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Fuzzy subquasigroups over a t-norm

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Abstract

In this paper, using a t-norm T, we introduce the notion of idempotent T-fuzzy subquasigroups of quasigroups, and investigate some of their properties. Also we describe fuzzy subquasigroups induced by t-norms in the direct product of quasigroups.

1. Introduction

Following the introduction of fuzzy sets by Zadeh [13], the fuzzy set theory developed by Zadeh himself and others have found many applications in the domain of mathematics and elsewhere. For example, in [7] Liu studied fuzzy subrings as well as fuzzy ideals in rings. Properties of some fuzzy ideals in semirings are investigated in [8]. Connections between fuzzy groups and so-called level subgroups are found in [3], [4] and [10]. The similar results for quasigroups are proved in [6].

In this paper, using a t-norm T, we introduce the notion of idempotent T-fuzzy subquasigroups of quasigroups, and investigate some of their properties. Next we use a t-norm to construct T-fuzzy subquasigroups in the finite direct product of quasigroups.

2. Preliminaries

As it is well known, a groupoid $(G,\cdot)$ is called a *quasigroup* if for any $a,b\in G$ each of the equations $ax=b,\ xa=b$ has a unique solution

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in G. A quasigroup may be also defined as an algebra $(G,\cdot,\backslash,/)$ with three binary operations $\cdot,\backslash,/$ satisfying the identities

$$(xy)/y=x,\quad x\backslash(xy)=y,\quad (x/y)y=x,\quad x(x\backslash y)=y$$

(cf. [2] or [9]). We say that such defined quasigroup $(G,\cdot,\backslash,/)$ is an *equasigroup* (i.e. *equationally definable quasigroup*) [9] or a *primitive quasigroup* [2]. Obviously, these two definitions are equivalent because

$$x\backslash y=z\Longleftrightarrow xz=y,\quad x/y=z\Longleftrightarrow zy=x.$$

A nonempty subset S of a quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ is called a *subquasigroup* if it is closed with respect to these three operations, i.e., if $x*y\in S$ for all $x,y\in S$ and $*\in\{\cdot,\backslash,/\}$.

The class of all equasigroups forms a variety. This means that a homomorphic image of an equasigroup is an equasigroup. Also every subset of an equasigroup closed with respect to these three operations is an equasigroup.

Note that in case when a quasigroup is defined as a set with only one operation, a homomorphic image is not in general a quasigroup. It is *only* a groupoid with division. Similarly a homomorphic preimage of a quasigroup $(G,\cdot)$ is not a quasigroup. Also a subset closed with respect to this multiplication is not a quasigroup (cf. [2]).

For the general development ot the theory of quasigroups the *unipotent quasigroups*, i.e., quasigroups with the identity $xx=yy$, play an important role. These quasigroups are connected with Latin squares which have one fixed element in the diagonal (cf. [5]). Such quasigroups may be defined as quasigroups G with the special element θ satisfying the identity $xx=\theta$. Obviously, θ is uniquely determined and it is an idempotent, but, in general, it is not the (left, right) neutral element.

To avoid repetitions we use the following conventions: *"a quasigroup $\mathcal{G}$"* always denotes an equasigroup $(G,\cdot,\backslash,/)$; G always denotes a nonempty set.

A function $\mu:G\to[0,1]$ is called *a fuzzy set* in a quasigroup $\mathcal{G}$. The set $\mu_\alpha=\{x\in G:\mu(x)\geqslant\alpha\}$, where $\alpha\in[0,1]$ is fixed, is called *a level subset of* μ. $Im(\mu)$ denotes the imege set of μ.

Let μ and ρ be two fuzzy sets defined on G. According to [13] we say that μ is contained in ρ, and denote this fact by $\mu\subseteq\rho$, iff

$\mu(x) \leqslant \rho(x)$ for all $x \in G$. Obviously $\mu = \rho$ iff $\mu(x) = \rho(x)$ for all $x \in G$.

According to [6], a fuzzy set μ in a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\min\{\mu(xy),\, \mu(x \setminus y),\, \mu(x/y)\} \geqslant \min\{\mu(x),\, \mu(y)\}$$

for all $x, y \in G$. It is clear, that this condition may be written as

$$\mu(x * y) \geqslant \min\{\mu(x),\, \mu(y)\}$$

for all $* \in \{\cdot, \backslash, /\}$ and $x, y \in G$.

A fuzzy subquasigroup μ of a quasigroup $\mathcal{G}$ is called *normal* if $\mu(xy) = \mu(yx)$ for all $x, y \in G$. It is not difficult to see that μ is normal iff $\mu(x \setminus y) = \mu(y/x)$ for all $x, y \in G$.

The following two results are proved in [6].

Proposition 2.1. *A fuzzy set μ of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is a fuzzy subquasigroup iff for every $\alpha \in [0,1]$, μ_α is either empty or a subquasigroup of G.* □

Proposition 2.2. *If μ is a fuzzy subquasigroup of a unipotent quasigroup $(G, \cdot, \backslash, /, \theta)$, then $\mu(\theta) \geqslant \mu(x)$ for any $x \in G$.* □

3. T-fuzzy subquasigroup

According to [1], by a *t-norm*, we mean a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(T_1) $T(\alpha, 1) = \alpha$,
(T_2) $T(\alpha, \beta) \leqslant T(\alpha, \gamma)$ whenever $\beta \leqslant \gamma$,
(T_3) $T(\alpha, \beta) = T(\beta, \alpha)$,
(T_4) $T(\alpha, T(\beta, \gamma)) = T(T(\alpha, \beta), \gamma)$

for all $\alpha, \beta, \gamma \in [0,1]$.

A simple example of a t-norm is a function $T(\alpha, \beta) = \min\{\alpha, \beta\}$. Generally, $T(\alpha, \beta) \leqslant \min\{\alpha, \beta\}$ and $T(\alpha, 0) = 0$ for all $\alpha, \beta \in [0,1]$.

Moreover, $([0,1];T)$ is a commutative semigroup with 0 as the neutral element. In particular it is *medial*, i.e.,

$$T(T(\alpha,\beta),\,T(\gamma,\delta)) = T(T(\alpha,\gamma),\,T(\beta,\delta))$$

holds for all $\alpha,\beta,\gamma,\delta\in[0,1]$.

Let T_1 and T_2 be two t-norms. We say that T_1 dominates T_2 and write $T_1 \gg T_2$ if

$$T_1(T_2(\alpha,\beta),\,T_2(\gamma,\delta)) \geqslant T_2(T_1(\alpha,\gamma),\,T_1(\beta,\delta))$$

for all $\alpha,\beta,\gamma,\delta\in[0,1]$ (cf. [1]). Obviously $T\gg T$ for all t-norms.

The set of all idempotents with respect to T, i.e. the set

$$E_T = \{\alpha\in[0,1] \;\mid\; T(\alpha,\alpha)=\alpha\}$$

is a subsemigroup of $([0,1],T)$. If $Im(\mu)\subseteq E_T$ then a fuzzy set μ is called an *idempotent with respect to a t-norm T* (briefly: *T-idempotent*).

Definition 3.1. A fuzzy set μ in G is called a *fuzzy subquasigroup of $\mathcal{G}$ with respect to a t-norm T* (briefly, a *T-fuzzy subquasigroup*) if

$$\mu(x*y)\geqslant T(\mu(x),\,\mu(y))$$

for all $x,y,z\in G$ and $*\in\{\cdot,\backslash,/\}$.

Since $\min\{\alpha,\,\beta\}\geqslant T(\alpha,\,\beta)$ for all $\alpha,\beta\in[0,1]$, every fuzzy subquasigroup is also a T-fuzzy subquasigroup, but not conversely as seen in the following example.

Example 3.2. Let $G=\{0,a,b,c\}$ be the Klein's group with the following Cayley table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a fuzzy set μ in G by $\mu(0) = 0,8$, $\mu(a) = 0,7$, $\mu(b) = 0,6$, $\mu(c) = 0,5$. It is not difficult to see that a function T_m defined by $T_m(\alpha, \beta) = \max\{\alpha + \beta - 1,\ 0\}$ for all $\alpha, \beta \in [0, 1]$ is a t-norm.

By routine calculations, we known that $\mu(x * y) \geqslant T_m(\mu(x),\ \mu(y))$ for all $x, y \in G$, which shows that μ is a T_m-fuzzy subquasigroup of $\mathcal{G}$, which is not T_m-idempotent. It is not a fuzzy subquasigroup since $\mu(c) = \mu(ab) < \min\{\mu(a),\ \mu(b)\}$.

But a fuzzy set ν defined by $\nu(0) = \nu(a) = 1$ and $\nu(b) = \nu(c) = 0$ is a T_m-idempotent fuzzy subquasigroup of G. It is also a fuzzy subquasigroup. □

Proposition 3.3. *If a fuzzy set μ is idempotent with respect to a t-norm T, then $T(\alpha, \beta) = \min\{\alpha, \beta\}$ for all $\alpha, \beta \in Im(\mu)$.*

Proof. Indeed, if α and β are in $Im(\mu)$, then

$$\min\{\alpha, \beta\} \geqslant T(\alpha, \beta) \geqslant T(\min\{\alpha, \beta\},\ \min\{\alpha, \beta\}) = \min\{\alpha, \beta\},$$

which completes the proof. □

Corollary 3.4. *Every T-idempotent fuzzy subquasigroup is also a fuzzy subquasigroup.* □

By application of Proposition 2.1 we obtain

Corollary 3.5. *Every nonempty level set of a T-idempotent fuzzy subquasigroup defined on a quasigroup $\mathcal{G}$ is a subquasigroup of $\mathcal{G}$.* □

Corollary 3.6. *Let T be an idempotent t-norm. Then a fuzzy set defined on a quasigroup $\mathcal{G}$ is a T-fuzzy subquasigroup iff it is a fuzzy subquasigroup.* □

Now we consider the converse of Corollary 3.4.

Theorem 3.7. *Let a fuzzy set μ on a quasigroup $\mathcal{G}$ be idempotent with respect to a t-norm T. If each nonempty level set μ_α is a subquasigroup of $\mathcal{G}$, then μ is a T-idempotent fuzzy subquasigroup.*

Proof. Assume that each nonempty level set μ_α is a subquasigroup of $\mathcal{G}$. Then μ is a fuzzy subquasigroup of $\mathcal{G}$ (by Proposition 2.1), and so

$$\mu(x * y) \geqslant \min\{\mu(x),\ \mu(y)\} = T(\,\mu(x), \mu(y)\,)$$

by Proposition 3.3. Hence μ is a T-idempotent fuzzy subquasigroup of a quasigroup $\mathcal{G}$. □

Theorem 3.8. *Let μ be a T-fuzzy subquasigroup of $\mathcal{G}$, where T is a t-norm and $\alpha \in [0,1]$. Then*

(i) *if $\alpha = 1$, then μ_α is either empty or is a subquasigroup of $\mathcal{G}$,*

(ii) *if $T = \min$, then μ_α is either empty or is a subquasigroup of $\mathcal{G}$.*

Proof. (i) Assume that $\alpha = 1$ and $\mu_\alpha \neq \emptyset$. Then there exist $x, y \in \mu_\alpha$ such that $\mu(x) \geqslant 1$ and $\mu(y) \geqslant 1$. Thus

$$\mu(x * y) \geqslant T(\,\mu(x), \mu(y)\,) \geqslant T(1,1) = 1$$

so that $x * y \in \mu_1$. Hence μ_1 is a subquasigroup of $\mathcal{G}$.

(ii) is a consequence of Proposition 2.1. □

Note that a fuzzy set μ defined in our Example 3.2 is a non-idempotent T_m-fuzzy subquasigroup in which μ_1 is empty and $\mu_{0{,}6}$ is not a subquasigroup of $\mathcal{G}$. This proves that the analog of Proposition 2.1 for T-fuzzy subquasigroups is not true.

4. Fuzzy sets induced by norms

Let T be a t-norm and let μ and ν be two fuzzy sets in G. Then the *T-product of μ and ν*, denoted by $[\mu \cdot \nu]_T$, is defined as

$$[\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))$$

for all $x \in G$.

Obviously $[\mu \cdot \nu]_T$ is a fuzzy set in G such that $[\mu \cdot \nu]_T = [\nu \cdot \mu]_T$. Moreover, if μ and ν are normal, then so is $[\mu \cdot \nu]_{T^*}$.

Theorem 4.1. *Let T be a t-norm and let μ and ν be T-fuzzy subquasigroups of $\mathcal{G}$. If a t-norm T^* dominates T, then T^*-product $[\mu \cdot \nu]_{T^*}$ is a T-fuzzy subquasigroup of $\mathcal{G}$.*

Proof. Indeed, for $x, y \in G$ we have

$$\begin{aligned}[\mu \cdot \nu]_{T^*}(x * y) &= T^*(\mu(x * y), \nu(x * y)) \\ &\geqslant T^*(T(\mu(x), \mu(y)),\, T(\nu(x), \nu(y)))\end{aligned}$$

$$\begin{aligned} &\geqslant T(T^*(\mu(x),\nu(x)),\, T^*(\mu(y),\nu(y))) \\ &= T([\mu\cdot\nu]_{T^*}(x),\, [\mu\cdot\nu]_{T^*}(y))\,, \end{aligned}$$

which proves that $[\mu\cdot\nu]_{T^*}$ is a T-fuzzy subquasigroup of $\mathcal{G}$. □

Corollary 4.2 *The T-product of T-fuzzy subquasigroups is a T-fuzzy subquasigroup.* □

Let G and H be nonempty sets and let $f: G \to H$ be an arbitrary mapping. If ν is a fuzzy set in $f(G)$ then $\mu = \nu \circ f$ is the fuzzy set in G, which is called the *preimage of ν under f*.

It is not difficult to see that the following lemma is true.

Lemma 4.3. *Let T be a t-norm and let $\mathcal{G}$ and $\mathcal{H}$ be two quasigroups. If $h: \mathcal{G} \to \mathcal{H}$ is an onto homomorphisms of quasigroups, ν is a fuzzy subquasigroup of $\mathcal{H}$ and μ the preimage of ν under h, then μ is a fuzzy subquasigroup of $\mathcal{G}$. Moreover, μ is normal iff ν is normal. If ν is T-idempotent, then so is μ.* □

Proposition 4.4. *Let T and T^* be t-norms in which T^* dominates T and let $\mathcal{G}$, $\mathcal{H}$ be two quasigroups. If $h: \mathcal{G} \to \mathcal{H}$ be an onto homomorphism of quasigroups, then for any T-fuzzy subquasigroups μ and ν of $\mathcal{H}$, we have*

$$h^{-1}([\mu\cdot\nu]_{T^*}) = [h^{-1}(\mu)\cdot h^{-1}(\nu)]_{T^*}.$$

Proof. By Lemma 4.3 $h^{-1}(\mu)$, $h^{-1}(\nu)$ and $h^{-1}([\mu\cdot\nu]_{T^*})$ are T-fuzzy subquasigroups of $\mathcal{G}$.

Moreover for $x \in G$ we have

$$\begin{aligned} {[h^{-1}([\mu\cdot\nu]_{T^*})](x)} &= [\mu\cdot\nu]_{T^*}(h(x)) = T^*(\mu(h(x)),\nu(h(x))) \\ &= T^*([h^{-1}(\mu)](x),\, [h^{-1}(\nu)](x)) = [h^{-1}(\mu)\cdot h^{-1}(\nu)]_{T^*}(x), \end{aligned}$$

which completes the proof. □

We say that a fuzzy set μ in G has a *sup property* if, for all subset $S \subseteq G$, there exists $s_0 \in S$ such that $\mu(s_0) = \sup\limits_{s\in S}\mu(s)$. In this case for any mapping f defined on G we can define in $f(G)$ the fuzzy set μ^f putting $\mu^f(y) = \sup\limits_{x\in f^{-1}(y)}\mu(x)$ for all $y \in f(G)$ (cf. [12]).

Let $f:\mathcal{G}\to\mathcal{H}$ be a homomorphisms of quasigroups and let T be a continuous t-norm (continuous with respect to the usual topology). Then sets $A_1=f^{-1}(y_1)$ and $A_2=f^{-1}(y_2)$, where $y_1,y_2\in f(G)$ are nonempty subsets of $f(G)$. Similarly, $A_3=f^{-1}(y_1*y_2)$, where $*\in\{\cdot,\backslash,/\}$ is a fixed operation.

Consider the set

$$A_1*A_2=\{a_1*a_2,\ \mid a_1\in A_1,\ a_2\in A_2\}.$$

If $x\in A_1*A_2$, then $x=x_1*x_2$ for some $x_1\in A_1$ and $x_2\in A_2$, and so

$$f(x)=f(x_1*x_2)=f(x_1)*f(x_2)=y_1*y_2\,,$$

which implies $x\in f^{-1}(y_1*y_2)=A_3$. Thus $A_1*A_2\subseteq A_3$ for any operation $*\in\{\cdot,\backslash,/\}$.

Therefore

$$\begin{aligned}\mu^f(y_1*y_2)&=\sup_{x\in f^{-1}(y_1*y_2)}\mu(x)\ =\ \sup_{x\in A_3}\mu(x)\\&\geqslant\sup_{x\in A_1*A_2}\mu(x)\ \geqslant\sup_{x_1\in A_1,\,x_2\in A_2}\mu(x_1*x_2)\\&\geqslant\sup_{x_1\in A_1,\,x_2\in A_2}T(\mu(x_1),\,\mu(x_2))\,.\end{aligned}$$

Since t-norm T is (by the assumption) continuous, for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\sup_{x_1\in A_1}\mu(x_1)-t_1\leqslant\delta\quad\text{and}\quad\sup_{x_2\in A_2}\mu(x_2)-t_2\leqslant\delta$$

implies

$$T\left(\sup_{x_1\in A_1}\mu(x_1),\,\sup_{x_2\in A_2}\mu(x_2)\right)-T(t_1,\,t_2)\leqslant\varepsilon\,.$$

This for $t_1=\mu(a_1)$, $t_2=\mu(a_2)$, where $a_1\in A_1$, $a_2\in A_2$, gives

$$T\left(\sup_{x_1\in A_1}\mu(x_1),\,\sup_{x_2\in A_2}\mu(x_2)\right)\leqslant T(\mu(a_1),\,\mu(a_2))+\varepsilon\,.$$

Consequently

$$\begin{aligned}\mu^f(y_1*y_2)&\geqslant\sup_{x_1\in A_1,\,x_2\in A_2}T(\mu(x_1),\,\mu(x_2))\\&\geqslant T\left(\sup_{x_1\in A_1}\mu(x_1),\,\sup_{x_2\in A_2}\mu(x_2)\right)=T(\mu^f(y_1),\,\mu^f(y_2))\,,\end{aligned}$$

which shows that μ^f is a T-fuzzy subquasigroup of $f(\mathcal{G})$.

Thus we have the following

Theorem 4.5. *Let T be a continuous t-norm and let f be a homomorphism on a quasigroup $\mathcal{G}$. If a T-fuzzy subquasigroup μ of $\mathcal{G}$ has the sup property, then μ^f is a T-fuzzy subquasigroup of $f(\mathcal{G})$.* □

Since the function "min" is a continuous t-norm, then, as a simple consequence of the above theorem, we obtain

Corollary 4.6. *If a fuzzy subquasigroup μ of $\mathcal{G}$ has the sup property, then μ^f is a fuzzy subquasigroup of $f(\mathcal{G})$ for every homomorphism f defined on $\mathcal{G}$.* □

5. Direct products of fuzzy subquasigroups

Let T be a fixed t-norm. If μ_1 and μ_2 are two fuzzy sets on G_1 and G_2 (respectively), then μ defined on $G_1 \times G_2$ by the formula

$$\mu(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)),$$

is a fuzzy set on $G_1 \times G_2$, which is denoted by $\mu_1 \times \mu_2$.

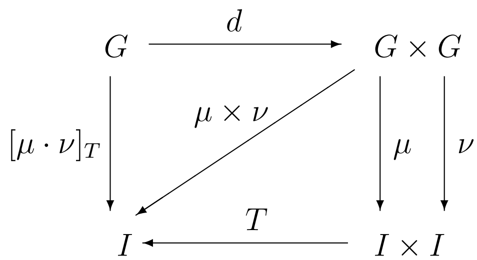
Proposition 5.1. *If μ_1 and μ_2 are T-fuzzy subquasigroup of quasigroups $\mathcal{G}_1$ and $\mathcal{G}_2$ (respectively), then $\mu_1 \times \mu_2$ is a T-fuzzy subquasigroup of the direct product $\mathcal{G}_1 \times \mathcal{G}_2$. Moreover, if μ_1 and μ_2 are T-idempotent, then so is $\mu_1 \times \mu_2$.*

Proof. Let (x_1, x_2), (y_1, y_2) be in $G_1 \times G_2$. Then

$$\begin{aligned}(\mu_1 \times \mu_2)((x_1, x_2) * (y_1, y_2)) &= (\mu_1 \times \mu_2)(x_1 * y_1,\, x_2 * y_2)\\ &= T(\mu_1(x_1 * y_1),\, \mu_2(x_2 * y_2))\\ &\geq T(T(\mu_1(x_1),\, \mu_1(y_1)),\, T(\mu_2(x_2),\, \mu_2(y_2)))\\ &= T(T(\mu_1(x_1),\, \mu_2(x_2)),\, T(\mu_1(y_1),\, \mu_2(y_2)))\\ &= T((\mu_1 \times \mu_2)(x_1, x_2),\, (\mu_1 \times \mu_2)(y_1, y_2)).\end{aligned}$$

Hence $\mu_1 \times \mu_2$ is a T-fuzzy subquasigroup of $\mathcal{G}_1 \times \mathcal{G}_2$. Obviously, if μ_1 and μ_2 are T-idempotent, then so is $\mu_1 \times \mu_2$. □

The relationship between T-fuzzy subquasigroups $\mu\times\nu$ and $[\mu\cdot\nu]$ can be viewed via the following diagram



where $I=[0,1]$ and $d: G\to G\times G$ is defined by $d(x)=(x,x)$.

Applying Lemma 3.2 from [1] it is not difficult to see that $[\mu\cdot\nu]_T$ *is the preimage of* $\mu\times\nu$ *under* d.

Note by the way, that our T-product is different from the product of fuzzy sets studied by Liu [7] and Sessa [11].

Now we generalize this idea to the product of $n\geqslant 2$ T-fuzzy subquasigroups. We first need to generalize the domain of t-norm T to $\prod\limits_{i=1}^{n}[0,1]$ as follows:

Definition 5.2. The function $T_n:\prod\limits_{i=1}^{n}[0,1]\to[0,1]$ is defined by

$$T_n(\alpha_1,\alpha_2,\ldots,\alpha_n)=T(\alpha_i,T_{n-1}(\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_n))$$

for all $1\leqslant i\leqslant n$, where $n\geqslant 2$, $T_2=T$ and $T_1=id$ (identity).

Using the induction on n, we have the following two lemmas.

Lemma 5.3. *For every t-norm* T *and every* $\alpha_i,\beta_i\in[0,1]$, *where* $1\leqslant i\leqslant n$ *and* $n\geqslant 2$, *we have*

$$T_n(T(\alpha_1,\beta_1),T(\alpha_2,\beta_2),\ldots,T(\alpha_n,\beta_n))$$
$$=T(T_n(\alpha_1,\alpha_2,\ldots,\alpha_n),T_n(\beta_1,\beta_2,\ldots,\beta_n)).\ \square$$

Lemma 5.4. *For a t-norm* T *and every* $\alpha_1,\ldots,\alpha_n\in[0,1]$, *where* $n\geqslant 2$, *we have*

$$T_n(\alpha_1,\ldots,\alpha_n)=T(\ldots T(T(T(\alpha_1,\alpha_2),\alpha_3),\alpha_4),\ldots,\alpha_n)$$
$$=T(\alpha_1,T(\alpha_2,T(\alpha_3,\ldots T(\alpha_{n-1},\alpha_n)\ldots))).\ \square$$

Theorem 5.5. *Let T be a t-norm and let $\mathcal{G} = \prod_{i=1}^{n} \mathcal{G}_i$ be the direct product of quasigroups $\{\mathcal{G}_i\}_{i=1}^{n}$. If μ_i is a T-fuzzy subquasigroup of $\mathcal{G}_i$, where $1 \leqslant i \leqslant n$, then $\mu = \prod_{i=1}^{n} \mu_i$ defined by*

$$\mu(x) = (\prod_{i=1}^{n} \mu_i)(x_1, x_2, \ldots, x_n) = T_n(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n))$$

for all $x = (x_1, x_2, \ldots, x_n) \in G$, is a T-fuzzy subquasigroup of $\mathcal{G}$. Moreover, if all μ_i are T-idempotent, then so is μ.

Proof. Now let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be any elements of $G = \prod_{i=1}^{n} G_i$. Then by Lemma 5.3 we have

$$\begin{aligned}
\mu(x * y) &= (\prod_{i=1}^{n} \mu_i)((x_1, x_2, \ldots, x_n) * (y_1, y_2, \ldots, y_n)) \\
&= (\prod_{i=1}^{n} \mu_i)((x_1 * y_1,\ x_2 * y_2, \ldots, x_n * y_n)) \\
&= T_n(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2), \ldots, \mu_n(x_n * y_n)) \\
&\geqslant T_n(T(\mu_1(x_1),\ \mu_1(y_1)),\ T(\mu_2(x_2),\ \mu_2(y_2)), \ldots, T(\mu_n(x_n),\ \mu_n(y_n))) \\
&= T(T_n(\mu_1(x_1),\ \mu_2(x_2), \ldots, \mu_n(x_n)),\ T_n(\mu_1(y_1),\ \mu_2(y_2), \ldots, \mu_n(y_n))) \\
&= T((\prod_{i=1}^{n} \mu_i)(x_1, x_2, \ldots, x_n),\ (\prod_{i=1}^{n} \mu_i)(y_1, y_2, \ldots, y_n)) \\
&= T(\mu(x),\ \mu(y))\,.
\end{aligned}$$

Therefore $\mu = \prod_{i=1}^{n} \mu_i$ is a T-fuzzy subquasigroup of $\mathcal{G}$.

Applying Lemma 5.3 it is not difficult to see that μ is T-idempotent if all μ_i are T-idempotent. □

References

[1] **M. T. Abu Osman**: *On some product of fuzzy subgroups*, Fuzzy Sets and Systems **24** (1987), 79 – 86.

[2] **V. D. Belousov**: *Foundations of the theory of quasigroups and loops*, Nauka, Moscow 1967.

[3] **P. Bhattacharya and N. P. Mukherjee**: *Fuzzy relations and fuzzy groups*, Inform. Sci. **36** (1985), 267 – 282.

[4] **P. S. Das**: *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **84** (1981), 264 – 269.

[5] **J. Dénes and A. D. Keedwell**: *Latin squares an their applications*, New York 1974.

[6] **W. A. Dudek**: *Fuzzy subquasigroups*, Quasigroups and Related Systems **5** (1998), 81 – 98.

[7] **W. J. Liu**: *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets and Systems **8** (1982), 133 – 139.

[8] **D. S. Malik and J. N. Mordeson**: *Extensions of fuzzy subring and Fuzzy ideals*, Fuzzy Sets and Systems **45** (1992), 245 – 251.

[9] **H. O. Pflugfelder**: *Quasigroups and loops: introduction*, Sigma Series in Pure Math., vol. **7**, Heldermann Verlag, Berlin 1990.

[10] **A. Rosenfeld**: *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512 – 517.

[11] **S. Sessa**: *On fuzzy subgroups and fuzzy ideals under triangular norm*, Fuzzy Sets and Systems **13** (1984), 95 – 100.

[12] **Y. Yu, J. N. Mordeson and S. C. Chen**: *Elements of L-algebras*, Lecture Notes in Fuzzy Math., Creighton Univ. Nebraska 1994.

[13] **L. A. Zadeh**: *Fuzzy sets*, Inform. Control **8** (1965), 338 – 353.

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