Quasigroups and Related Systems **8** (2001), 37 – 44

Transversals in groups. 3. Semidirect product of a transversal operation and subgroup

Eugene A. Kuznetsov

Abstract

The investigation of transversals in groups beginned in [6, 7] is continued in a present article. The main aim of this article is a demonstration of a natural way of a construction of a semidirect product of a left quasigroup with two-sided unit and some group by the help of transversals.

The present article is a continuation of a cycle of works about the investigations of transversals in groups, beginned in [6, 7]. As it is known, the concept of transversal is introduced for investigation of left (right) cosets in a group by its proper subgroup. The case when this subgroup is not normal, is the most interesting one.

In [6] it was proved that the operation of $\langle E, \overset{(T)}{\bullet} \rangle$, corresponding to the left transversal T in a group G to its subgroup H, is a left quasigroup with two-sided unit 1. So, by the natural way, the following problem is appears define correctly such product of the left quasigroup $\langle E, \overset{(T)}{\bullet}, 1\rangle$ with two-sided unit 1 and a subgroup H that the result of this product will be isomorphic to the initial group G.

The analogous investigations took place in [8, 9, 2] and especially in [10]. In these works we may see some formula (formula (7) in the present article) of a semidirect product mentioned above. But in these works the way of a construction of this formula is not clear and, moreover, the uniqueness of this formula as a formula of a semidirect product satisfying the conditions mentioned above was not shown.

2000 Mathematics Subject Classification: 20N15
Keywords: group, loop, transversal, permutation, semidirect product

The author of this article want to show the natural way of constructing of above mentioned product by the help of the concept of transversal in a group. The uniqueness of the formula (7) of semidirect product satisfying the conditions mentioned above immediately follows from the method of constructing.

1. Necessary definitions and notations

Definition 1. A system $\langle E, \cdot \rangle$ is called a *left* (*right*) *quasigroup*, if for arbitrary $a, b \in E$ the equation $x \cdot a = b$ (respectively: $a \cdot y = b$) has a unique solution in the set E. If $\langle E, \cdot \rangle$ in the same times is a left and right quasigroup, then it is called a *quasigroup*. A quasigroup containing an element e satisfying the identity $x \cdot e = e \cdot x = e$ is called a *loop*.

Definition 2. (cf. [1]) Let G be a group and H its subgroup. A complete system $T = \{t_i\}_{i \in E}$ of representatives of the left (right) cosets of H in G $(e = t_1 \in H)$ is called a *left* (*right*) *transversal in G to H*.

Let $T = \{t_i\}_{i \in E}$ be a left transversal in G to H. We can define correctly (see [1, 7]) the following operation on the set E (E is an index set; left cosets in G to H are numbered by indexes from E):

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \quad h \in H.$$

In [7] it is proved that $\langle E, \cdot \rangle$ is a left quasigroup with two-sided unit 1.

Below we assume (for simplicity) that $Core_G(H) = e$ and we study a permutation representation $\hat{G}$ of a group G by its left cosets of a subgroup H. According to [5], we have $\hat{G} \cong G$, where

$$\hat{g}(x) = y \quad \overset{def}{\Longleftrightarrow} \quad g t_x H = t_y H.$$

Note that $\hat{H} = St_1(\hat{G})$.

Lemma 1. ([6], Lemma 4) *Let T be an arbitrary left transversal in G to H. Then the following statements are true:*

1. $\hat{h}(1) = 1 \quad \forall h \in H$.

2. *For any $x, y \in E$* $\hat{t}_x(y) = x \overset{(T)}{\cdot} y; \quad \hat{t}_1(x) = \hat{t}_x(1) = x;$

$$\hat{t}_x^{-1}(y) = x \overset{(T)}{\backslash} y; \quad \hat{t}_x^{-1}(1) = x \overset{(T)}{\backslash} 1; \quad \hat{t}_x^{-1}(x) = 1,$$

where $\overset{(T)}{\backslash}$ *is the left division in* $< E, \overset{(T)}{\cdot}, 1 >$ (*i.e.* $x \overset{(T)}{\backslash} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$).

Since an arbitrary element $g \in G$ is contained in some left coset of H in G, then it can be a uniquely represented in the form:

$$g = t_u h, \tag{1}$$

where $t_u \in T$, $h \in H$.

Let $g_1 g_2 = g_3$ be the product of two arbitrary elements of G. According to the representation (1) we have

$$t_x h_1 t_y h_2 = t_z h_3. \tag{2}$$

Let $x \overset{(T)}{\cdot} y = x \bullet y$. In view of Lemma 1 we have

$$z = \hat{t}_z(1) = \hat{t}_z \hat{h}_3(1) = \hat{t}_x \hat{h}_1 \hat{t}_y \hat{h}_2(1) = \hat{t}_x \hat{h}_1 \hat{t}_y(1) = \hat{t}_x \hat{h}_1(y) = x \bullet \hat{h}_1(y). \tag{3}$$

Applying (2) and (3) we obtain

$$h_3 = t_z^{-1} t_x h_1 t_y h_2 = t^{-1}_{x\bullet \hat{h}_1(y)} t_x h_1 t_y h_2 = (t^{-1}_{x\bullet \hat{h}_1(y)} t_x t_{\hat{h}_1(y)})(t^{-1}_{\hat{h}_1(y)} h_1 t_y h_1^{-1}) h_1 h_2,$$

which implies

$$t_x h_1 t_y h_2 = t_{x\bullet \hat{h}_1(y)} (t^{-1}_{x\bullet \hat{h}_1(y)} t_x t_{\hat{h}_1(y)})(t^{-1}_{\hat{h}_1(y)} h_1 t_y h_1^{-1}) h_1 h_2\,. \tag{4}$$

Now let

$$\begin{aligned} l_{a,b} &\rightleftharpoons t^{-1}_{a\bullet b} t_a t_b, \\ \varphi(u,h) &\rightleftharpoons t^{-1}_{\hat{h}(u)} h t_u h^{-1} \end{aligned}$$

(for details see [10]).

Lemma 2. *The following sentences are true:*

1) $\hat{l}_{a,b} \in \hat{H}$ *for any* $a, b \in E$.

2) $\hat{\varphi}(u,h) \in \hat{H}$ *for any* $u \in E$ *and* $h \in H$.

Proof. 1) For any $a, b \in E$ we have

$$\hat{l}_{a,b}(1) = \hat{t}^{-1}_{a\bullet b} \hat{t}_a \hat{t}_b(1) = \hat{t}^{-1}_{a\bullet b} \hat{t}_a(b) = \hat{t}^{-1}_{a\bullet b}(a \bullet b) = (a \bullet b)\backslash(a \bullet b) = 1,$$

i.e. $\hat{l}_{a,b} \in St_1(\hat{G}) = \hat{H}$.

2) For any $u \in E$ and $h \in H$ we obtain

$$\begin{aligned} \hat{\varphi}(u,h)(1) &= \hat{t}^{-1}_{\hat{h}(u)} \hat{h}\, \hat{t}_u \hat{h}^{-1}(1) = \hat{t}^{-1}_{\hat{h}(u)} \hat{h}\, \hat{t}_u(1) \\ &= \hat{t}^{-1}_{\hat{h}(u)} \hat{h}\,(u) = (\hat{h}\,(u))\backslash(\hat{h}\,(u)) = 1, \end{aligned}$$

i.e. $\hat{\varphi}(u,h) \in St_1(\hat{G}) = \hat{H}$. □

Remark 1. All permutations $\hat{l}_{a,b}$ generate the group

$$LI(\langle E, \cdot, 1\rangle) \rightleftharpoons \langle \hat{l}_{a,b} | a, b \in E\rangle,$$

which is called a *left inner mapping group* of operation $\langle E, \cdot, 1\rangle$. In view of Lemma 2 we have

$$LI(\langle E, \cdot, 1\rangle) \subseteq \hat{H}. \tag{5}$$

Remark 2. In [10] it is shown that $\hat{\varphi}(u, LI(\langle E, \cdot, 1\rangle)) \subset LI(\langle E, \cdot, 1\rangle)$, for any $u \in E$, i.e. all elements of the group $LI(\langle E, \cdot, 1\rangle)$ satisfy both the conditions of Lemma 2.

2. Semidirect products

The investigations in the previous chapter lead us in the natural way to the definition of a product of the left quasigroup $\langle E, \bullet, 1\rangle$ with two-sided unit 1 and a group H (satisfying some conditions connected with the operation in $\langle E, \bullet, 1\rangle$).

Let $\langle E, \bullet, 1\rangle$ be a left quasigroup with two-sided unit 1 and let H be a permutation group on the set E $(H \subseteq St_1(S_E))$ such that

$$\begin{aligned} \forall a, b \in E \quad & l_{a,b} = L^{-1}_{a \bullet b} L_a L_b \in H, \\ \forall u \in E,\ \forall h \in H \quad & \varphi(u, h) = L^{-1}_{h(u)} h L_u h^{-1} \in H, \end{aligned} \tag{6}$$

where L_a is the left translation by a in $\langle E, \bullet, 1\rangle$. In the set

$$E \times H = \{(u, h) |\, u \in E,\ h \in H\}$$

we define the operation

$$(u, h_1) * (v, h_2) \overset{def}{=} (u \bullet h_1(v), l_{u, h_1(v)} \varphi(v, h_1) h_1 h_2) \tag{7}$$

(see [10]). In view of (6) this definition is correct.

On the set E we define the function:

$$\begin{aligned} (u, h) :\ & E \to E, \\ (u, h)(x) & \overset{def}{=} u \bullet h(x). \end{aligned} \tag{8}$$

Lemma 3. *The following sentences are true:*

1. *The function $(u, h) : E \to E$ is an action, i.e.*
   - (*a*) *it is a permutation on the set E,*

(b) *if* $(u,h_1)(x)=(v,h_2)(x)$ *for all* $x\in E$, *then* $u=v$ *and* $h_1=h_2$,

2. $(u,h_1)((v,h_2)(x))=((u,h_1)*(v,h_2))(x)$ *for any* $x\in E$, *where* $*$ *is defined by* (7),
3. $(1,id)$ *is an unit of* $\langle E\times H,*\rangle$,
4. $(\,h^{-1}(u\backslash 1),\,(L_uhL_{h^{-1}(u\backslash 1)})^{-1}\,)$ *is an inverse element of* (u,h) *in* $\langle E\times H,*,(1,id)\rangle$,
5. $G=\langle E\times H,*,(1,id)\rangle$ *is a group.*

Proof. 1a. According to (8), we have $(u,h)(x)=u\bullet h(x)=L_uh(x)$. Because L_u is a permutation of the set E, then also L_uh is a permutation of E. Obviously $(1,id)(x)=x$.

1b. If $(u,h_1)(x)=(v,h_2)(x)$ for all $x\in E$, then $L_uh_1(x)=L_vh_2(x)$. This for $x=1$ gives $L_u=L_v$. Thus $u=v$. Hence $h_1(x)=h_2(x)$ for all $x\in E$, and, in the consequence, $h_1=h_2$.

2. For $\alpha=(u,h_1)$ and $\beta=(v,h_2)$ we have

$$\begin{aligned}\alpha(\beta(x))&=(u,h_1)((v,h_2)(x))=(u,h_1)(v\bullet h_2(x))\\&=u\bullet h_1(v\bullet h_2(x))=L_uh_1L_vh_2(x)\,.\end{aligned}$$

But on the other hand

$$\begin{aligned}((u,h_1)*(v,h_2))(x)&=(u\bullet h_1(v),l_{u,h_1(v)}\varphi(v,h_1)h_1h_2)(x)\\&=(u\bullet h_1(v))\bullet L^{-1}_{u\bullet h_1(v)}L_uL_{h_1(v)}L^{-1}_{h_1(v)}h_1L_vh_1^{-1}h_1h_2(x)\\&=L_{u\bullet h_1(v)}L^{-1}_{u\bullet h_1(v)}L_uh_1L_vh_2(x)=L_uh_1L_vh_2(x)\,.\end{aligned}$$

So $(u,h_1)((v,h_2)(x))=((u,h_1)*(v,h_2))(x)$.

3. According to the previous paragraph we have

$$((u,h_1)*(v,h_2))(x)=L_uh_1L_vh_2(x),\tag{9}$$

which gives

$$((1,id)*(u,h))(x)=L_1id\,L_uh(x)=L_uh(x)=(u,h)(x)$$

for any $x\in E$. Thus $(1,id)*(u,h)=(u,h)$.

We obtain also

$$((u,h)*(1,id))(x)=L_uhL_1id\,(x)=L_uh(x)=(u,h)(x)\,.$$

Hence $(u,h)*(1,id)=(u,h)$, which proves that $(1,id)$ is a two-sided unit.

4. According to the general properties of $\{L_u\}_{u\in E}$ in $LM(\langle E,\bullet,1\rangle)$ to $LI(\langle E,\bullet,1\rangle)\subseteq H$ we have $L_u^{-1}=L_{u\backslash 1}h'$ for some $h'\in H$. So

$$(h^{-1}(u\backslash 1),(L_uhL_{h^{-1}(u\backslash 1)})^{-1})=(h^{-1}(u\backslash 1),L^{-1}_{h^{-1}(u\backslash 1)}h^{-1}L_u^{-1})$$
$$=(h^{-1}(u\backslash 1),L^{-1}_{h^{-1}(u\backslash 1)}h^{-1}L_{u\backslash 1}hh'')=(h^{-1}(u\backslash 1),\varphi(u\backslash 1,h^{-1})h'')\in E\times H.$$

But by (9) we have also

$$((u,h)*(h^{-1}(u\backslash 1),(L_uhL_{h^{-1}(u\backslash 1)})^{-1})(x)$$
$$=L_uhL_{h^{-1}(u\backslash 1)}L^{-1}_{h^{-1}(u\backslash 1)}h^{-1}L_u^{-1}(x)=x,$$

which proves $(u,h)*(h^{-1}(u\backslash 1),(L_uhL_{h^{-1}(u\backslash 1)})^{-1}=(1,id)$.

In the same way

$$((h^{-1}(u\backslash 1),(L_uhL_{h^{-1}(u\backslash 1)})^{-1}*(u,h))(x)$$
$$=L_{h^{-1}(u\backslash 1)}L^{-1}_{h^{-1}(u\backslash 1)}h^{-1}L_u^{-1}L_uh(x)=x$$

implies $(h^{-1}(u\backslash 1),(L_uhL_{h^{-1}(u\backslash 1)})^{-1}*(u,h)=(1,id)$.

5. It is a simple consequence of 2, 3 and 4. □

Lemma 4. *Let $G=\langle E\times H,*,(1,id)\rangle$ be a group. Then*

1. *$\hat H=\langle H^*,*,(1,id)\rangle$ (where $H^*\rightleftharpoons\{(1,h)|\,h\in H\}$) is a subgroup of G and it is isomorphic to the group H.*
2. *$\hat T\rightleftharpoons\{(u,id)|\,u\in E\}$ is a left transversal in G to its subgroup $\hat H$ and the operation of $\langle E,\overset{(\hat T)}{\bullet},1\rangle$ coincides with the operation of $\langle E,\bullet,1\rangle$.*

Proof. 1. According to (9) we have

$$((1,h_1)*(1,h_2))(x)=h_1h_2(x)=(1,h_1h_2)(x),$$

which proves that $\hat H=\langle H^*,*,(1,id)\rangle$ is a subgroup of G. Moreover, the bijection $\psi: H^*\to H,\quad \psi((1,h))=h$ defines an isomorphism between groups $\hat H$ and H.

2. In view of (9) we have

$$((u,id)*(1,h))(x)=L_u id\,L_1h(x)=L_uh(x)=(u,h)(x),$$

which gives $(u,id)*(1,h)=(u,h)$. Then for any $u\in E$ the set

$$H_u\rightleftharpoons(u,id)*H^*=\{(u,h)|\,h\in H\}$$

is a left coset of $\hat{H}$ in G. Obviously $(u,id)\in H_u$ and $(1,id)\in H_1=\hat{H}$.

So, $\hat{T}\rightleftharpoons\{(u,id)|\,u\in E\}$ is a left transversal in G to its subgroup $\hat{H}$. Moreover, for $\langle E,\overset{(\hat{T})}{\bullet},1\rangle$ we have $u\overset{(\hat{T})}{\bullet}v=z$, $(u,id)*(v,id)=(z,h)$, $z=u\bullet v$ and $h=l_{u,v}$, which implies $u\overset{(\hat{T})}{\bullet}v=u\bullet v$. □

3. The case of a left A_l-loop

Note that if in the previous part of this work the permutation $h\in H$ is an automorphism of $\langle E,\bullet,1\rangle$, then any $u,x\in E$ we have

$$hL_uh^{-1}(x)=h(u\bullet h^{-1}(x))=h(u)\bullet x=L_{h(u)}(x).$$

Thus $hL_uh^{-1}=L_{h(u)}$ and $\varphi(u,h)=L^{-1}_{h(u)}hL_uh^{-1}=L^{-1}_{h(u)}L_{h(u)}=id.$

This means that the study of the general construction of a semidirect product from the previous chapter can be interesting in the case when

$$LI(\langle E,\bullet,1\rangle)\subseteq H\subseteq Aut\,(\langle E,\bullet,1\rangle).$$

In this case the left loop $\langle E,\bullet,1\rangle$ is a left special loop (left A_l-loop) and (7) can be written in the form

$$(u,h_1)*(v,h_2)=(u\bullet h_1(v),l_{u,h_1(v)}h_1h_2). \tag{10}$$

Obviously such defined product has all properties mentioned in Lemmas 3 and 4.

Remark 3. By Lemma 3, for any $u\in E$ and $h\in H$ we have

$$(u,h)^{-1}=(h^{-1}(u\backslash 1),(L_uL_{u\backslash 1}h)^{-1}),$$

and, in the consequence, $(u,id)^{-1}=(u\backslash 1,(L_uL_{u\backslash 1})^{-1})$.

Remark 4. Formula (10) coincides with the formula of a gyrosemidirect product of a left gyrogroup and its gyroautomorphism group (see [11, 4]).

References

[1] **R. Baer**: *Nets and groups. 1.* Trans. Amer. Math. Soc. **46** (1939), 110 – 141.

[2] **I. Burdujan**: *Some remarks about geometry of quasigroups*, (Russian), Mat. Issled. **39** (1976), 40 – 53.

[3] **V. D. Belousov**: *Foundations of quasigroup and loop theory*, (Russian), Nauka, Moscow 1967.

[4] **T. Foguel and A. Ungar**: *Transversals, loops and gyrogroups*, Preprint of NDSY, 1998.

[5] **M. I. Kargapolov and Yu. I. Merzlyakov**: *Foundations of group theory*, (Russian), 3rd edition, Nauka, Moscow 1982.

[6] **E. A. Kuznetsov**: *Transversals in groups. 1. Elementary properties*, Quasigroups and Related Systems **1** (1994), 22 – 42.

[7] **E. A. Kuznetsov**: *Transversals in groups. 2. Loop transversals in a group by the same subgroup*, Quasigroups and Related Systems **6** (1999), 1 – 12.

[8] **L. V. Sabinin**: *About equivalence of the categories of loops and homogeneous spaces*, (Russian), Dokl. AN SSSR, **205** (1972), 533 – 537.

[9] **L. V. Sabinin**: *About geometry of loops*, (Russian), Mat. zametki, **12** (1972), 605 – 616.

[10] **L. V. Sabinin and O. I. Mikheev**: *Quasigroups and differential geometry*, Chapter XII in the book *Quasigroups and loops: Theory and Applications*, Helderman-Verlag, Berlin 1990, 357 – 430.

[11] **A. Ungar**: *Thomas Precession: Its underlying gyrogroup axioms and their use in hyperbolic geometry and relativistic physics*, Foundations of Physics, **27** (1997), 881 – 951.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
str. Academiei 5
MD-2028 Chishinau
MOLDOVA
e-mail: ecuz@math.md

Received August 20, 2001