ (in particular, this implies that $\mathrm{char} F \neq 2, 3$). The identity $J(x_\alpha, y_\alpha, hx_{2\alpha}) = J(x_\alpha, y_\alpha, x_{2\alpha})h$ implies that $u = J(x_\alpha, y_\alpha, x_{2\alpha}) \in A_{-2\alpha}$. On the other hand,

$$J(x_{2\alpha}, x_\alpha, hy_\alpha) = J(x_{2\alpha}, x_\alpha, y_\alpha)h + z_\alpha,$$

where $z_\alpha = J(h, x_\alpha, y_\alpha)x_{2\alpha} \in A_\alpha$. Consequently, $u \in A_\alpha + A_{-\alpha}$. Taking into account that $-2\alpha \neq \alpha, -\alpha$, we conclude that $u = 0$.

Assume that $\mathrm{char} F \neq 2$. It follows from

$$J(x_\alpha, y_\alpha, hx_{-\alpha}) = J(x_\alpha, y_\alpha, x_{-\alpha})h,$$

for any $x_\alpha, y_\alpha \in A_\alpha$, $x_{-\alpha} \in A_{-\alpha}$ $(\alpha \neq 0)$, $h \in H$, that $J(x_\alpha, y_\alpha, x_{-\alpha}) \in A_\alpha$. Moreover, $J(x_\alpha, y_\alpha, hz_\alpha) = J(x_\alpha, y_\alpha, z_\alpha)h + u_0$, where $u_0 = J(h, y_\alpha, z_\alpha)x_\alpha \in A_{-\alpha}A_\alpha \subseteq A_0$. Therefore $J(x_\alpha, y_\alpha, z_\alpha) \in A_{-\alpha} + A_0$. On the other hand, expanding the Jacobian $J(x_\alpha, y_\alpha, z_\alpha)$ and taking into account formulas (23) and (27) for multiplication of root spaces we note that $J(x_\alpha, y_\alpha, z_\alpha) \in A_{3\alpha} + A_0$. Since $3\alpha \neq \alpha$, $J(x_\alpha, y_\alpha, z_\alpha) \in A_0$.

To sum up, we state the above results in the following lemma:

Lemma 1.8. *Let H be a nilpotent subalgebra of a Malcev algebra A over a field F. Assume that the regular representation of H in A is split and $A = A_0 + A_1 + \dots$ is the corresponding decomposition of A into root spaces. Then*

$$A_\alpha A_\beta \subseteq A_{\alpha+\beta}\ (\alpha \neq \beta), \qquad A_\alpha^2 \subseteq A_{2\alpha} + A_{-\alpha}, \tag{29}$$
$$J(A_\alpha, A_\beta, A_\gamma) = 0\ (\alpha \neq \beta \neq \gamma \neq \alpha), \tag{30}$$
$$J(A_\alpha, A_\alpha, A_\beta) = 0\ (\beta \neq 0, \alpha, -\alpha), \tag{31}$$
$$J(A_\alpha, A_\alpha, A_0) \subseteq A_{-\alpha}. \tag{32}$$

If $\mathrm{char}F \neq 2$ *then*

$$J(A_\alpha, A_\alpha, A_{-\alpha}) \subseteq A_\alpha, \tag{33}$$
$$J(A_\alpha, A_\alpha, A_\alpha) \subseteq A_0. \tag{34}$$

1.6. Let us introduce the important notion of a Cartan subalgebra of Malcev algebra.

Definition 1.9. A subalgebra H of a Malcev algebra A is said to be a *Cartan subalgebra* if it is nilpotent and coincides with the Fitting component A_0 of A with respect to H.

The definition above is similar to the usual definition of Cartan subalgebra of a Lie algebra. Any Cartan subalgebra of A is obviously a maximal nilpotent subalgebra of A. If Ω is an extension of the base field F, then $A_\Omega = A_F \otimes \Omega$ is the corresponding tensor extension of A, and if H is a Cartan subalgebra of A then $H_\Omega = H_F \otimes \Omega$ is a Cartan subalgebra of A_Ω (to prove this it suffices to note that $H = A_0$ coincides with the intersection of root subspaces A_0^h for a finite number of $h \in H$).

The *normalizer* $N(H)$ of a subalgebra $H \subseteq A$ is the set of elements $x \in A$ such that $xH \subseteq H$.

Proposition 1.10. *A subalgebra H of a Malcev algebra A is a Cartan subalgebra of A if and only if it is nilpotent and coincides with its normalizer.*

Proof. For any nilpotent subalgebra H of A we have $H \subseteq N(H) \subseteq A_0$. If H is a Cartan subalgebra then these inclusions become equalities. To prove the other implication, let $H \subset A_0$. Since the regular representation of H in A_0 is nilpotent, by Theorem 1.1 we have H has an induced nilpotent representation in A_0/H. Therefore there exists an element $\xi \neq 0$ in A_0/H annihilated by all operators R_h $(h \in H)$. The preimage x of ξ in A_0 is an element of $N(H)$. Moreover, $x \in H$. □

As for Lie algebras, there exists a simple way of constructing a Cartan subalgebra of a Malcev algebra A if the base field F is sufficiently large, say $|F| \geq \mathrm{dim}A$.

Definition 1.11. An element $x \in A$ is said to be *regular* if the dimension of the Fitting 0 component of A with respect to the operator R_x is minimal.

Proposition 1.12. *If A is a finite dimensional Malcev algebra over a field F with $\dim A \leq |F|$ and x is a regular element of A, then the Fitting 0 component A_0^x of A with respect to R_x is a Cartan subalgebra. Conversely, if H is a Cartan subalgebra of A that contains a regular element h then $H = A_0^h$.*

This can be proved in the same way as in the case of Lie algebras [3]. Note that in the case of binary Lie algebras the proposition does not make sense because A_0^x is not a subalgebra of A. In [13] the definition of a Cartan subalgebra of a binary Lie algebra is more restrictive than that given in Definition 1.9. However, it is not very good because it requires too many conditions to hold; following this definition, Cartan subalgebras might not even exist for Malcev algebras or binary Lie algebras.

2. Generalization of Lie's theorem. Criteria for solvability and semisimplicity of Malcev algebras

2.1. In this section, unless otherwise stated, we assume that the base field F has characteristic 0.

To every representation ρ of a Malcev algebra A we associate the bilinear trace form $(x, y) = \operatorname{tr}(R_x R_y)$. It is clear that (x, y) is symmetric, that is $(x, y) = (y, x)$. It follows from (4) after canceling the 2s that $(yx, x) = 0$. Linearizing this expression in x gives $(yx, z) + (yz, x) = 0$ or

$$(xy, z) = (x, yz), \tag{35}$$

for any $x, y, z \in A$. We call a bilinear form (x, y) satisfying this condition *invariant.* The bilinear form (x, y) associated to the regular representation of a Malcev algebra is called the Killing form. Using the trace technique we can obtain a number of results about Malcev algebras over fields of characteristic 0. The following lemma generalizes Jacobson's well-known lemma [3] about nilpotent elements of a Lie algebra of linear transformations.

Lemma 2.1. *Let A be a Malcev algebra over a field of characteristic 0 such that for some $c \in A$ this relation holds:*

$$c = \sum_{i=1}^{r} a_i b_i, \quad c a_i = 0 \quad (i = 1, \ldots, r).$$

Then the operator R_c is nilpotent in any representation $\rho\colon x \mapsto R_x$ in A.

Proof. Let us show that $ac = 0$ for some $a, c \in A$ implies $\operatorname{tr} R_c^k R_{ab} = 0$ for some $k \geq 1$ and for all $b \in A$. Setting $a = a_i$, $b = b_i$ and summing over i we obtain $\operatorname{tr} R_c^{k+1} = 0$ $(k \geq 1)$, which implies nilpotency of R_c.

Note that by (12), $\operatorname{tr} R_{xyz} = 0$ for all $x, y, z \in A$. Taking this into account and comparing traces of operators on both sides of (17) we obtain $\operatorname{tr} R_x^n R_{yx} = 0$ $(n \geq 1)$. In particular,

$$\operatorname{tr} R_c^n R_{bac} = 0 \quad (n \geq 0). \tag{36}$$

It follows from (9) that $R_{bac} = R_b R_a R_c - R_c R_b R_a + R_a R_{cb}$. Substituting this into (36) we obtain

$$\operatorname{tr} R_c^n R_a R_{cb} = 0 \quad (n \geq 0). \tag{37}$$

On the other hand,

$$0 = R_{cab} = R_c R_a R_b - R_b R_c R_a + R_a R_{bc} - R_{ab} R_c.$$

Multiplying this relation by R_c^n on the left and taking into account (37) we obtain

$$\begin{aligned}\operatorname{tr} R_c^n R_{ab} R_c &= \operatorname{tr}(R_c^{n+1} R_a R_b - R_c^n R_b R_c R_a) \\ &= \operatorname{tr}(R_c^{n+1} R_a - R_c R_a R_c^n) R_b \quad (n \geq 0).\end{aligned}$$

This remains to show that $R_c^{n+1} R_a - R_c R_a R_c^n = 0$ when $n \geq 0$. It follows easily from (16) so the proof is complete. □

In the case of Malcev algebras, the notion of solvability defined for arbitrary nonassociative algebras admits a useful modification. We remark that it follows from (10) that if $I \lhd A$ then $L(I) = I^2 + I^2 \cdot A \lhd A$. For an arbitrary ideal I of a Malcev algebra A we define the chain of ideals $I_k = L_k(I)$, $k \geq 0$, by setting $I_0 = I$ and $I_k = L(I_{k-1})$, $k \geq 1$. We also define the derived series $I^{(k)}$ by $I^{(0)} = I$ and $I^{(k)} = I^{(k-1)} \cdot I^{(k-1)}$, $k \geq 1$. The ideal I is said to be *solvable* (resp. *L-solvable*) if $I^{(k)} = 0$ (resp. $I_k = 0$) for some $k \geq 0$. Since $I_k \supseteq I^{(k)}$ for any k, it follows that any L-solvable ideal of a Malcev algebra A is solvable. The converse is also true.

Proposition 2.2. *[5] Every solvable ideal of a Malcev algebra A is also L-solvable.*

Proof. Yamaguti [15] gives a similar definition of solvability for Malcev algebras. However he did not note that this definition is equivalent to the usual definition of solvability. For the sake of completeness we prove Proposition 2.2. Let $I \lhd A$. Let us show that $I_2 \subseteq I^{(1)} = I^2$. Since $I_1 \subseteq I$, it suffices to show that $I_1^2 \cdot A \subseteq I^2$ or $(I^2 + I^2 A)^2 A \subseteq I^2$, which can be reduced to the proof of $(I^2 \cdot I)A \subseteq I^2$ and $((I^2 A)I)A \subseteq I^2$. Obviously, the first inclusion follows from (10). If $c_1 \in I^2$, $c_2 \in I$, $a_1, a_2 \in A$ then

$$c_1 a_1 c_2 a_2 + a_1 c_2 a_2 c_1 + c_2 a_2 c_1 a_1 + a_2 c_1 a_1 c_2 = (c_1 c_2)(a_1 a_2).$$

Moreover, $a_1 c_2 a_2 c_1, a_2 c_1 a_1 c_2 \in I^2$ and $c_2 a_2 c_1 a_1, c_1 c_2 \cdot a_1 a_2 \in I^3 \cdot A$. Note that we have already seen that $I^3 \cdot A \subseteq I^2$. Suppose that $I_{2k} \subseteq I^{(k)}$ for some $k \geq 1$. Then $I_{2k+2} = L_2(I_{2k}) \subseteq I_{2k}^2 \subseteq I^{(k+1)}$. Consequently, $I^{(n)} = 0$ implies $I_{2n} = 0$, i.e., we have L-solvability of the ideal I. □

Since all elements of the sequence $\{I_k \mid k \geq 0\}$ are ideals of A, it follows from Proposition 2.2 that:

Corollary 2.3. *In any nonzero solvable ideal of a Malcev algebra there exists a nonzero abelian ideal of the algebra.*

A maximal solvable ideal $S(A)$ of an algebra A is said to be the *radical* of the algebra A. If $S(A) = 0$ then A is called *semisimple*. According to the previous remarks, semisimple Malcev algebras can be equivalently defined as Malcev algebras without nontrivial abelian ideals.

In some sense, *reductive Lie algebras*, i.e., Lie algebras whose regular representation is completely reducible, are close to semisimple Lie algebras. More generally, they are defined as algebras with a faithful completely reducible representation. Theorem 8 in [4] gives a description of such algebras. Results about them are similar to results about Lie algebras.

Theorem 2.4. *Let A be a Malcev algebra which has a nearly faithful representation ρ with semisimple enveloping algebra A^*_ρ. Then $A = A_1 + C$ where A_1 is a semisimple subalgebra and C is the center (annihilator) of the algebra A.*

Proof. Let S be the radical of the algebra A. We show that S coincides with the center of A. Otherwise, $S_1 = S \cdot A \subseteq S$ is a nonzero solvable ideal of A. Let S_2 be a nonzero abelian ideal in S_1 (which exists by Corollary 2.3) and set $S_3 = S_2 \cdot A \subseteq S_2$. By Lemma 2.1 each element of the ideal S_3 can be represented by a nilpotent operator, and by Theorem 1.2, S^*_3 is in the radical of A^*_ρ, so $S^*_3 = 0$. Then $S_3 \subseteq \ker\rho$ and thus, since ρ is nearly faithful, $S_3 = 0$. Hence S_2 lies in the center of A. Also $S_2 \subseteq S \cdot A$. Using again Lemma 2.1 and repeating the reasoning we can show that $S_2 = 0$. This contradicts the assumption that S_2 is nonzero and thus $S \cdot A = 0$. For the same reason $S \cap A^2 = 0$ and therefore $A = S + A_1$ where A_1 is a complementary subspace of S containing A^2. Since $A_1 \supseteq A^2$ we immediately have $A_1 \lhd A$. Moreover, $A_1 \cong A/S$ so A_1 is semisimple. □

Definition 2.5. The Lie algebra generated by the operators R_x, $x \in A$ is said to be the *Lie enveloping algebra $L_\rho(A)$ of a representation ρ.*

Identity (12) shows that $L_\rho(A)$ and $\rho(A) + [\rho(A), \rho(A)]$ coincide as vector spaces. If the algebra A is abelian then $L_\rho(A)$ is at least metabelian. The associative enveloping algebra of $L_\rho(A)$ coincides with A^*_ρ.

Corollary 2.6. *Under the hypothesis of Theorem 2.4, if A is a solvable Malcev algebra then A is abelian and the algebra A^*_ρ is commutative. More generally, if ρ is a nearly faithful representation of a solvable Malcev algebra A, and R is the radical of A^*_ρ, then the quotient algebra A^*_ρ/R is commutative.*

Proof. The algebra A is abelian by Theorem 2.4, so $L_\rho(A)$ is at least metabelian. However, since the associative enveloping algebra A^*_ρ of $L_\rho(A)$ is semisimple, then

$L_\rho(A)$ is indeed abelian. Therefore A_ρ^* is commutative. To prove the second claim by analogy with Lie algebras [3] we consider the sequence

$$A \to A_\rho^* \to A_\rho^*/R,$$

which is a representation of a solvable Malcev algebra, so its associative enveloping algebra is the semisimple algebra A_ρ^*/R. □

The following theorem based on Theorem 2.4 and Corollary 2.6 has an important application.

Theorem 2.7. *Let ρ be a nearly faithful representation of a Malcev algebra A in a vector space V, let S be the radical of A, let R be the radical of A_ρ^*, and let $\bar\rho$ be the induced representation $A \to A_\rho^*/R$ with $I = \widetilde{\ker}\bar\rho$. Then I is a nilpotent ideal of A which coincides with the set S_0 consisting of all the elements of S which are nilpotent with respect to ρ. Moreover, $S \cdot A \subseteq S_0$.*

Proof. Let R_0 be the radical of the subalgebra $S^* \leq A_\rho^*$. Then by Corollary 2.6, S^*/R_0 is semisimple and commutative. The set S_0 coincides with the kernel of the representation $S \to S^*/R_0$, so S_0 is a subspace of S. Consider the representation $\bar\rho$; its enveloping algebra is the semisimple algebra A_ρ^*/R. Elements of the ideal I are represented by nilpotent operators with respect to ρ. Then by Theorem 1.2, I is a nilpotent ideal in A, i.e., $I \subseteq S$ and by definition of S_0, $I \subseteq S_0$. The radical of the algebra $\bar A = A/I$ equals S/I and the induced representation $\bar A \to A_\rho^*/R$ is nearly faithful. By Theorem 2.4, the radical of $\bar A$ coincides with its center, so $S \cdot A \subseteq I \subseteq S_0$, where S_0 is an ideal of A. Again by Theorem 1.2 we have $S_0^* \subseteq R$ and therefore $S_0 \subseteq \widetilde{\ker}\bar\rho = I$. The other inclusion was already proved. □

Corollary 2.8. *If S is the radical and N is the nil-radical of an algebra A then $S \cdot A \subseteq N$. In particular, if A is solvable then A^2 is nilpotent.*

Lemma 2.9. *Let ρ be a split representation of a solvable algebra A and let V be irreducible. Then V is one-dimensional.*

Proof. The algebra A_ρ^* is semisimple and owing to solvability of A it is also commutative. The rest of the proof is obvious. □

Theorem 2.10. *Let ρ be a split representation of a Malcev algebra A. Then all matrices R_x can be simultaneously reduced to triangular form. In other words, in the vector space V there exists an A-invariant flag of subspaces.*

The same is true for split representations of nilpotent Malcev algebras. However, in this case the subspace of a representation is a direct sum of weight spaces by Theorem 1.7 and therefore the matrices R_x have a more specific form, as in the case of Lie algebras.

Theorem 2.11. *Let ρ be a representation of a nilpotent Malcev algebra A in a vector space V. Then V can be decomposed into the direct sum of weight spaces V_α, and all matrices corresponding to the restriction of R_x to V_α can be simultaneously reduced to triangular form with $\alpha(x)$ on the main diagonal.*

Corollary 2.12. *Under the hypothesis of Theorem 2.11 the weights $\alpha\colon A \to F$ are linear maps that are 0 on A^2.*

This last corollary implies that elements from A^2 are represented by nilpotent operators. Moreover, this is true even in the case of a solvable algebra A. Indeed, by Theorem 2.7 we have $S \cdot A = A^2 \subseteq S_0$.

2.2. The following is the proof of solvability and semisimplicity criteria for Malcev algebras over fields of characteristic 0, which is similar to the well-known Cartan criteria for Lie algebras [3].

Let F be an algebraically closed field, let H be a Cartan subalgebra of the Malcev algebra A over F, and let ρ be a representation of A in V. Then V can be decomposed into the sum of weight spaces V_α with respect to the representation of H in V induced by ρ. On the other hand, A has a decomposition into the sum of subspaces A_β with respect to the subalgebra H ($A_0 = H$). Let us show that

$$V_\alpha A_\beta \subseteq V_{\alpha+\beta}\ (\alpha \neq \beta), \quad V_\alpha A_\alpha \subseteq V_{2\alpha} + V_{-\alpha}, \tag{38}$$

where as usual we assume that $V_\alpha = 0$ if α is not a weight of H in V. Consider the semidirect extension $E = V + A$ of A given by ρ and the regular representation of H in E. Since H is a nilpotent subalgebra of E, we can decompose E into the sum of root spaces with respect to H. These subspaces are of the form $V_\alpha + A_\alpha$ where one of the terms can be absent (for example V_α, if a root α of H in A is not a weight of H in V). Indeed, a system of such spaces satisfies the conditions of Theorem 1.7. By Lemma 1.8 we have

$$V_\alpha A_\beta \subseteq E_{\alpha+\beta} \cap V = (V_{\alpha+\beta} + A_{\alpha+\beta}) \cap V = V_{\alpha+\beta}.$$

The second formula of (38) can be proved in a similar way.

Lemma 2.13. *If α, β, γ are pairwise distinct weights then the identity $v_\alpha(x_\beta x_\gamma) = (v_\alpha x_\beta)x_\gamma$ holds for any $v_\alpha \in V_\alpha$, $x_\beta \in A_\beta$ and $x_\gamma \in A_\gamma$. The same is true if $\alpha \neq 0$, $\beta = \gamma = 0$.*

Proof. The proof is similar to that of (38). For the algebra $E = V + A$ this lemma claims that $J(V_\alpha, A_\beta, A_\gamma) = 0$, $J(V_\alpha, A_0, A_0) = 0$. It suffices to apply Lemma 1.8. □

Note that $A^2 = \sum A_\alpha A_\beta$. Formulas for multiplication of root spaces show that

$$H \cap A^2 = \sum_\alpha A_\alpha A_{-\alpha}.$$

Lemma 2.14. *Let A be a Malcev algebra over an algebraically closed field F of characteristic 0, let H be a Cartan subalgebra of A, and let ρ be a representation of A in a vector space V. Suppose that $\alpha, -\alpha$ are roots of H, $e_\alpha \in A_\alpha$, $e_{-\alpha} \in A_{-\alpha}$ and $h_\alpha = e_\alpha \cdot e_{-\alpha}$. Then for any weight φ of H in V the value of $\varphi(h_\alpha)$ is a rational multiple of $\alpha(h_\alpha)$.*

Proof. If φ is an integer multiple of α then the claim is obviously true for any $h \in H$, in particular, for any h_α. Let φ be a non-multiple of α. Consider the direct sum U of subspaces of the form $V_{\varphi+k\alpha}$, where k runs over the set of integers (of course, we assume that this sum has a finite number of nonzero terms). The subspace U is invariant with respect to e_α and $e_{-\alpha}$. The hypothesis of Lemma 2.13 holds for the weights $\varphi + k\alpha$, α and $-\alpha$, hence R_{h_α} restricted to U equals $[R_{e_\alpha}, R_{e_{-\alpha}}]$, and therefore the trace of R_{h_α} restricted to U equals 0. The rest of the proof is similar to that of Lemma 1.3 in [3]. □

Note that if $n_\alpha = \dim V_\alpha$ then

$$0 = \mathrm{tr}_U R_{h_\alpha} = \sum_k n_{\varphi+k\alpha} \cdot (\varphi + k\alpha)(h_\alpha),$$

$$\varphi(h_\alpha) = r_{\varphi,\alpha} \cdot \alpha(h_\alpha),$$

$$\text{where } r_{\varphi,\alpha} = -\sum_k k n_{\varphi+k\alpha} \Big/ \sum_k n_{\varphi+k\alpha}.$$

Theorem 2.15. *Let A be a Malcev algebra over a field of characteristic 0, and let ρ be a nearly faithful representation of A such that the bilinear form on $A' = A^2$ associated to ρ is trivial. Then A is solvable.*

Proof. Replacing the base field F by an algebraic extension if necessary, we use induction on the dimension of A. As in [3], it can be shown that A' is strictly contained in A. If $A = A^2$ then $H = \sum_\alpha A_\alpha \cdot A_{-\alpha}$ and by Lemma 2.14 the condition $\mathrm{tr}\, R^2_{h_\alpha} = 0$ implies $\varphi(h_\alpha) = 0$ for any weight φ of H in V. It follows from linearity of weights that $\varphi = 0$ is the only weight of H, that is, $V = V_0$. Then $VA_\alpha = 0$ for any $\alpha \neq 0$, and the representation ρ of A can be reduced to a representation of H with weight 0, i.e., ρ is a representation of A by nilpotent operators. By Theorem 1.1 A is nilpotent, but this contradicts $A = A^2$. Let ρ' be the restriction of ρ to A', $I = \widetilde{\ker\rho'} \subseteq \ker\rho$. Then A'/I satisfies the induction hypothesis and is solvable. By Proposition 2.2 it is also L-solvable, i.e., $L_m(A') \subseteq I \subseteq \ker\rho$ for some $m \geq 0$. Since $L_m(A') \lhd A$ and the representation of ρ is nearly faithful, $L_m(A') = 0$ and it follows that A is solvable. □

Corollary 2.16. *A Malcev algebra A over a field of characteristic 0 is solvable if and only if $\mathrm{tr}\, R_x^2 = 0$ for all $x \in A^2$ (here R_x is the operator of right multiplication by $x \in A$).*

To prove the necessary condition it suffices to note that in the regular representation of a Malcev algebra A the operators R_x, for $x \in A^2 \subseteq N$, are nilpotent.

Theorem 2.17. *Let ρ be a nearly faithful representation of a semisimple Malcev algebra A. Then the form associated with ρ is non-degenerate. If the Killing form of an algebra A is non-degenerate then A is semisimple.*

Proof. The proof of the first claim, like the proof of Theorem 2.15, uses L-solvability specifically. It follows from invariance of the form associated to the representation ρ that its kernel B is an ideal of A. Assume that $B \neq 0$ and let ρ' be the restriction of ρ to B and let $I = \widetilde{\ker\rho'}$. Then B/I satisfies the hypothesis of Theorem 2.15 and thus it is solvable; therefore, it is L-solvable: $L_m(B) \subseteq I \subseteq \ker\rho$. However, $L_m(B) \lhd A$ so $L_m(B) = 0$ and B is a solvable ideal of A, a contradiction.

The second claim of the theorem was proved by Sagle [11] and it is clearly a consequence of Dieudonné's theorem [3] (it follows from this theorem that a nonassociative algebra with non-degenerate invariant Killing form can be decomposed into the direct sum of simple ideals; therefore, this algebra is semisimple). However, taking into account Corollary 2.6, it is possible to prove this second claim by repeating the arguments from the Lie algebra case: if A is not semisimple then A contains a nonzero abelian ideal and such an ideal is contained in the kernel of the Killing form. □

Corollary 2.18. *Any nearly faithful representation of a semisimple Malcev algebra is faithful.*

Since the non-degeneracy of the Killing form does not depend on extensions of the base field, the following holds:

Corollary 2.19. *A Malcev algebra A over a field F of characteristic 0 is semisimple if and only if A_Ω is semisimple over any extension Ω of the field F.*

Below are a few more facts whose proofs are standard.

Structure Theorem. If A is a finite dimensional semisimple Malcev algebra over a field of characteristic 0 then A can be decomposed into the direct sum of ideals which are simple algebras.

Corollary 2.20. *If A is a semisimple algebra then any ideal of A is a semisimple subalgebra.*

Corollary 2.21. *If A is semisimple then $A = A^2$.*

Corollary 2.22. *If S is the radical of an algebra A and $B \lhd A$ then $B \cap S$ is the radical of B.*

Proposition 2.23. *If N is the nil-radical of an algebra A and $B \lhd A$ then $B \cap N$ is the nil-radical of B.*

Proof. If N_1 is the nil-radical of B and S_1 is the radical of B then $N_1 \subseteq S_1 \subseteq S$ and $N_1 A \subseteq S \cdot A \cap B \subseteq N \cap B \subseteq N_1$. Therefore N_1 is a nilpotent ideal of A and $N_1 \subseteq N \cap B$. □

Proposition 2.24. *The radical S of a Malcev algebra A coincides with the orthogonal complement in A of the subalgebra A^2 with respect to the Killing form of A.*

Corollary 2.25. *Any solvable (resp. nilpotent) subinvariant subalgebra of an algebra A lies in the radical (resp. nilradical) of A.*

Remark 1. The solvability and semisimplicity criteria for Malcev algebras are similar to the Cartan criteria. (Theorems 2.15 and 2.17 were first obtained only for the regular representation in [9] by using the connection between Malcev algebras and Lie triple systems (LTS) and their embeddings into Lie algebras.)

In §§4 and 5 we will return to the study of Malcev algebras of characteristic 0.

3. Simple Malcev algebras over a field of arbitrary characteristic

In this section we assume that the base field F has either characteristic 0 or $p > 3$. We consider the classification of non-Lie simple Malcev algebras over F.

3.1. Let A be a non-Lie simple Malcev algebra, let H be a Cartan subalgebra of A, and assume that the regular representation of H in A is split. (If such a subalgebra exists, then it is called a *split Cartan subalgebra* and A is called *split*. Proposition 1.12 shows that in order for Cartan subalgebras to exist the base field F must be infinite; if F is algebraically closed then any Lie subalgebra is split.) Note that there exist nonzero roots α of H in A. Indeed, otherwise we would have $A = A_0 = H$ and A would be nilpotent, which is not possible. Identity (11) shows that the subspace $J(A, A, A)$ is an ideal of A. Thus

$$A = J(A, A, A). \tag{39}$$

Lemma 3.1. *[12] If for some $x, y \in A$ we have*

$$J(x, y, A) = 0, \tag{40}$$

then $xy = 0$.

Proof. Equation (40) can be written as $R_{xy} = [R_x, R_y]$. Then $D(x, y) = 2R_{xy}$ and the identity $R_{xD(x,y)} = [R_z, D(x, y)]$ implies that either $R_{z(xy)} = [R_z, R_{xy}]$ for any $z \in A$ or

$$J(xy, A, A) = 0. \tag{41}$$

This argument shows, in particular, that the set of elements $x \in A$ such that $J(x, A, A) = 0$ (the so-called *center* of A) is a Lie ideal in A. In a simple algebra A this ideal must be equal to 0 and, in particular, $xy = 0$. □

Lemma 3.2. *[12] For any nonzero root α of a subalgebra H in A we have $A_\alpha^2 \subseteq A_{-\alpha}$. Moreover, $A = A_0 + A_\alpha + A_{-\alpha}$ and $A_0 = A_\alpha A_{-\alpha}$.*

Proof. Let $x_\alpha y_\alpha = z_{2\alpha} + z_{-\alpha}$; see (27). Then by (21) we have $J(h, x_\beta, z_{2\alpha}) = 0$ for all $\beta \neq 2\alpha$. If $\beta = 2\alpha$, then by Lemma 1.8 we have

$$\begin{aligned} J(h, x_{2\alpha}, z_{2\alpha}) &= J(h, x_{2\alpha}, x_\alpha y_\alpha) \\ &= -J(x_\alpha, x_{2\alpha}, hy_\alpha) + J(h, x_{2\alpha}, y_\alpha)x_\alpha + J(x_\alpha, x_{2\alpha}, y_\alpha)h = 0. \end{aligned}$$

Therefore $J(h, z_{2\alpha}, A) = 0$ and $hz_{2\alpha} = 0$, and since $h \in H$ was chosen arbitrarily, we have $z_{2\alpha} = 0$. Using what was just proved, the subspace

$$B = A_\alpha A_{-\alpha} + A_\alpha + A_{-\alpha} \subseteq A_0 + A_\alpha + A_{-\alpha},$$

is invariant under multiplications by A_α and $A_{-\alpha}$. Invariance of $A_\alpha A_{-\alpha}$ with respect to multiplication by A_0 follows from the relation $J(A_0, A_\alpha, A_{-\alpha}) = 0$. Thus B is a subalgebra. Let us show that B is an ideal of A. By (30) and (31), for any $\beta \neq 0, \alpha, -\alpha$ we have $J(A, A_\alpha, A_\beta) = 0$ and $A_\alpha A_\beta = 0$. Similarly, $A_{-\alpha} A_\beta = 0$. It follows from $J(A_\alpha, A_{-\alpha}, A_\beta) = 0$ that $(A_\alpha A_{-\alpha})A_\beta = 0$. Hence $BA \subseteq B$ and $B \lhd A$. Therefore $B = A$ and, in particular, $A_0 = A_\alpha A_{-\alpha}$. □

Lemma 3.2 shows that the system of roots of A has a very simple structure.

Lemma 3.3. *The subalgebra $H = A_0$ is abelian. A root $\alpha\colon A \to F$ is a linear map.*

Proof. Using for example (11) we can show that the subspace $J(A_0, A_0, A_0)$ is invariant under multiplications by A_0, A_α and $A_{-\alpha}$, i.e., it is an ideal of A. Therefore

$$J(A_0, A_0, A_0) = 0, \quad J(A_0, A_0, A) = 0, \quad A_0^2 = 0. \tag{42}$$

By (42), for any $x, y \in H$ we have $R_{xy} = R_x R_y - R_y R_x = 0$. Therefore, the operators R_x and R_y have a common eigenvector e_α in A_α: $e_\alpha(x+y) = [\alpha(x), \alpha(y)]e_\alpha$. However, the operator R_{x+y} has the unique eigenvalue $\alpha(x+y)$. Thus $\alpha(x+y) = \alpha(x) + \alpha(y)$ and the lemma is proved. □

Let us choose an element $h_0 \in H$ such that $\alpha(h_0) = 1$. Then any element $h \in H$ can be represented in the form $h = \alpha(h)h_0 + h_1$ where $\alpha(h_1) = 0$. For any $x \in A_\alpha$, $y \in A_{-\alpha}$, $h \in H$ we have

$$\begin{aligned} &0 = J(h, x, y) = hx \cdot y + yh \cdot x, \qquad xh \cdot y = -x \cdot yh, \\ &x[\alpha(h) - R_h] \cdot y = x \cdot y[\alpha(h) + R_h]. \end{aligned} \tag{43}$$

Lemma 3.4. *Let $h \in H$, $h \neq 0$ and let U be any cyclic subspace of A_α (or $A_{-\alpha}$) with respect to R_h. Then for any $u_1, u_2 \in U$ we have $u_1 u_2 = 0$.*

Proof. Let us choose any element u of maximal height in U. For all $h' \in H$ we have $J(h', h, u) = 0$, i.e., the triple of elements $\{h', h, u\}$ is Lie. By [10] it generates a Lie subalgebra $B \leq A$. In particular, $J(U, U, h') = 0$. Therefore, the operator $R_{h'}$ is a derivation of the linear subspace $U \cdot U$ and, since $h' \in H$ was arbitrary, $U \cdot U \subseteq A_{2\alpha}$. However, 2α is not a root, so $U \cdot U = 0$. □

Formula (39) shows that

$$A_{-\alpha} = J(A_0, A_\alpha, A_\alpha) + J(A_{-\alpha}, A_{-\alpha}, A_\alpha). \tag{44}$$

Using identity (11) and the known relations for root subspaces we can show that

$$\begin{aligned} &A_0 J(A_0, A_\alpha, A_\alpha) \subseteq J(A_0, A_\alpha, A_\alpha), \\ &A_0 J(A_{-\alpha}, A_{-\alpha}, A_\alpha) = J(A_0, A_\alpha, A_{-\alpha}^2) \subseteq J(A_0, A_\alpha, A_\alpha). \end{aligned}$$

Multiplying both sides of (44) on the left by A_0 we obtain $A_{-\alpha} \subseteq J(A_0, A_\alpha, A_\alpha)$. Since the converse inclusion also holds we have

$$A_{-\alpha} = J(A_0, A_\alpha, A_\alpha) \subseteq A_\alpha^2 + A_\alpha^2 \cdot A_0.$$

Similarly $A_\alpha = J(A_0, A_{-\alpha}, A_{-\alpha})$. In particular, $A_\alpha^2 \neq 0$ and $A_{-\alpha}^2 \neq 0$.

Lemma 3.5. *For all $x, y \in A_\alpha$, $h \in A_0$ we have*

$$yx \cdot x = 0, \quad hx \cdot x = 0. \tag{45}$$

Proof. Set $y = J(a_0, a_{-\alpha}, b_{-\alpha})$. Then by (6)

$$\begin{aligned} yx &= J(b_{-\alpha}, a_0, a_{-\alpha})x \\ &= -J(x, a_0, a_{-\alpha})b_{-\alpha} + J(b_{-\alpha}, a_0, xa_{-\alpha}) + J(x, a_0, b_{-\alpha}a_{-\alpha}) \\ &= J(x, a_0, b_{-\alpha}a_{-\alpha}) = J(x, a_0, c_\alpha), \\ yx \cdot x &= J(x, a_0, xc_\alpha) \in J(A_0, A_\alpha, A_{-\alpha}) = 0. \end{aligned}$$

The second claim follows from Lemma 3.4. □

Let us denote the system of roots of H in A by Δ; then $\Delta = \{0, \alpha, -\alpha\}$. We denote by (x, y) the symmetric bilinear form on A given by

$$(x, y) = \begin{cases} 0 & x \in A_\beta;\ y \in A_\gamma;\ \beta, \gamma \in \Delta;\ \beta + \gamma \neq 0; \\ \alpha(x)\alpha(y) & x, y \in A_0; \\ \alpha(x \cdot y_1) & x \in A_\alpha;\ y_1 \in A_{-\alpha};\ y = y_1 h_0. \end{cases} \tag{46}$$

Since the restriction of R_{h_0} to $A_{-\alpha}$ is non-degenerate, the form (46) is well-defined. In all previous lemmas the expressions were symmetric in α and $-\alpha$; however, in the definition of the form (46) this symmetry is lost. Let us show that this apparent asymmetry does not in fact hold. We change α to $\alpha' = -\alpha$ and h_0

to $h'_0 = -h_0$ so that $\alpha'(h'_0) = 1$. Then for $x, y \in A_0$ we have $(x, y) = \alpha(x)\alpha(y) = \alpha'(x)\alpha'(y)$. For $x \in A_\alpha$, $y \in A_{-\alpha}$ the definition of the form (46) can be written as $(xh_0, yh_0) = \alpha(xh_0 \cdot y)$. Let us check that $(yh'_0, xh'_0) = \alpha'(yh'_0 \cdot x)$. Indeed,

$$(yh'_0, xh'_0) = (yh_0, xh_0) = (xh_0, yh_0) = \alpha(xh_0 \cdot y) = \alpha(-x \cdot yh_0) = \alpha(yh_0 \cdot x)$$
$$= \alpha'(yh'_0 \cdot x).$$

Lemma 3.6. *The form* (46) *is invariant; i.e., for all* $x, y \in A$ (35) *holds.*

Proof. Taking into account the linearity of (35) in x, y, z, it suffices to consider the cases when x, y, z are in the root subspaces. Omitting the trivial relations, we need to check that $(xh, y) = (x, hy)$ and $(xy, h) = (x, yh)$ only when $x \in A_\alpha$, $y \in A_{-\alpha}$, $h \in H$, and the cases $x, y, z \in A_\alpha$ and $x, y, z \in A_{-\alpha}$.

(a) Let $x \in A_\alpha$, $y \in A_{-\alpha}$. Setting $y = y_1 h_0$ ($y_1 \in A_{-\alpha}$) we obtain by definition $(xh, y) = \alpha(xh \cdot y_1)$ and $(x, hy) = (x, hy_1 \cdot h_0) = \alpha(x \cdot hy_1)$; then the equality $(xh, y) = (x, hy)$ follows from $xh \cdot y_1 = x \cdot hy_1$.

(b) For the same x, y, h, y_1 we have $(xy, h) = \alpha(xy)\alpha(h) = \alpha(x \cdot y_1 h_0)\alpha(h)$ and $(x, yh) = \alpha(x \cdot y_1 h)$. Let us show that the following identity holds:

$$\alpha(x \cdot yh_0)\alpha(h) = \alpha(x \cdot yh), \ x \in A_\alpha, y \in A_{-\alpha}. \tag{47}$$

Note that (47) is linear in h; if $h = h_0$ then it is trivial. It remains to consider the case $\alpha(h) = 0$. Write $x \cdot yh = h_1$. Since $\alpha(h) = 0$, the operator R_h is nilpotent. Let $xR_h^{m-1} = x_1 \neq 0$, $x_1 h = 0$ ($m \geq 1$). It follows from $J(x, y, h) = 0$ that x, y, h, x_1 belong to the same Lie subalgebra of A. In particular,

$$0 = J(x, x_1, yh) = xx_1 \cdot yh + (x_1 \cdot yh)x + x_1 h_1 = x_1 h_1,$$

since $xx_1 = 0$ by Lemma 3.4, and $x_1 \cdot yh = -x_1 h \cdot y = 0$ since $x_1 h = 0$. It follows from $x_1 h_1 = 0$ that $\alpha(h_1) = 0$.

(c) Let $x, y, z \in A_\alpha$. We rewrite identity (35) in the form $(yx, z) + (yz, x) = 0$, so it suffices to prove that $(yx, x) = 0$ ($x, y \in A_\alpha$) and then linearize in x. Setting $x = x_1 h_0$ ($x_1 \in A_\alpha$) and using the previous arguments we get $(yx, x) = (yx \cdot x_1, h_0) = \alpha(yx \cdot x_1)$. Let us prove that $yx \cdot x_1 = 0$. Linearizing the second identity in (45) we obtain $yx = y \cdot x_1 h_0 = -x_1 \cdot yh_0 = yh_0 \cdot x_1$. Using the first relation in (45) we have $yx \cdot x_1 = (hy_0 \cdot x_1)x_1 = 0$ as desired. The case $x, y, z \in A_{-\alpha}$ is immediate owing to the symmetry of the roots α and $-\alpha$, so the lemma is proved. □

The form (x, y) is non-trivial since, for example $(h_0, h_0) = 1$. It follows from its invariance and the simplicity of A that the form is non-degenerate. If $\alpha(h) = 0$ for some $h \in H$ then by (46) we have $(h, A) = 0$ and therefore $h = 0$. Consequently the subalgebra H is one-dimensional: $H = (h_0)$. The subspaces A_α and $A_{-\alpha}$ are dual to each other with respect to (x, y); in particular, $\dim A_\alpha = \dim A_{-\alpha}$. If $x \in A_\alpha$ and $y \in A_{-\alpha}$ then $xy = \lambda h_0$ where $\lambda = (xy, h_0) = (x, yh_0)$. Hereafter we will denote the element h_0 simply by h.

Lemma 3.7. *All cyclic subspaces with respect to R_h in A_α (and $A_{-\alpha}$) are one-dimensional.*

Proof. Let U be a cyclic subspace in A_α with $\dim U = n > 0$, let $x_1, \dots x_n$ be a cyclic basis of U (here x_k is a vector of height k), and let y be an eigenvector (with respect to R_h) from $A_{-\alpha}$. Then it follows from (43) that $xy = 0$ for any vector $x \in U$ of height less than n; in particular, $x_1 \cdot y = 0$. Let V be an arbitrary cyclic subspace of $A_{-\alpha}$. Let us show that $x_1 \cdot V = 0$. If $\dim V = 1$ then this is already known, so let $\dim V = m > 1$ and let $y_1, \dots, y_m$ be a cyclic basis of V. Then $x_1 y_i = 0$ for $i = 1, \dots, m-1$. If $x_1 y_m \neq 0$ then, without loss of generality, $x_1 y_m = h$. Since A is binary Lie, the elements x_1 and y_m generate a Lie subalgebra in A with basis $x_1, y_1, \dots, y_m, h$. Then $0 = J(x_1, y_m, y_1) = x_1 y_m \cdot y_1 = y_1$, which is impossible. Consequently, $x_1 A_{-\alpha} = 0$ and $(x_1, A_{-\alpha}) = 0$, which contradicts the non-degeneracy of (x, y). The lemma is then proved. □

Lemma 3.7 shows that the operator R_h acts diagonally on A_α and $A_{-\alpha}$. Its restriction to A_α is the identity operator 1 and its restriction to $A_{-\alpha}$ is -1. In particular, for all $x \in A_\alpha$, $y \in A_{-\alpha}$ we have $xy = -(x, y)h$.

Further arguments can be made as in the case of characteristic 0 [16]. For all $x, y, z \in A_\alpha$ we have $xy \cdot z = yz \cdot x = zx \cdot y = (xy, z)h$; furthermore, $J(x, y, h) = -3xy$. If $x, y \in A_\alpha$, $z' \in A_{-\alpha}$ then

$$J(x, y, z'h) + J(z', y, xh) = J(x, y, h)z' = -3xy \cdot z'. \tag{48}$$

Also, the left side of (48) equals $-2J(x, y, z')$; therefore, $3xy \cdot z' = 2J(x, y, z')$ or

$$xy \cdot z' = 2(yz' \cdot x + z'x \cdot y). \tag{49}$$

According to (49), for any elements $x, y, z, t \in A_\alpha$ we have

$$xz \cdot yt = 2\left[(z \cdot yt)x + (yt \cdot x)z\right] = 2yztx + 2txyz.$$

Comparing this identity with (10) we obtain

$$xyzt = yztx - ztxy + txyz. \tag{50}$$

We now have enough identities to construct a basis and a multiplication table for A. Taking into account that $A_\alpha^2 \neq 0$, we choose two arbitrary elements $x, y \in A_\alpha$ for which $xy = z' \neq 0$. Then $xz' = yz' = 0$. If $z \in A_\alpha$ such that $zz' = \frac{1}{2}h$ then x, y, z are linearly independent and (50) shows that any element $t \in A_\alpha$ is a linear combination of x, y, z. Therefore, $\dim A_\alpha = \dim A_{-\alpha} = 3$. Write $yz = x'$ and $zx = y'$. Then $xx' = yy' = zz' = \frac{1}{2}h$, and it follows from the orthogonality of elements $\{x, y, z\}$ and $\{x', y', z'\}$ that $\{x', y', z'\}$ is a basis of $A_{-\alpha}$. In order to find the multiplication formulas for A_α we use identity (49):

$$x'y' = yz \cdot zx = 2\left[(z \cdot zx)y + (zx \cdot y)z\right] = 2(zx \cdot y)z = 2y'y \cdot z = -hz = z.$$

Similarly $y'z' = x$ and $z'x' = y$. Thus the multiplication table for A is complete. Note that $\dim A = 7$. We can find an explicit automorphism of order 2 which interchanges A_α and $A_{-\alpha}$. This automorphism sends x to x', y to y', z to z' and $h = 2xx'$ to $2x'x = -h$.

There is a close relation between A and a split Cayley-Dickson algebra C over F. Recall that C is a simple alternative algebra whose elements are matrices

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix},$$

where $\alpha, \beta \in F$ and a, b are arbitrary vectors of a 3-dimensional vector space over F. If $a \times b$ is the ordinary vector product and (a, b) is the dot product with the identity matrix as the Gram matrix for the chosen basis, then the product of two elements of C is given by the formula

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma - (a,d) & \alpha c + \delta a + b \times d \\ \gamma b + \beta d + a \times c & \beta\delta - (b,c) \end{pmatrix}.$$

We define a new multiplication in C by $x * y = \frac{1}{2}[x, y]$, slightly different from the commutator; C becomes a Malcev algebra $C^{(-)}$ with respect to this operation. Elements of the form $\operatorname{diag}(\alpha, \alpha)$ form the 1-dimensional center of $C^{(-)}$. The complementary subspace for the center consists of the matrices of trace 0. In fact, this subspace is a subalgebra denoted by $C^{(-)}/F$. Multiplication in $C^{(-)}/F$ is given by

$$\begin{pmatrix} \alpha & a \\ b & -\alpha \end{pmatrix} * \begin{pmatrix} \beta & c \\ d & -\beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}[(b,c) - (a,d)] & \alpha c - \beta a + b \times d \\ \beta b - \alpha d + a \times c & \frac{1}{2}[(a,d) - (b,c)] \end{pmatrix}. \tag{51}$$

Comparing (51) with the known multiplication table of the algebra A shows that A is isomorphic to $C^{(-)}/F$. To the element h corresponds the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and to the element $\alpha_1 x + \alpha_2 y + \alpha_3 z + \beta_1 y' + \beta_2 y' + \beta_3 z'$ corresponds the matrix

$$\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}, \ a = (\alpha_1, \alpha_2, \alpha_3), b = (\beta_1, \beta_2, \beta_3).$$

This correspondence is the isomorphism $A \to C^{(-)}/F$.

Theorem 3.8. *If F is an arbitrary field of characteristic not 2 or 3, then there exists a unique non-Lie split simple Malcev algebra A over F. This algebra is isomorphic to the algebra $C^{(-)}/F$ obtained from the Cayley-Dickson algebra C over F using the operation $x * y = \frac{1}{2}(xy - yx)$ and factoring out the center.*

The following proposition clarifies the meaning of the bilinear form (46) on A.

Proposition 3.9. *For all $x, y \in A$ we have*

$$xy \cdot y = (y, y)x - (x, y)y. \tag{52}$$

Proof. The proof is based on the multiplication table for A. Using the isomorphism $A \cong C^{(-)}/F$, computations can be performed using the matrix form. It should be noted that if

$$x \rightsquigarrow \begin{pmatrix} \alpha & a \\ b & -\alpha \end{pmatrix}, \qquad y \rightsquigarrow \begin{pmatrix} \beta & c \\ d & -\beta \end{pmatrix},$$

then by the above isomorphism we have $(x, y) = \alpha\beta - \frac{1}{2}[(a, d) + (b, c)]$. □

Identity (52) shows that the bilinear form (46) on A can be defined independently of the choice of the Cartan subalgebra H. Moreover, it follows from (52) that for all $x, y \in A$ the subspace spanned by x, y, xy is a subalgebra, i.e., any two elements $x, y \in A$ generate a subalgebra which is at most 3-dimensional.

Lemma 3.10. *We have*

$$(xy, xy) = (x, y)^2 - (x, x)(y, y). \tag{53}$$

Proof. This claim is trivial if $x = 0$, so let $x \neq 0$. Replacing y by xy in (52) we get $(x \cdot xy)(xy) = (xy, xy)x$. On the other hand,

$$(x \cdot xy)(xy) = [(x, x)y - (x, y)x]\,(xy) = \left[(x, y)^2 - (x, x)(y, y)\right] x,$$

hence the assertion follows. □

Linearizing (53) on y we obtain

$$(xy, xz) = (x, y)(x, z) - (x, x)(y, z). \tag{54}$$

It is well known that the problem of the classification of finite dimensional simple algebras over the field F can be reduced to the description of central simple algebras over F and over finite extensions of F. Let us describe central simple non-Lie Malcev algebras over a field F of characteristic not 2 or 3. Let A be an algebra as above. If F is algebraically closed then A is split and its structure is well known. In general, let $\bar{F}$ be the algebraic closure of F and $\bar{A} = A_F \otimes \bar{F}$ be the corresponding extension of A. Then $\bar{A}$ is a central simple Malcev algebra over $\bar{F}$ and $\dim_F A = \dim_{\bar{F}} \bar{A} = 7$. Let (x, y) be the bilinear form (46) defined on $\bar{A}$. Identity (52) shows that the restriction of this form to A is defined over F, and it is a non-singular bilinear form, which we also denote by (x, y). We construct a basis $\{e_1, \dots, e_7\}$ of the algebra A as follows. We choose e_1, e_2 to be two arbitrary non-isotropic elements of A which are orthogonal with respect to (x, y) and write $(e_1, e_1) = -\alpha$, $(e_2, e_2) = -\beta$, $e_1e_2 = e_3$. Then e_1, e_2, e_3 are pairwise orthogonal and it follows from (52) and (53) that $e_2e_3 = \beta e_1$, $e_3e_1 = \alpha e_2$ and $(e_3, e_3) = -\alpha\beta \neq 0$. The subspace (e_1, e_2, e_3) is non-singular. Its orthogonal complement $(e_1, e_2, e_3)^{\perp}$ has the same property. We choose as e_4 any non-isotropic element of $(e_1, e_2, e_3)^{\perp}$ and write $(e_4, e_4) = -\gamma$, $e_1e_4 = e_5$, $e_2e_4 = e_6$, $e_3e_4 = e_7$. Then by (54) for any $i, j = 1, 2, 3$ we have $(e_i, e_je_4) = (e_ie_j, e_4) = 0$ and $(e_ie_4, e_je_4) = -(e_4, e_4)(e_i, e_j)$, which implies that e_5, e_6e_7 are non-isotropic and $e_1, \ldots, e_7$ are mutually orthogonal. Therefore e_i $(i = 1, \ldots, 7)$ form a basis of A. Using the linearization of (52) we obtain for $i, j = 1, 2, 3$ that

$$e_ie_4 \cdot e_j + e_ie_j \cdot e_4 = -(e_i, e_j)e_4, \qquad e_4e_i \cdot e_4e_j + (e_4 \cdot e_4e_j)e_i = 0.$$

As a result, the multiplication table of A in the chosen basis is as follows, where $i, j = 1, 2, 3$ $(i \neq j)$:

$$\begin{array}{lll} e_1e_2 = e_3, & e_2e_3 = \beta e_1, & e_3e_1 = \alpha e_2, \\ e_ie_4 = e_{i+4}, & e_ie_{i+4} = (e_i, e_i)e_4, & e_4e_{i+4} = \gamma e_i, \\ e_{i+4}e_j = -e_ie_j \cdot e_4, & e_{i+4}e_{j+4} = -\gamma e_ie_j; & \end{array} \tag{55}$$

we write $(e_1, e_1) = -\alpha$, $(e_2, e_2) = -\beta$ and $(e_3, e_3) = -\alpha\beta$.

We denote by $M(\alpha, \beta, \gamma)$ any anticommutative algebra with multiplication table (55). It can be defined over a field of arbitrary characteristic and it is a Malcev algebra (i.e., it satisfies the identity (10)) for any $\alpha, \beta, \gamma \in F$. If $\mathrm{char} F \neq 3$ then $M(\alpha, \beta, \gamma)$ is a non-Lie algebra. If $\alpha\beta\gamma \neq 0$ then it is central simple. Hence we have proved the following theorem.

Theorem 3.11. *The class of non-Lie central simple Malcev algebras over an arbitrary field F of characteristic not 2 or 3 coincides with the class $M(\alpha, \beta, \gamma)$ for any $\alpha, \beta, \gamma \neq 0 \in F$.*

If, for example, A is the split simple Malcev algebra with basis h, x, y, z, x', y', z' constructed above, then we can set

$$\begin{array}{llll} e_1 = h, & e_2 = x + x', & e_3 = e_1e_2 = x' - x, & e_4 = y + y', \\ e_5 = e_1e_4 = y' - y, & e_6 = e_2e_4 = z + z', & e_7 = e_3e_4 = z - z'. & \end{array}$$

The parameters α, β and γ take the following values: $\alpha = -1$, $\beta = 1$ and $\gamma = 1$, i.e., $A = M(-1, 1, 1)$.

Isomorphic algebras $M(\alpha, \beta, \gamma)$ may correspond to different values $\alpha, \beta, \gamma \in F$ $(\alpha\beta\gamma \neq 0)$. The solution to the isomorphism problem for $M(\alpha, \beta, \gamma)$ follows from the method of constructing the basis described above and the Witt theorem on extension of partial isometries of bilinear metric spaces.

Theorem 3.12. *Two algebras of type $M(\alpha, \beta, \gamma)$ $(\alpha\beta\gamma \neq 0)$ over the same field F of characteristic not 2 are isomorphic if and only if their corresponding quadratic forms $f(x) = (x, x)$ are equivalent.*

Note that if $x = \sum_i t_ie_i$ $(t_i \in F)$ then

$$(x, x) = -\alpha t_1^2 - \beta t_2^2 - \alpha\beta t_3^2 - \gamma t_4^2 - \alpha\gamma t_5^2 - \beta\gamma t_6^2 - \alpha\beta\gamma t_7^2. \tag{56}$$

To every $M(\alpha, \beta, \gamma)$ over F we can associate $C(\alpha, \beta, \gamma) = F + M(\alpha, \beta, \gamma)$, whose multiplication is given by

$$(\alpha + x) \cdot (\beta + y) = \alpha\beta + \alpha y + \beta x + x \cdot y,$$

for any $\alpha, \beta \in F$ and $x, y \in M(\alpha, \beta, \gamma)$ where $x \cdot y = (x, y) + xy$. If $\alpha\beta\gamma \neq 0$ and char$F \neq 2$ then $C(\alpha, \beta, \gamma)$ is a simple alternative algebra (Cayley-Dickson algebra) which is related to $M(\alpha, \beta, \gamma)$ in the same way as $C^{(-)}/F$ is related to the split Cayley-Dickson algebra $M(-1, 1, 1)$. Clearly, two algebras $M(\alpha, \beta, \gamma)$ and $M(\alpha', \beta', \gamma')$ are isomorphic if and only if the corresponding alternative algebras $C(\alpha, \beta, \gamma)$ and $C(\alpha', \beta', \gamma')$ are isomorphic.

Let us discuss the question of Cartan subalgebras of a central simple Malcev algebra $A = M(\alpha, \beta, \gamma)$. Let y be an arbitrary nonzero element in A. If $(y, y) \neq 0$ then the subspace $V = (y)^{\perp}$ is invariant with respect to R_y and identity (52) shows that for all $x \in V$ we have

$$xy \cdot y = (y, y)x, \tag{57}$$

that is, R_y restricted to V is non-degenerate, $A_0^y = (y)$ and y is a regular element in the sense of Definition 1.11. If $(y, y) = 0$ then it follows from (52) that $R_y^3 = 0$ and $A_0^y = A$. Therefore, an element $y \in A$ is regular if and only if $(y, y) \neq 0$, and hence any Cartan subalgebra H of A coincides with the intersection of subspaces A_0^y $(y \in H)$; then H contains a regular element y and therefore coincides with the 1-dimensional subalgebra generated by y. Conversely, any regular element in A generates a (1-dimensional) Cartan subalgebra of A, independently of the cardinality of the field F.

It follows from (57) that nonzero characteristic roots of R_y coincide with quadratic roots of (y, y), and that a Cartan subalgebra $H = (y)$ is split if and only if (y, y) is the square of a nonzero element of F. Therefore, the following holds.

Proposition 3.13. *An algebra $M(\alpha, \beta, \gamma)$ is split, thus isomorphic to $M(-1, 1, 1)$, if and only if the quadratic form* (56) *represents the identity in F.*

Theorem 3.12 shows that the classification of central simple Malcev algebras over F is related to the theory of quadratic forms over F. For example, let F be the field $\mathbb{Q}$ of rational numbers. If not all α, β, γ are positive then (56) is undefined. Since an indefinite (or positive definite) quadratic form of rank $n \geq 4$ over $\mathbb{Q}$ represents 1, the form $-(x, x)$ is also positive definite, and using the above properties of quadratic forms over $\mathbb{Q}$ we have $M(\alpha, \beta, \gamma) \equiv M(1, 1, 1)$. Therefore, there are only two distinct non-Lie central simple Malcev algebras over $\mathbb{Q}$. The same is true if $F = \mathbb{R}$, the field of the real numbers. If the base field F is the field of p-adic numbers $\mathbb{Q}_p$ then any algebra of the form $M(\alpha, \beta, \gamma)$ over F is split, as in the case of an algebraically closed field, although $\mathbb{Q}_p$ is not algebraically closed.

4. Conjugacy of Cartan subalgebras of Malcev algebras

If A is an arbitrary (nonassociative) algebra over a field of characteristic 0, and D is a nilpotent derivation of A, then $\exp D$ is an automorphism of A. A derivation D is said to be *inner* if it belongs to the algebra A^* of multiplications of A, where A^* is generated by the operators of left and right multiplication. Consider the group Φ of all automorphisms of A generated by all $\exp D$ where D is an inner nilpotent derivation. Elements of Φ will be called *special automorphisms of* A.

4.1. Let F be an algebraically closed field of characteristic 0, let A be a Malcev algebra over F, let H be a Cartan subalgebra of A, and let $\alpha_1, \ldots, \alpha_n$ be the nonzero roots of H in A. To each pair of elements $x, y \in A$ we associate the inner derivation $D(x, y) = R_{xy} + [R_x, R_y]$; see equation (13). Let us show that any element $e_\alpha \in A_\alpha$ ($\alpha \neq 0$) and any element $h \in H$ define a nilpotent derivation $D(e_\alpha, h)$. Indeed, if $e_\beta \in A_\beta$ and $\beta \neq k\alpha$ for any integer k, then for every $k > 0$ we have $e_\beta D^h(e_\alpha, h) = 0$ for k sufficiently large. The same is true for $\beta = k\alpha$ for $k \geq 2$. The case $\beta = -\alpha$ is of special interest; then $J(h, e_\alpha, e_{-\alpha}) = 0$. It follows that the elements $h, e_\alpha, e_{-\alpha}$ generate a Lie subalgebra in A. The restriction of $D(e_\alpha, h)$ to this subalgebra coincides with $R_{e'_\alpha}$, where $e'_\alpha = 2e_\alpha h \in A_\alpha$. Thus

$$e_{-\alpha} D^{k+1}(e_\alpha, h) = [(e_{-\alpha} \underbrace{e'_\alpha) \cdots] \, e'_\alpha}_{k+1}. \tag{58}$$

For any $h_1 \in H$ the elements $e_\alpha, e_{-\alpha}, h_1$ form a Lie triple, i.e., $J(e_\alpha, e_{-\alpha}, h_1) = 0$. Therefore, the right side of (58) belongs to $A_{k\alpha}$ for any $k \geq 0$, and since $\alpha \neq 0$ we conclude that $e_{-\alpha} D^h(e_\alpha, h) = 0$ for $k > 0$ sufficiently large. By (29) the remaining cases can be reduced to the cases considered above.

We choose a basis $\{h_1, \ldots, h_s, e_{s+1}, \ldots, e_m\}$ of A in such a way that the elements $\{h_1, \ldots, h_s\}$ form a basis of H and $\{e_{s+1}, \ldots, e_m\}$ lie in root spaces A_α, $\alpha \neq 0$. We choose an element $h_0 \in H$ such that $\alpha_i(h_0) \neq 0$ for all $i = 1, \ldots, n$. This can be done owing to the linearity of the roots: the product $\alpha_1 \alpha_2 \ldots \alpha_n$ is a polynomial function $H \to F$ which is not identically 0. Let $\lambda_1, \ldots, \lambda_m$ be independent variables and let

$$x = \lambda_1 h_1 + \cdots + \lambda_s h_s + \lambda_{s+1} e_{s+1} + \cdots + \lambda_m e_m,$$

be an element of A. Then the element

$$xP = \left(\sum_{i=1}^{s} \lambda_i h_i \right) \exp D(\lambda_{s+1} e_{s+1}, h_0) \cdots \exp D(\lambda_m e_m, h_0),$$

defines a polynomial map P of the algebra A into itself (the coordinates of xP are polynomial functions of the coordinates of x). Let us compute the differential

$d_{h_0}P$ of this map at the point h_0. Set

$$x = h + e, \qquad h \in H, \qquad e \in \sum_{\alpha \neq 0} A_\alpha.$$

Then

$$\begin{aligned}(h_0 + tx)P = [(h_0 + th) + te]\,P &\equiv (h_0 + th)\,[1 + tD(e, h_0)] \pmod{t^2}\\ &\equiv h_0 + t\,[h + h_0 D(e, h_0)] \pmod{t^2},\end{aligned}$$

which implies that $d_{h_0}P$ is a map

$$h + e \mapsto h + h_0 D(e, h_0) = h - 2(eh_0)h_0.$$

Since $h \mapsto h$ and $e \mapsto -2(eh_0)h_0$ are non-degenerate, we see that $d_{h_0}P$ is an epimorphism. Arguing in the same way as [3] we can show the following:

Theorem 4.1. *If H_1 and H_2 are Cartan subalgebras of a finite dimensional Malcev algebra A over an algebraically closed field of characteristic 0 then there exists a special automorphism η of A such that $H_1^\eta = H_2$.*

In the proof it is shown that a Zariski open set consisting of regular elements of A is the image of the regular elements from an arbitrary Cartan subalgebra $H \leq A$ with respect to a special automorphism. In particular, all Cartan subalgebras of A have the same dimension and contain regular elements. When extending the base field $F \subset \Omega$, the Fitting 0 component A_0^x of A with respect to R_x for any $x \in A$ becomes the Fitting 0 component $A_0^x \otimes \Omega$ of $A_\Omega = A \otimes \Omega$ with respect to the same operator, and a Cartan subalgebra $H \leq A$ becomes a Cartan subalgebra $H_\omega = H \otimes \Omega$ of A_Ω. Therefore, the following holds:

Corollary 4.2. *If A is a finite dimensional Malcev algebra over an arbitrary field of characteristic 0 then all Cartan subalgebras of A have the same dimension. Moreover, each Cartan subalgebra contains a regular element.*

Proof. We only need to prove the second claim. Let $x = \lambda_1 h_1 + \cdots + \lambda_s h_s$ be an element of a Cartan subalgebra H, and let $f(\lambda, x) = \det(\lambda - R_x)$ be the characteristic polynomial of R_x. If the multiplicity of 0 as eigenvalue of R_x (i.e., the dimension of A_0^x) is greater than $\dim H = s$ for any specialization of $\lambda_1, \ldots, \lambda_s$ in the base field F then $f(\lambda, x)$ has the form

$$f(\lambda, x) = \lambda^m - \tau_1(x)\lambda^{m-1} + \cdots + (-1)^{m-1}\tau_{m-1}(x)\lambda^\ell,$$

where $\ell > s$. However, the same is true for any extension Ω of F; this contradicts the existence of a regular element in $H_\Omega = H \otimes \Omega$ when Ω is algebraically closed. □

5. Representations of semisimple Malcev algebras of characteristic 0

The results of this section are based on the connection between Malcev algebras and Lie triple systems pointed out by Loos [9]. The main result is the theorem about complete reducibility of representations of semisimple Malcev algebras (Theorem 5.5) which is similar to Weyl's theorem for Lie algebras.

5.1. We recall the definition and basic properties of Lie triple systems (LTS) [8, 2]. A vector space T over a field F is called an LTS if a ternary operation $[xyz]$ defined on it is linear in each variable and satisfies the following identities:

$$[aab] = 0,$$
$$[abc] + [bca] + [cab] = 0,$$
$$[ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]].$$

The last identity shows that the map $D_{a,b}\colon x \mapsto [abx]$ is a derivation of T. Such derivations are called *inner* and they generate a Lie algebra $D_0(T)$ which is called the *algebra of inner derivations.* Any Lie algebra L with triple product $[xyz] = xy\cdot z$ (or any subspace of L closed under the iterated product) is an example of an LTS. On the other hand, any LTS can be realized as a subspace of a Lie algebra with the iterated product; in this case we say that the LTS is embedded into the Lie algebra. If an LTS T is embedded into a Lie algebra L then the subalgebra of L generated by T is called the *enveloping Lie algebra of the embedding.* For an arbitrary LTS we can define the notions of ideal, solvability, radical, and semisimplicity. If an LTS T is semisimple then its enveloping Lie algebra is also semisimple for any embedding $T \to L$. Among all embeddings of an LTS into a Lie algebra there are two special ones: the standard and the universal. The underlying vector space of the standard enveloping algebra $L_s(T)$ has the form $T + D_0(T)$ and the multiplication in $L_s(T)$ is defined in the obvious way. In particular, if $a, b \in T$ then $ab = D_{a,b}$. The universal Lie enveloping algebra $L_u(T)$ is characterized by the property that any homomorphism $T \to L$, where L is an arbitrary Lie algebra, can be uniquely extended to a homomorphism $L_u(T) \to L$. If an LTS T is semisimple then its standard and universal enveloping algebra coincide.

We now assume that the characteristic of the base field F is 0. If A is a semisimple algebra then T_A is also semisimple[†]; in general, the radical of A coincides with the radical of T_A [9]. The set of inner derivations of T_A is generated by the operators of the form $R(x,y) = 2R_{xy} + [R_x, R_y]$. Identities (15) show that each operator R_x is a derivation of the LTS T_A. Therefore, the Lie enveloping algebra $L(A)$ of the regular representation of A is a subalgebra of the algebra $D(T_A)$ for all derivations of $T(A)$:

$$D_0(T_A) \subseteq L(A) \subseteq D(T_A). \tag{59}$$

[†] Translator's note: T_A is the Lie triple system associated to the Malcev algebra A as in the paper by Loos [9].

Since all derivations of a semisimple LTS are inner [8], for any semisimple Malcev algebra the inclusions in (59) become equalities [9].

Proposition 5.1. *If A is a simple (respectively semisimple) Malcev algebra then the Lie enveloping algebra $L(A)$ of its regular representation is also simple (respectively semisimple).*

Proof. To the decomposition of A into a direct sum of ideals A_i corresponds a decomposition of $L(A)$ into a direct sum of ideals isomorphic to $L(A_i)$. If A_i is a simple Lie algebra then $L(A_i)$ is also a simple Lie algebra isomorphic to A_i. Let A be a simple non-Lie Malcev algebra; we show that $L(A)$ is again a simple Lie algebra. It suffices to consider the case when A is a central simple algebra. Indeed, if A is not central then A can be regarded as a central simple algebra A_Γ over its centroid $\Gamma \supset F$ [3]. Since all operators R_x ($x \in A$) are Γ-linear and $R_{\gamma a} = \gamma R_a$ for $a \in A$, $\gamma \in \Gamma$, we see that the Lie algebra $L(A)$ can be regarded as an algebra (of smaller dimension) over the field Γ, which, obviously, coincides with $L(A_\Gamma)$. If we prove that $L(A_\Gamma)$ is a central simple algebra (over Γ), it would imply that $L(A)$ is also simple and its centroid is isomorphic to Γ. Using the same arguments we can restrict our attention to the case of an algebraically closed field F. In the case that the algebra has dimension 7 its structure is known (see §3). Inner derivations $D(x,y) = R_{xy} + [R_x, R_y]$ generate a subalgebra L_0 of dimension 14 in $L(A)$ which is a simple Lie algebra of type G_2 [2]. The underlying vector space of $L(A)$ can be decomposed into the sum of the subspaces L_0 and $R(A)$, where $R(A)$ is the subspace generated by the operators R_x; the sum is direct since R_x is a derivation of A if and only if x lies in the Lie center of A, which is 0 in a simple non-Lie Malcev algebra (compare Lemma 3.1). Therefore, $\dim L(A) = 21$. The Killing form on A is non-degenerate and each operator R_x ($x \in A$) is skew-symmetric with respect to this form. Therefore, $L(A)$ is a subalgebra of a simple Lie algebra of type B_3 (the orthogonal algebra of a 7-dimensional vector space). Comparing the dimensions of $L(A)$ and B_3 we see that $L(A) = B_3$. The proof is complete. □

Corollary 5.2. *If A is a simple (respectively semisimple) Malcev algebra over a field of characteristic 0 then the algebra $D(T_A)$ of derivations of the Lie triple system T_A is also simple (respectively semisimple). In particular, if $A = C^{(-)}/F$ then $D(T_A) = L(A) = B_3$.*

Theorem 5.3. *Let A be a Malcev algebra over a field of characteristic 0, let S be its radical and N its nil-radical. Then every derivation D of A maps S to N.*

Proof. As shown in [9], S coincides with the radical of T_A. However, for any LTS T, the radical of $L_s(T)$ is generated as an ideal by the radical of T; if R is the radical of T then the radical of $L_s(T)$ equals $R + [R,T]$ [8]. In particular, S lies in the radical of $L_s(T_A)$. A derivation D of the algebra A is also a derivation of the LTS T_A, i.e., D can be regarded as an element of the algebra $D(T_A)$. Since $L_s(T_A)$ is an ideal of the Lie algebra $T_A + D(T_A)$, $(S)D$ lies in the nil-radical of $L_s = L_s(T_A)$. In order to distinguish the operators of right multiplication by x ($x \in A$) in L_s

from the operators R_x in A, we will denote them by $\operatorname{ad} x$. Thus, for any $x \in (S)D$, $\operatorname{ad} x$ is a nilpotent operator. Furthermore, $(\operatorname{ad} x)^2$ leaves the subspace $T_A \subset L_s$ invariant, and since $[[ax]x] = [axx] = 3(ax)x$ for any $a \in A$, $(\operatorname{ad} x)^2$ coincides with $3R_x^2$ in T_A. Therefore, R_x is a nilpotent operator. However, it follows from Theorem 2.7 that the nil-radical of A coincides with the set of all elements from S which are nilpotent with respect to the regular representation. Hence $x \in N$. (We assume that it is known that the radical S is closed under all derivations of A. Any solvable radical of a finite dimensional algebra of characteristic 0 has this property.) □

The following theorem gives important information about the structure of the representations of semisimple Malcev algebras.

Theorem 5.4. *Let A be a semisimple Malcev algebra of characteristic 0, let ρ be a representation of A in a vector space V, and let $L_\rho(A)$ be the enveloping algebra of the representation ρ. Then $L_\rho(A)$ is a semisimple algebra.*

Proof. Let $E = V + A$ be the semidirect extension of A by means of V defined by ρ. If $\tilde{\rho}$ is the regular representation of A in E, and $\tilde{L}(A)$ is the enveloping algebra of $\tilde{\rho}$, then V is invariant under the action of $\tilde{L}(A)$ $(V \lhd E)$; the restriction of ρ to V induces an epimorphism $\pi\colon \tilde{L}(A) \to L_\rho(A)$. Consider the LTS T_A and T_E; there exists a unique embedding $\iota\colon T_A \to T_E \subset L_s(T_E)$. Since the LTS T_A is semisimple, the standard embedding for T_A coincides with the universal embedding; therefore ι can be extended to a homomorphism $\iota^*\colon L_s(T_A) \to L_s(T_E)$. The operators $\tilde{R}(x,y) = 2\tilde{R}_{xy} + [\tilde{R}_x, \tilde{R}_y] \in \tilde{L}(A)$ are the images of the elements $[x,y] = R(x,y) \in D_0(T_A)$ under ι^*. The restriction of ι^* to $D_0(T_A) = D(T_A)$ defines a homomorphism $\iota'\colon D(T_A) \to \tilde{L}(A)$ and the composition of ι' and π defines a homomorphism from $D(T_A)$ onto the subalgebra $I \subseteq L_\rho(A)$ generated by the operators $\rho(x,y) = 2\rho(xy) + [\rho(x), \rho(y)]$, $x, y \in A$. Identity (15), which is true for arbitrary representations, shows that I is an ideal of $L_\rho(A)$. By Corollary 5.2, $D(T_A)$ is a semisimple algebra, therefore its homomorphic image I is also semisimple. Consider the quotient algebra $\bar{L} = L_\rho(A)/I$, and denote by $\bar{\rho}(x)$ the image of $\rho(x) \in L_\rho(A)$ under the canonical homomorphism $L_\rho(A) \to \bar{L}$. The underlying vector space of $\bar{L}$ is generated by the elements $\bar{\rho}(x)$ and they satisfy

$$\text{either} \quad 2\bar{\rho}(xy) + [\bar{\rho}(x), \bar{\rho}(y)] = 0, \quad \text{or} \quad -\frac{1}{2}\bar{\rho}(xy) = [-\tfrac{1}{2}\bar{\rho}(x), -\tfrac{1}{2}\bar{\rho}(y)].$$

Then the map $x \mapsto -\frac{1}{2}\rho(x) \mapsto -\frac{1}{2}\bar{\rho}(x)$ is a homomorphism of A onto $\bar{L}$. Since A is a semisimple algebra, it follows from the structural theorem (§2) that $\bar{L}$ is semisimple (or trivial). Then $L_\rho(A)$ is also a semisimple Lie algebra because the extension of a semisimple Lie algebra by a semisimple algebra is also semisimple. The proof is complete. □

Since each representation ρ of a semisimple algebra A in a vector space V can be regarded as the natural representation of the Lie algebra $L_\rho(A)$ in the same vector space, the next theorem follows directly from Theorem 5.4.

Theorem 5.5. *Any representation of a semisimple Malcev algebra of characteristic 0 is completely reducible.*

Corollary 5.6. *If the radical of a Malcev algebra A coincides with its center C then $A = A_1 + C$, where A_1 is a semisimple subalgebra which coincides with A^2.*

Proof. It suffices to consider the regular representation of A and note that it induces a completely reducible representation of A/C in A. An invariant subspace A_1 complementary to C is the desired subalgebra (even ideal). □

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Editors' Comments on Recent Developments

In this section we briefly summarize research on Malcev algebras since the publication of Kuzmin's paper [6] in 1968 which contained the first statement (in some cases without detailed proofs) of the results in the present English translation.

Kuzmin's papers provide a complete theory for finite-dimensional semisimple Malcev algebras and their finite-dimensional representations over a field $\mathbb{F}$ of characteristic 0; in particular, such representations are completely reducible. With these assumptions, a simple Malcev algebra is either a Lie algebra or a 7-dimensional non-Lie Malcev algebra. Gavrilov [G] has recently given a detailed proof of the classification by Kuzmin [K] of 5-dimensional Malcev algebras.

If we regard a simple Lie algebra L as a Malcev algebra, then Carlsson [C1] showed that every Malcev module for L is a Lie module, with one exception: there is an irreducible 2-dimensional non-Lie Malcev module for $\mathfrak{sl}(2,\mathbb{F})$. The same author gave a different proof [C2] of the Wedderburn decomposition of a Malcev algebra into a semisimple subalgebra and the solvable radical. She also showed [C3] that in every characteristic any finite-dimensional Malcev module over a 7-dimensional central simple non-Lie Malcev algebra is completely reducible.

Elduque [E1] classified the maximal subalgebras of central simple non-Lie Malcev algebras over a field of characteristic not 2. The same author studied [E2] the lattice of subalgebras of a Malcev algebra, and showed that two semisimple Malcev algebras over an algebraically closed field are isomorphic if and only if their lattices are isomorphic. He also extended Carlsson's result on Malcev modules to characteristic not 2 or 3, and obtained a new 4-dimensional irreducible non-Lie Malcev module over a nonsplit simple 3-dimensional Lie algebra. The classification of non-Lie Malcev modules was completed by Elduque and Shestakov [ES] in the more general setting of Malcev superalgebras with no restriction on the dimension of the modules and only the condition that $\frac{1}{6} \in \mathbb{F}$.

In 2004, an important breakthrough was made by Pérez-Izquierdo and Shestakov [PS], who constructed universal nonassociative enveloping algebras for Malcev algebras. For any Malcev algebra M over a field $\mathbb{F}$ of characteristic not 2 or 3, there exists a nonassociative algebra $U(M)$ and an injective map from M to $U(M)$ such that the image of M lies in the generalized alternative nucleus of $U(M)$, and $U(M)$ is universal with respect to such maps. The algebra $U(M)$ has a basis of Poincaré-Birkhoff-Witt type, so $U(M)$ is linearly isomorphic to the polynomial algebra $P(M)$; moreover, $U(M)$ has a natural (nonassociative) Hopf algebra structure, and the image of M can be characterized as the primitive elements of $U(M)$ with respect to the diagonal homomorphism

$\Delta\colon U(M) \to U(M) \otimes U(M)$. The paper [PS] also proved an analogue of the Ado-Iwasawa theorem: every finite-dimensional Malcev algebra is isomorphic to a subalgebra of the generalized alternative nucleus of a finite-dimensional unital nonassociative algebra. Zhelyabin and Shestakov [ZS] established analogues for Malcev algebras of the Chevalley and Kostant theorems on centers of universal enveloping algebras of Lie algebras. The nonassociative bialgebra structure of the enveloping algebras $U(M)$ has been studied by Zhelyabin [Z]; see also [M]. Structure constants for $U(M)$ when $\dim M \leq 5$ have been obtained by various authors; see [B1,B2,TB] and the survey [B3].

The theory of free Malcev algebras has been developed primarily by Filippov, who showed (over a field of characteristic not 2 or 3) that free Malcev algebras have zero-divisors [F1]; that free Malcev algebras with 5 or more generators are not semiprime [F3], have nonzero annihilator, and are not separated [F4]; and that the base rank of the variety of Malcev algebras is infinite [F4]. Shestakov and Kornev [SK] showed that the prime radical of a free Malcev algebra on two or more generators coincides with the set of all its universally Engelian elements.

Simple Malcev superalgebras have been studied by Shestakov [S1], who showed that a prime Malcev superalgebra of characteristic not 2 or 3 with a nontrivial odd part is a Lie superalgebra. The same author in collaboration with Zhukavets has developed the theory of free Malcev superalgebras; see [S3,SZ1,SZ2].

The speciality problem for Malcev algebras asks whether every Malcev algebra is "special"; that is, isomorphic to a subalgebra of the commutator algebra of some alternative algebra. Filippov [F2] proved that over a field containing $\frac{1}{2}$ every semiprime Malcev algebra is special. Sverchkov [Sv] proved that every Malcev algebra in the variety generated by the 7-dimensional simple non-Lie Malcev algebra is special. Recent progress on this problem, and the corresponding problem for Malcev superalgebras, is primarily the work of Shestakov and Zhukavets. There is a close relation between this problem and the deformation theory of algebras [S2]. It has been shown that the free Malcev superalgebra on one odd generator is special [SZ3]; more generally, this holds for any Malcev superalgebra generated by one odd element.

For a generalization of Malcev algebras to the setting of dialgebras; see [B4,Sa].

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