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On two-sided bases of an ordered semigroup

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Abstract. We introduce the concept of two-sided base of an ordered semigroup, and study the structure of an ordered semigroup containing two-sided bases.

1. Preliminaries

Given a semigroup S, a subset A of S is called a *two-sided base* of S if it satisfies the following conditions: $S = A \cup SA \cup AS \cup SAS$, and if B is a subset of A such that $S = B \cup SB \cup BS \cup SBS$ then $B = A$. This notion was introduced and studied by Fabrici [2]. Indeed, the author described the structure of semigroups containing two-sided bases. The purpose of this paper is to introduce the concept of two-sided base of an ordered semigroup, and extend the Fabrici's results to ordered semigroups.

A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ that is *compatible* with the semigroup operation, meaning that, for any x, y, z in S,

$$x \leqslant y \text{ implies } zx \leqslant zy \text{ and } xz \leqslant yz,$$

is called a *partially ordered semigroup*, or simply an *ordered semigroup* [1]. A nonempty subset T of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *subsemigroup* of S if, for any x, y in T, $xy \in T$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. For A, B nonempty subsets of S, we write AB for the set of all elements xy in S where $x \in A$ and $y \in B$, and write $(A]$ for the set of all elements $x \in S$ such that $x \leqslant a$ for some $a \in A$, i.e.,

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}.$$

In particular, we write Ax for $A\{x\}$, and xA for $\{x\}A$. It was shown in [7] that the following hold:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $(A \cup B] = (A] \cup (B]$;

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(5) $((A]] = (A]$.

The concepts of left, right and two-sided ideals of an ordered semigroup can be found in [3]. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (respectively, *right*) *ideal* of S if it satisfies the following conditions:

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) $A = (A]$, that is, for any x in A and y in S, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. If A and B are ideals of S, then the union $A \cup B$ is an ideal of S.

If A is a nonempty subset of an ordered semigroup $(S, \cdot, \leqslant)$, then the intersection of all ideals containing A of S, denoted by $I(A)$, is an ideal containing A of S, and it is of the form

$$I(A) = (A \cup SA \cup AS \cup SAS].$$

In particular, we write $I(\{a\})$ by $I(a) = (a \cup Sa \cup aS \cup SaS]$ (this is called the *principal ideal generated by* a).

A proper ideal M of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *maximal* if there is no a proper ideal A of S such that $M \subset A$. The symbol $\subset$ stands for proper inclusion for sets.

2. Ordered semigroups containing two-sided bases

We begin this section with the definition of two-sided base of an ordered semigroup; it is more general than that of a two-sided base of a semigroup (without order).

Definition 2.1. Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A subset A of S is called a *two-sided base* of S if it satisfies the following conditions:

(i) $S = I(A)$;

(ii) if B is a subset of A such that $S = I(B)$, then $B = A$.

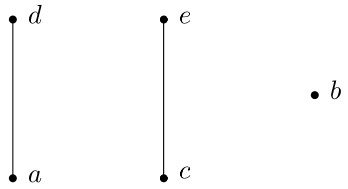
Example 2.2. ([6]) Let $(S, \cdot, \leqslant)$ be an ordered semigroup such that the multiplication and the partial order are defined by:

$\cdot$	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

$$\leqslant = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (c, e)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a,d),(c,e)\}$$



We have $\{b\}$ is the only one two-sided base of S.

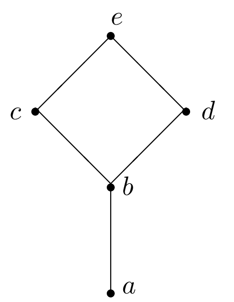
Example 2.3. ([5]) Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\begin{aligned}\leqslant= \{&(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),\\ &(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.\end{aligned}$$

The covering relation and the figure of S are given by:

$$\prec = \{(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,e),(d,e)\}$$



The two-sided bases of S are $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ and $\{e\}$.

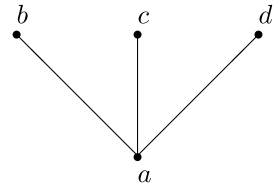
Example 2.4. ([9]) Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leqslant = \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(d,d)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a,b),(a,c),(a,d)\}$$



The two-sided base of S is $\{c,d\}$.

Beside the partial order $\leqslant$ on an ordered semigroup $(S,\cdot,\leqslant)$, we define $\preceq_I$ a quasi-order on S as follows: for any a,b in S, let

$$a \preceq_I b \text{ if and only if } I(a) \subseteq I(b).$$

The symbol $a \prec_I b$ stands for $a \preceq_I b$, but $a \neq b$. It is clear that, for any a,b in S, $a \leqslant b$ implies $a \preceq_I b$. The following example shows that the converse statement is not valid in general.

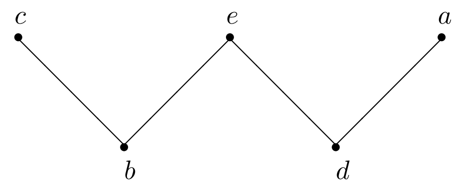
Example 2.5. ([4]) Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	b	d	a	b	e
b	d	b	b	d	e
c	d	b	c	d	e
d	b	d	d	b	e
e	e	e	e	e	e

$$\leqslant = \{(a,a),(b,b),(b,c),(b,e),(c,c),(d,a),(d,d),(d,e),(e,e)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(b,c),(b,e),(d,a),(d,e)\}$$



We have $b \preceq_I a$, but $b \leqslant a$ is false.

Theorem 2.6. *A subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is a two-sided base of S if and only if it satisfies the following conditions*:

(i) *for any x in S there exists a in A such that $x \preceq_I a$;*

(ii) *for any $a, b \in A$, if $a \neq b$, then neither $a \preceq_I b$ nor $b \preceq_I a$.*

Proof. Assume that A is a two-sided base of S. If $x \in S$, then $x \in I(A)$; hence $x \preceq_I a$ for some a in A. This shows that (i) holds. Let a, b be elements of A such $a \neq b$. Suppose $a \preceq_I b$. We set $B = A \setminus \{a\}$. Then $b \in B$. Let x be an element of S. By (i), there exists c in A such that $x \preceq_I c$. There are two cases to consider. If $c \neq a$, then $c \in B$; thus $I(x) \subseteq I(c) \subseteq I(B)$. Hence $S = I(B)$. This is a contradiction. If $c = a$, then $x \preceq_I b$; hence $x \in I(B)$ since $b \in B$. We have $S = I(B)$. This is a contradiction. The case $b \preceq_I a$ is proved similarly. Thus (ii) holds true.

Conversely, assume that the conditions (i) and (ii) hold. It follows from (i) that $S = I(A)$. Suppose that $S = I(B)$ for some a proper subset B of A. Let a be an element of $A \setminus B$. We have $a \in I(B)$. By (iii), $a \in (SB \cup BS \cup SBS]$. This implies that $a \preceq_I b$ for some b in B. This contradicts to (ii). Hence A is a two-sided base of S. □

Lemma 2.7. *Let A be a two-sided base of an ordered semigroup $(S, \cdot, \leqslant)$. For any a, b in A, if $a \in (Sb \cup bS \cup SbS]$, then $a = b$.*

Proof. Let a, b be any elements of A such that $a \in (Sb \cup bS \cup SAS]$ and $a \neq b$. We set $B = A \setminus \{a\}$. Then $b \in B$. Since

$$I(a) \subseteq (Sb \cup bS \cup SbS] \subseteq I(b) \subseteq I(B),$$

it follows that $I(A) \subseteq I(B)$, and so $S = I(B)$. This is a contradiction since A is a two-sided base of S. Hence $a = b$. □

Theorem 2.8. *Let A be a two-sided base of an ordered semigroup $(S, \cdot, \leqslant)$ such that $I(a) = I(b)$ for some a in A and b in S. If $a \neq b$, then S contains at least two two-sided bases.*

Proof. Assume that $a \neq b$. Suppose $b \in A$. Since $a \preceq_I b$, it follows by Theorem 2.6 that $a \in (Sb \cup bS \cup SbS]$; hence $a = b$ by Lemma 2.7. This is a contradiction. Thus $b \in S \setminus A$. We set $A_1 = (A \setminus \{a\}) \cup \{b\}$. If A_1 is a two-sided base of S, then we obtain the assertion since $A_1 \neq A$. This is proved using Theorem 2.6 as follows.

Let $x \in A \setminus \{a\}$. If $x \preceq_I b$, then by $I(b) = I(a)$ it follows that $I(x) \subseteq I(a)$. By Lemma 2.7, $x = a$. This is a contradiction. Thus $x \preceq_I b$ is false. Similarly, if $b \preceq_I x$, then $b \preceq_I x$ is false. Let x be an element of S. Then there exists $c \in A$ such that $x \preceq_I c$. If $c \neq a$, then $c \in A_1$. If $c = a$, then $I(c) = I(b)$; hence $x \preceq_I b$. □

The following corollary follows directly from Theorem 2.8.

Corollary 2.9. *Let A be a two-sided base of an ordered semigroup $(S, \cdot, \leqslant)$, and let $a \in A$. If $I(a) = I(x)$ for some x in S, then x is in a two-sided base which is different from A.*

Theorem 2.10. *Any two two-sided bases of an ordered semigroup $(S, \cdot, \leqslant)$ have the same cardinality.*

Proof. Let A and B be two two-sided bases of an ordered semigroup $(S, \cdot, \leqslant)$. Let $a \in A$. Since B is a two-sided base of S, we have $a \preceq_I b$ for some b in B. Similarly, since A is a two-sided base of S we have $b \preceq_I a'$ for some a' in A. By $a \preceq_I a'$, $a = a'$. This implies $I(a) = I(b)$. Define a mapping

$$\varphi : A \to B \text{ by } \varphi(a) = b \text{ for all } a \text{ in } A.$$

If $a_1, a_2 \in A$ such that $\varphi(a_1) = \varphi(a_2)$, then $I(a_1) = I(a_2)$; hence $a_1 = a_2$ by Theorem 2.6. This shows that φ is one to one. Let $b \in B$. Then there exists a in A such that $b \preceq_I a$. Similarly, there exists b' in B such that $a \preceq_I b'$. Then $b \preceq_I b'$. Since $I(b) = I(b')$, so $I(a) = I(b)$. Thus φ is onto. □

In Example 2.3, it is easy to see that $\{b\}$ is a two-sided base of S, but it is not a subsemigroup of S. This shows that a two-sided base of an ordered semigroup need not to be a subsemigroup in general. An element e of an ordered semigroup $(S, \cdot, \leqslant)$ is called an *idempotent* if $e^2 = e$. The following theorem gives necessary and sufficient conditions of a two-sided base of S to be a subsemigroup of S.

Theorem 2.11. *Let A be a two-sided base of an ordered semigroup $(S, \cdot, \leqslant)$. Then A is a subsemigroup of S if and only if $A = \{a\}$ where $a^2 = a$.*

Proof. Assume that A is a subsemigroup of S. Let a, b be elements of A. Then $ab \in A$. Since $ab \in (Sb \cup bS \cup SbS]$, it follows by Lemma 2.7 that $a = b$. By $ab \in (Sa \cup aS \cup SaS]$, we have $ab = a$. Hence $a = b$. The converse statement is obvious. □

This is a consequence of Theorem 2.11.

Corollary 2.12. *Any ordered semigroup $(S, \cdot, \leqslant)$ containing a two-sided base which is a subsemigroup contains an idempotent element.*

Theorem 2.13. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup, and let A be the union of all two-sided bases of S. If $S \setminus A$ is nonempty, then it is an ideal of S.*

Proof. Assume that $S \setminus A$ is nonempty. Let $a \in S \setminus A$, and let $x \in S$. To show that $xa \in S \setminus A$, we assume that $xa \in A$. Then $xa \in A_1$ for some a two-sided base A_1 of S. Let $xa = b$ for some b in A_1. Then $b \in Sa$; thus $I(b) \subseteq I(a)$. If $I(b) = I(a)$, then by Corollary 2.9 we have $a \in A$. This is a contradiction. Hence $b \prec_I a$. Since A_1 is a two-sided base, there exists c in A_1 such that $a \preceq_I c$. We have $b \prec_I a \preceq_I c$. This is a contradiction. Hence $xa \in S \setminus A$. Similarly, we have $ax \in S \setminus A$. Let $x \in S \setminus A$ and $y \in S$ such that $y \leqslant x$. If $y \in A$, then $y \in A_2$ for some a two-sided base A_2 of S. Let $z \in A_2$ be such that $x \preceq_I z$. Since $y \preceq_I x$, so $y \preceq_I z$. This is a contradiction. Hence $S \setminus A$ is an ideal of S. □

Theorem 2.14. *Let A be the union of all two-sided bases of an ordered semigroup $(S, \cdot, \leqslant)$ such that $\emptyset \neq A \subset S$. Let M^* be a proper ideal of S containing every proper ideal of S. The following statements are equivalent*:

(1) *$S \setminus A$ is a maximal ideal of S*;

(2) *$A \subseteq I(a)$ for every a in A*;

(3) *$S \setminus A = M^*$*;

(4) *Every two-sided bases of S is a singleton set.*

Proof. The proof is a modification of the proof of Theorem 6 in [2].

(1) $\Leftrightarrow$ (2). If there is an element a of A such that $A \subseteq I(a)$ is false, then $(S \setminus A) \cup I(a)$ is a proper two-sided ideal of S. This contradicts to the maximality of $S \setminus A$. Conversely, assume that for every element a in A, $A \subseteq I(a)$. By Theorem 2.13, $S \setminus A$ is an ideal of S. Let M be an ideal of S such that $S \setminus A \subset M \subset S$. Then $M \cap A$ is nonempty, i.e., there is an element c in $M \cap A$. We have

$$(Sc] \subseteq (SM] \subseteq M,\ (cS] \subseteq (MS] \subseteq M,\ (ScS] \subseteq (SMS] \subseteq (SM] \subseteq M.$$

Thus

$$S = (S \setminus A) \cup A \subseteq (S \setminus A) \cup I(c) \subseteq M.$$

This is a contradiction. Hence $S \setminus A$ is a maximal ideal of S.

(3) $\Leftrightarrow$ (4). Assume that $S \setminus A = M^*$. Let $a \in A$. Then $S \setminus A \subseteq I(a)$. Since $A \subseteq I(a)$, so $S = I(a)$. Hence $\{a\}$ is a two-sided base of S. Conversely, assume that every two-sided base of S is a singleton set. Then $S = I(a)$ for all a in A. Let M be an ideal of S such that M is not contained in $S \setminus A$. Then there exists x in $A \cap M$. Since

$$(Sx] \subseteq (SM] \subseteq M,\ (xS] \subseteq (MS] \subseteq M \text{ and } (SxS] \subseteq (SMS] \subseteq M,$$

we have $S = I(x) \subseteq M$, and so $S = M$.

(1) $\Leftrightarrow$ (3). Assume that $S \setminus A$ is a maximal ideal of S. Let M be an ideal of S such that M is not contained in $S \setminus A$. Then $M = A \cup X$ for some $X \subseteq S \setminus A$. This implies that $M = S$. Thus $S \setminus A = M^*$. The converse is obvious. $\square$

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