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Commutants of middle Bol loops

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Abstract. The commutant of a loop is the set of all its elements that commute with each element of the loop. It is known that the commutant of a left or right Bol loop is not a subloop in general. Below we prove that the commutant of a middle Bol loop is an AIP-subloop, i.e., a subloop for which the inversion is an automorphism. A necessary and sufficient condition when the commutant is invariant under the existing isostrophy between middle Bol loops and the corresponding right Bol loops is given.

1. Introduction

Recall that a loop $(Q, \cdot)$ is a *right* (*left*) *Bol loop* if it satisfies the identity $(zx \cdot y)x = z(xy \cdot x)$ (resp. $x(y \cdot xz) = (x \cdot yx)z$). We say that a quasigroup $(Q, \cdot)$ satisfies the *right* (*left*) *inverse property*, if there exists a mapping $\varphi : Q \mapsto Q$, such that $yx \cdot \varphi(x) = y$ (resp. $\varphi(x) \cdot xy = y$), for every $x, y \in Q$. If a loop satisfies the right (left) inverse property then the left inverse of each element coincides with the right inverse ${}^{-1}x = x^{-1}$ and $yx \cdot x^{-1} = y$ (resp., $x^{-1} \cdot xy = y$), $\forall x, y \in Q$. Right (left) Bol loops satisfy the right (resp. left) inverse property. A loop $(Q, \circ)$ is called a *middle Bol loop* if the condition $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$, $\forall x, y \in Q$, called the *anti-automorphic inverse property*, is universal in $(Q, \circ)$, i.e., if every loop isotope of $(Q, \circ)$ satisfies the anti-automorphic inverse property. V. D. Belousov proved in [1] that a loop $(Q, \cdot)$ is middle Bol if and only if the corresponding e-loop $(Q, \cdot, /, \backslash)$ (the operation "/" and, resp. "\", is the left, resp. right, division in $(Q, \cdot)$), satisfies the identity:

$$x \cdot ((y \cdot z) \backslash x) = (x/z) \cdot (y \backslash x). \tag{1}$$

Middle Bol loops are studied in [1, 2, 3, 6]. It was proved in [3] that middle Bol loops are isostrophes of right (left) Bol loops. We will consider below the isostrophy between right Bol loops and middle Bol loops. Left Bol loops can be characterized analogously, by a "mirror reflection".

According to [3], a loop $(Q, \circ)$ is middle Bol if and only if there exists a right Bol loop $(Q, \cdot)$ such that, for $\forall x, y \in Q$, the following equality holds:

$$x \circ y = (y \cdot xy^{-1})y, \tag{2}$$

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which is equivalent to

$$x \circ y = y^{-1} \backslash x, \tag{3}$$

and to

$$x \cdot y = y // x^{-1}, \tag{4}$$

where "\" is the right division in the right Bol loop $(Q, \cdot)$, and "//" is the left division in the middle Bol loop $(Q, \circ)$.

The connection between middle and left Bol loops is analogous [6].

A middle Bol loop satisfies the right or left inverse property if and only if it is a Moufang loop (see [3]). It is known (see [2, 6]) that two middle Bol loops are isotopic (resp. isomorphic) if and only if the corresponding right Bol loops are isotopic (resp. isomorphic). Note also that a middle Bol loop $(Q, \circ)$ and its corresponding right Bol loop $(Q, \cdot)$ have a common unit and that the inverse of each element x in $(Q, \circ)$ is equal to the inverse of x in $(Q, \cdot)$. Moreover, middle Bol loops, as well as their corresponding right Bol loops, are power-associative (i.e., every subloop generated by one element is associative).

The *commutant of a loop* $(Q, \cdot)$ is the set of all elements that commute with each element of the loop $(Q, \cdot)$. This notion is known also as: *centrum, commutative center, semicenter*, etc. In groups the commutant is the center and a normal subgroup. In loops the commutant is not always a subloop. But it is known, for example, that the commutant of a Moufang loop is a subloop.

The commutants of left Bol loops are studied in [4] and [5] where examples of finite left Bol loops with non-subloop commutants are given and necessary conditions when the commutants of finite left Bol loops are subloops are found.

Below is proved that the commutants of middle Bol loops are AIP-subloops, i.e., subloops with automorphic inverse property: $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$. Also, necessary and sufficient conditions when the commutant of a middle Bol loop and the commutant of the corresponding right Bol loop coincide are found. In the last section the "normality" of the commutants in middle Bol loops is partially examined.

2. The commutants of middle Bol loops

Let's denote the commutant of a loop $(Q, \cdot)$ by $C^{(\cdot)}$, so:

$$C^{(\cdot)} = \{a \in Q \mid a \cdot x = x \cdot a, \forall x \in Q\}.$$

Lemma 2.1. *Let $(Q, \circ)$ be a middle Bol loop. If $a \in C^{(\circ)}$, then $a^{-1} \in C^{(\circ)}$.*

Proof. If $a \in C^{(\circ)}$, then $a \circ x = x \circ a$, $\forall x \in Q$. So, as $(Q, \circ)$ satisfies the anti-automorphic inverse property, the last equality implies $x^{-1} \circ a^{-1} = a^{-1} \circ x^{-1}$, $\forall x \in Q$, i.e., $a^{-1} \in C^{(\circ)}$. □

Lemma 2.2. *Let $(Q, \circ)$ be a middle Bol loop and let $(Q, \cdot)$ be the corresponding right Bol loop. For $a \in Q$, the following statements hold:*

1. $a \in C^{(\circ)}$ *if and only if, for* $\forall z \in Q$*:*

$$(a \cdot z)^{-1} = a^{-1} \cdot z^{-1}; \tag{5}$$

2. $a \in C^{(\circ)}$ *if and only if, for* $\forall z \in Q$*:*

$$(z \cdot a)^{-1} = z^{-1} \cdot a^{-1}. \tag{6}$$

Proof. 1. According to the definition of $C^{(\circ)}$, $a \in C^{(\circ)}$ if and only if $a \circ x = x \circ a$, $\forall x \in Q$. So, using (3), we get $x^{-1} \backslash a = a^{-1} \backslash x$, $\forall x \in Q$, where "$\backslash$" is the right division in the corresponding right Bol loop $(Q, \cdot)$. Denoting $a^{-1} \backslash x$ by z and applying the right inverse property of $(Q, \cdot)$, we obtain:

$$(a^{-1} \cdot z)^{-1} \backslash a = z \Leftrightarrow (a^{-1} \cdot z)^{-1} \cdot z = a \Leftrightarrow (a^{-1} \cdot z)^{-1} = a \cdot z^{-1}.$$

The last equality is equivalent to (5).

2. Using (2), the right Bol identity, the right inverse property and the power-associativity of $(Q, \cdot)$, we have:

$$a \in C^{(\circ)} \Leftrightarrow a \circ x = x \circ a \Leftrightarrow (x \cdot ax^{-1}) \cdot x = (a \cdot xa^{-1}) \cdot a \Leftrightarrow$$
$$x^{-1} \cdot [(x \cdot ax^{-1}) \cdot x] = x^{-1}[(a \cdot xa^{-1}) \cdot a] \Leftrightarrow$$
$$ax^{-1} \cdot x = (x^{-1}a \cdot xa^{-1}) \cdot a \Leftrightarrow a = (x^{-1}a \cdot xa^{-1}) \cdot a \Leftrightarrow$$
$$e = x^{-1}a \cdot xa^{-1} \Leftrightarrow (x \cdot a^{-1})^{-1} = x^{-1} \cdot a \Leftrightarrow (x \cdot a)^{-1} = x^{-1} \cdot a^{-1},$$

for every $x \in Q$, where e is the common unit of $(Q, \circ)$ and $(Q, \cdot)$. $\square$

Remark 2.3. (I). According to Lemma 2.2,

$$C^{(\circ)} = \{a \in Q \mid (a \cdot x)^{-1} = a^{-1} \cdot x^{-1}, \forall x \in Q\} = \{a \in Q \mid (x \cdot a)^{-1} = x^{-1} \cdot a^{-1}, \forall x \in Q\},$$

where $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right Bol loop.

(II). Let $(Q, \circ)$ be a middle Bol loop and let $a \in C^{(\circ)}$. Using (1), we have:

$$a \circ [(y \circ z) \backslash\backslash a] = (a // z) \circ (y \backslash\backslash a),$$

for every $y, z \in Q$, where "$\backslash\backslash$" ("$//$") is the right (respectively, left) division in $(Q, \circ)$. Puting $y = e$, where e is the unit of $(Q, \circ)$, and using the fact that $a \in C^{(\circ)}$, the previous equality implies:

$$a \circ (z \backslash\backslash a) = (a // z) \circ a = a \circ (a // z),$$

for every $z \in Q$, so $(Q, \circ)$ satisfies the equality

$$z \backslash\backslash a = a // z, \tag{7}$$

for $\forall z \in Q$ and $\forall a \in C^{(\circ)}$.

(III). Recall that the inversion is a left semi-automorphism of right Bol loops, i.e., $(xy \cdot x)^{-1} = (x^{-1} \cdot y^{-1}) \cdot x^{-1}$. This fact was observed by D.A. Robinson. Note that it can be easily obtained if we denote $(xy \cdot x)^{-1}$ by z. In other words, if $e = z \cdot (xy \cdot x) = (zx \cdot y) \cdot x$. Then, applying three times the right inverse property, we obtain $(xy \cdot x)^{-1} = (x^{-1} \cdot y^{-1}) \cdot x^{-1}$.

Theorem 2.4. *The commutant of a middle Bol loop is a subloop.*

Proof. Let $(Q, \circ)$ be a middle Bol loop and let $(Q, \cdot)$ be the corresponding right Bol loop. If $a, b \in C^{(\circ)}$ then, using the equalities (6) and (5), the right Bol identity and the fact that $x \mapsto x^{-1}$ is a left semi-automorphism of $(Q, \cdot)$, we have:

$$(ba \cdot y)^{-1} \cdot a^{-1} = [(ba \cdot y) \cdot a]^{-1} = [b \cdot (ay \cdot a)]^{-1} = b^{-1} \cdot (ay \cdot a)^{-1}$$
$$= b^{-1} \cdot (a^{-1}y^{-1} \cdot a^{-1}) = (b^{-1}a^{-1} \cdot y^{-1}) \cdot a^{-1},$$

so

$$(ba \cdot y)^{-1} = b^{-1}a^{-1} \cdot y^{-1} = (ba)^{-1} \cdot y^{-1},$$

for every $y \in Q$.

According to Lemma 2.2, the condition $(ba \cdot y)^{-1} = (ba)^{-1} \cdot y^{-1}$, $\forall y \in Q$, is equivalent to $b \cdot a \in C^{(\circ)}$. Thus, using (2) and Lemma 2.1, we can see that $a \circ b = (b \cdot ab^{-1}) \cdot b \in C^{(\circ)}$, which means that " $\circ$ " is an algebraic operation on $C^{(\circ)}$.

Moreover, using Lemma 2.1, (4) and (7), we get: $a, b \in C^{(\circ)} \Rightarrow a, b^{-1} \in C^{(\circ)} \Rightarrow b^{-1} \cdot a = a//b = b\backslash\backslash a \in C^{(\circ)}$, i.e., $y = b\backslash\backslash a = a//b \in C^{(\circ)}$, where y is the solution of the equations $b \circ y = y \circ b = a$. Hence $(C^{(\circ)}, \circ)$ is a subloop of $(Q, \circ)$. □

Corollary 2.5. *The commutant of a middle Bol loop $(Q, \circ)$ is its AIP-subloop.*

Proof. Indeed, if $a \in C^{(\circ)}$ then $a^{-1} \in C^{(\circ)}$, so $(a \circ x)^{-1} = x^{-1} \circ a^{-1} = a^{-1} \circ x^{-1}$, $\forall x \in Q$. □

Corollary 2.6. *If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right Bol loop, then $(C^{(\circ)}, \cdot)$ is an AIP-subloop of $(Q, \cdot)$.*

Proof. It was shown in the proof of Theorem 2.4 that $a \cdot b, a \circ b, a\backslash\backslash b \in C^{(\circ)}$ for $a, b \in C^{(\circ)}$. This means that " $\cdot$ " is an algebraic operation on $C^{(\circ)}$. So, if $a, b \in C^{(\circ)}$ then, using (3), (4) and Lemma 2.1, we have: $b\backslash a = a \circ b^{-1} \in C^{(\circ)}$ and $a/b = c \Leftrightarrow c \cdot b = a \Leftrightarrow b//c^{-1} = a \Leftrightarrow a \circ c^{-1} = b \Leftrightarrow a\backslash\backslash b = c^{-1} \Leftrightarrow (a\backslash\backslash b)^{-1} = c$, so $a/b = (a\backslash\backslash b)^{-1} \in C^{(\circ)}$, i.e., $(C^{(\circ)}, \cdot)$ is a subloop of $(Q, \cdot)$.

Moreover, according to (5), Lemma 2.2, $(a \cdot z)^{-1} = a^{-1} \cdot z^{-1}$, $\forall a \in C^{(\circ)}$ and $\forall z \in Q$, so $(C^{(\circ)}, \cdot)$ is an AIP-subloop of $(Q, \cdot)$. □

3. A criterion for $C^{(\circ)} = C^{(\cdot)}$

If $(Q, \cdot)$ is a Moufang loop and $(Q, \circ)$ is the corresponding middle Bol loop, then "$\circ$" = "$\cdot$" and $C^{(\circ)} = C^{(\cdot)}$. The examples below show that both cases $C^{(\circ)} = C^{(\cdot)}$ and $C^{(\circ)} \neq C^{(\cdot)}$ are possible for an arbitrary middle Bol loop $(Q, \circ)$ and its corresponding right Bol loop $(Q, \cdot)$. The right Bol loops, used in these examples, can be found at http://www.uwyo.edu/moorhouse/pub/bol/mult8.txt (the loop 8.1.4.0 of order 8) and at http://www.uwyo.edu/moorhouse/pub/bol/mult12.txt (the loop 12.9.1.0 of order 12)

Example 3.1. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Consider the right Bol loop $(Q, \cdot)$ and the corresponding middle Bol loop $(Q, \circ)$, given by the tables:

$(\cdot)$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	8	6	1	7	3	5	4
3	3	7	8	6	1	4	2	5
4	4	1	7	8	6	5	3	2 .
5	5	6	1	7	8	2	4	3
6	6	3	4	5	2	8	1	7
7	7	5	2	3	4	1	8	6
8	8	4	5	2	3	7	6	1

$(\circ)$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	8	6	1	7	3	5	4
3	3	7	8	6	1	4	2	5
4	4	1	7	8	6	5	3	2 .
5	5	6	1	7	8	2	4	3
6	6	5	2	3	4	8	1	7
7	7	3	4	5	2	1	8	6
8	8	4	5	2	3	7	6	1

The commutants of the given loops are $C^{(\cdot)} = \{1, 6, 7, 8\}$ and $C^{(\circ)} = \{1, 8\}$, respectively, so $C^{(\circ)} \neq C^{(\cdot)}$.

Example 3.2. In this example $C^{(\circ)} = C^{(\cdot)} = \{1\}$, $(Q, \circ)$ is a right Bol loop and $(Q, \cdot)$ is the corresponding middle Bol loop.

$(\cdot)$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	3	6	5	8	10	11	9	12	7
3	3	5	6	2	4	1	10	9	12	11	7	8
4	4	6	5	1	3	2	9	11	7	12	8	10
5	5	3	2	6	1	4	12	7	10	8	9	11
6	6	4	1	5	2	3	11	12	8	7	10	9
7	7	9	11	8	12	10	1	5	4	6	3	2
8	8	10	12	7	11	9	2	1	6	5	4	3
9	9	7	8	11	10	12	4	3	1	2	5	6
10	10	8	7	12	9	11	3	2	5	1	6	4
11	11	12	10	9	8	7	6	4	2	3	1	5
12	12	11	9	10	7	8	5	6	3	4	2	1

$(\circ)$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	5	6	3	4	12	7	10	8	9	11
3	3	4	6	5	2	1	11	12	8	7	10	9
4	4	3	2	1	6	5	9	11	7	12	8	10
5	5	6	4	3	1	2	8	10	11	9	12	7
6	6	5	1	2	4	3	10	9	12	11	7	8
7	7	12	10	9	8	11	1	4	2	3	6	5
8	8	7	9	11	10	12	4	1	3	2	5	6
9	9	10	12	7	11	8	2	6	1	5	4	3
10	10	8	11	12	9	7	6	2	5	1	3	4
11	11	9	7	8	12	10	3	5	4	6	1	2
12	12	11	8	10	7	9	5	3	6	4	2	1

The following theorem gives a necessary and sufficient condition for $C^{(\circ)} = C^{(\cdot)}$.

Theorem 3.3. *If $(Q,\circ)$ is a middle Bol loop and $(Q,\cdot)$ is the corresponding right Bol loop, then $C^{(\circ)} = C^{(\cdot)}$ if and only if the following conditions are satisfied:*

1. $x^{-1} \cdot xa = a$, $\forall x \in Q$, $\forall a \in C^{(\cdot)}$;

2. $x^{-1} \circ (x \circ b) = b$, $\forall x \in Q$, $\forall b \in C^{(\circ)}$.

Proof. Let $a \in C^{(\cdot)}$ and let $x^{-1} \cdot xa = a$, $\forall x \in Q$. Then, using (2) and the right inverse property of $(Q,\cdot)$, we obtain:

$$a \circ x = (x \cdot ax^{-1})x = (x \cdot x^{-1}a)x = a \cdot x = x \cdot a = (xa^{-1} \cdot a)a = (a \cdot xa^{-1})a = x \circ a,$$

for every $x \in Q$, i.e., $a \in C^{(\circ)}$. So, $C^{(\cdot)} \subseteq C^{(\circ)}$.

Conversely, let $C^{(\cdot)} \subseteq C^{(\circ)}$ and let $a \in C^{(\cdot)}$. Since $a \cdot x = x \cdot a$ and $a \circ x = x \circ a$, $\forall x \in Q$, we have:

$$\begin{aligned} x \cdot x^{-1}a &= x \cdot ax^{-1} = (x \cdot ax^{-1})x \cdot x^{-1} = (a \circ x) \cdot x^{-1} = (x \circ a) \cdot x^{-1} \\ &= (a \cdot xa^{-1})a \cdot x^{-1} = (xa^{-1} \cdot a)a \cdot x^{-1} = xa \cdot x^{-1} = ax \cdot x^{-1} = a, \end{aligned}$$

for every $x \in Q$, hence $x \cdot x^{-1}a = a$, $\forall x \in Q$.

Now, let $b \in C^{(\circ)}$. Then $b \in C^{(\cdot)}$ if and only if $b \cdot x = x \cdot b$, $\forall x \in Q$, which according to (4), is equivalent to $x//b^{-1} = b//x^{-1}$, $\forall x \in Q$, i.e., to $b = (x//b^{-1}) \circ x^{-1}$, $\forall x \in Q$, where "//" is the left division in $(Q,\circ)$. Making the substitution $x//b^{-1} \to y$ in the last equality and using the anti-automorphic inverse property of $(Q,\circ)$, we get:

$$b = y \circ (y \circ b^{-1})^{-1} = y \circ (b \circ y^{-1}) = y \circ (y^{-1} \circ b),$$

for every $y \in Q$. So, the second condition is equivalent to $C^{(\circ)} \subseteq C^{(\cdot)}$. $\square$

Corollary 3.4. *If $C^{(\circ)} = C^{(\cdot)}$, then $(C^{(\circ)}, \circ) = (C^{(\cdot)}, \cdot)$ is a commutative Moufang subloop.*

Proof. If $C^{(\circ)} = C^{(\cdot)}$ then, for $\forall x, y \in C^{(\circ)} = C^{(\cdot)}$, have:

$$x \circ y = (y \cdot xy^{-1}) \cdot y = (xy^{-1} \cdot y) \cdot y = x \cdot y,$$

hence, " $\circ$ " = " $\cdot$ " on the set $C^{(\circ)} = C^{(\cdot)}$. So, as $(C^{(\circ)}, \circ)$ is a commutative middle Bol IP-loop, it is a commutative Moufang loop. □

4. When the commutant is a normal subloop?

In this section, for simplicity the operation of a middle Bol loop will be denoted by " $\cdot$ ".

Lemma 4.1. *If $(Q, \cdot)$ is a middle Bol loop and H is a subloop in $(Q, \cdot)$, then the following conditions are equivalent:*

1. *$L_{x,y}(H) = H$, $\forall x, y \in Q$;*

2. *$R_{x,y}(H) = H$, $\forall x, y \in Q$,*

where $L_{x,y} = L_{xy}^{-1} L_x L_y$ and $R_{x,y} = R_{xy}^{-1} R_y R_x$.

Proof. Let $L_{x,y}(H) = H$, $\forall x, y \in Q$. Then, for $x^{-1}, y^{-1} \in Q$ and $h^{-1} \in H$, there exists $h_1^{-1} \in H$, such that $L_{x^{-1},y^{-1}}(h^{-1}) = h_1^{-1}$. Hence $L_{x^{-1}y^{-1}}^{-1} L_{x^{-1}} L_{y^{-1}}(h^{-1}) = h_1^{-1}$, and consequently $x^{-1} \cdot y^{-1} h^{-1} = x^{-1} y^{-1} \cdot h_1^{-1}$.

Since the loop $(Q, \cdot)$ is power-associative and satisfies the anti-automorphic property, the last equtION implies $hy \cdot x = h_1 \cdot yx$, so $R_{yx}^{-1} R_x R_y(h) = h_1 \in H$, i.e., $R_{y,x}(H) \subseteq H$.

Analogously, for $h_1 \in H$ there exists $h^{-1} \in H$ such that $L_{x^{-1},y^{-1}}(h^{-1}) = h_1^{-1}$, which implies $R_{y,x}(h) = h_1$, so $H \subseteq R_{y,x}(H)$. In a similar way we can prove that the condition $R_{x,y}(H) = H$, $\forall x, y \in Q$ implies $L_{x,y}(H) = H$, $\forall x, y \in Q$. □

Theorem 4.2. *The commutant $C^{(\cdot)}$ of a middle Bol loop $(Q, \cdot)$ is a normal subloop if and only if $L_{x,y}(C^{(\cdot)}) = C^{(\cdot)}$ (or, equivalently, if and only if $R_{x,y}(C^{(\cdot)}) = C^{(\cdot)}$), for every $x \in Q$.*

Proof. The subloop $C^{(\cdot)}$ is normal if and only if $L_{x,y}(C^{(\cdot)}) = C^{(\cdot)}$, $R_{x,y}(C^{(\cdot)}) = C^{(\cdot)}$, and $T_x(C^{(\cdot)}) = C^{(\cdot)}$, where $T_x = R_x^{-1} L_x$, $\forall x, y \in Q$. If $c \in C^{(\cdot)}$ then, denoting $T_x(c)$ by b, we have $b = T_x(c) = R_x^{-1} L_x(c)$. Thus $R_x(b) = L_x(c)$, i.e., $bx = xc$, and consequently, $bx = cx$. Therefore $b = c$, so $T_x(c) = c$, $\forall c \in C^{(\cdot)}$. □

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