Quasigroups and Related Systems **6** (1999), 81 – 86

On TS-n-groups

Janez Ušan

Abstract

In this article totally simmetric n-group is described as an n-groupoid (Q,B) in which the following laws hold: $B(x,y,a_1^{n-2})=B(y,x,a_1^{n-2})$,
$B(a,c_1^{n-2},B(B(B(z,c_1^{n-2},z),c_1^{n-2},b),c_1^{n-2},B(B(z,c_1^{n-2},z),c_1^{n-2},a)))=b$,
$B(x,a_1^{n-2},y)=B(x,a_1^{n-2},B(B(y,a_1^{n-2},y),a_1^{n-2},y))$ and
$B(B(x,z,b_1^{n-2}),B(y,a_1^{n-2},z),a_1^{n-2})=B(x,y,b_1^{n-2})$.

1. Introduction

Definition 1.1. Let (Q,A) be an n-quasigroup and $n\geqslant 2$. Also let α be a permutation in the set $\{1,2,...,n+1\}$. Moreover, let

$$A^{\alpha}(x_1^n)=a_{n+1}\iff A(x_{\alpha(1)},...,x_{\alpha(n)})=x_{\alpha(n+1)}$$

for all $x_1^{n+1}\in Q$. We say that (Q,A) is a *totally simmetric n-quasigroup* (briefly: *TS-n-quasigroup*) iff for any permutation α on $\{1,2,...,n+1\}$ we have $A^{\alpha}=A$. In the case when $\alpha=(1,n+1)$ instead of A^{α} we write ${}^{-1}\!A$. Similarly in the case $\alpha=(n,n+1)$ instead of A^{α} we write A^{-1}.

Proposition 1.2. *Let (Q,A) be an n-group, ${}^{-1}$ its inversing operation,* **e** *its $\{1,n\}$-neutral operation and $n\geqslant 2$. Also let*

(*a*) ${}^{-1}\!A(x,a_1^{n-2},y)=z\iff A(z,a_1^{n-2},y)=x,$

(*b*) $A^{-1}(x,a_1^{n-2},y)=z\iff A(x,a_1^{n-2},z)=y$

1991 Mathematics Subject Classification: 20N15

Keywords: $n-$groupoid, $n-$semigroup, $n-$quasigroup, $n-$group, $\{i,j\}$-neutral operation on $n-$groupoid, inversing operation on $n-$group, TS-$n-$group

for all $x, y, z \in Q$ *and for every* $a_1^{n-2} \in Q$. *Then, for all* $x, y \in Q$ *and for every* $a_1^{n-2} \in Q$ *the following equalities hold*

(1) $\;{}^{-1}\!A(x, a_1^{n-2}, y) = A(x, a_1^{n-2}, (a_1^{n-2}, y)^{\,-1})$,

(2) $A^{-1}(x, a_1^{n-2}, y) = A((a_1^{n-2}, x)^{\,-1}, a_1^{n-2}, y)$,

(3) $\mathbf{e}(a_1^{n-2}) = {}^{-1}\!A(x, a_1^{n-2}, x)$,

(4) $(a_1^{n-2}, x)^{\,-1} = {}^{-1}\!A({}^{-1}\!A(x, a_1^{n-2}, x), a_1^{n-2}, x)$,

(5) $A(x, a_1^{n-2}, y) = {}^{-1}\!A(x, a_1^{n-2}, {}^{-1}\!A({}^{-1}\!A(y, a_1^{n-2}, y), a_1^{n-2}, y))$.

Proof. To prove (2) observe that

$$\begin{aligned}
&A^{-1}(x, a_1^{n-2}, y) = z \Longleftrightarrow A(x, a_1^{n-2}, z) = y \\
&\Longleftrightarrow A((a_1^{n-2}, x)^{\,-1}, a_1^{n-2}, A(x, a_1^{n-2}, z)) = A((a_1^{n-2}, x)^{\,-1}, a_1^{n-2}, y) \\
&\Longleftrightarrow A(A((a_1^{n-2}, x)^{\,-1}, a_1^{n-2}, x), a_1^{n-2}, z) = A((a_1^{n-2}, x)^{\,-1}, a_1^{n-2}, y) \\
&\Longleftrightarrow A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, z) = A((a_1^{n-2}, x)^{\,-1}, a_1^{n-2}, y) \\
&\Longleftrightarrow z = A((a_1^{n-2}, x)^{\,-1}, a_1^{n-2}, y).
\end{aligned}$$

The rest is proved in [7]. □

As a simple consequence of [2], [3] and [4] (see also [6]) we obtain:

Proposition 1.3. *Let* $n \geqslant 2$. *An* n*-group* (Q, A) *is a TS-*n*-group iff there exist a boolean group* $(Q, \cdot)$ *and element* $b \in Q$ *such that*

$$A(x_1^n) = x_1 \cdot \ldots \cdot x_n \cdot b$$

for all $x_1^n \in Q$.

2. Results

From the above we conclude that the following proposition holds.

Proposition 2.1. *Let* (Q, B) *be a TS-*n*-group with* $n \geqslant 2$. *Then*

(i) $B(B(x, z, b_1^{n-2}), B(y, a_1^{n-2}, z), a_1^{n-2}) = B(x, y, b_1^{n-2})$,

(ii) $B(a, c_1^{n-2}, B(B(B(z, c_1^{n-2}, z), c_1^{n-2}, b), c_1^{n-2}, B(B(z, c_1^{n-2}, z), c_1^{n-2}, a))) = b$,

(iii) $B(x, a_1^{n-2}, y) = B(x, a_1^{n-2}, B(B(y, a_1^{n-2}, y), a_1^{n-2}, y))$,

(iv) $B(x, y, a_1^{n-2}) = B(y, x, a_1^{n-2})$.

Theorem 2.2. *If the following laws*

$$(i)\quad B(B(x,z,b_1^{n-2}),B(y,a_1^{n-2},z),a_1^{n-2})=B(x,y,b_1^{n-2}),$$

$$(ii)\quad B(a,c_1^{n-2},B(B(B(z,c_1^{n-2},z),c_1^{n-2},b),c_1^{n-2},B(B(z,c_1^{n-2},z),c_1^{n-2},a)))=b,$$

$$(iii)\quad B(x,a_1^{n-2},y)=B(x,a_1^{n-2},B(B(y,a_1^{n-2},y),a_1^{n-2},y)),$$

$$(iv)\quad B(x,y,a_1^{n-2})=B(y,x,a_1^{n-2})$$

hold in an n-groupoid (Q,B), $n\geqslant 2$, then (Q,B) is a TS-n-group.

Proof. For $n\geqslant 2$ the following statements hold.

1° Let (Q,B) be an n-groupoid. If the following two laws

$$B(B(x,z,b_1^{n-2}),B(y,a_1^{n-2},z),a_1^{n-2})=B(x,y,b_1^{n-2}),$$

$$B(a,c_1^{n-2},B(B(B(z,c_1^{n-2},z),c_1^{n-2},b),c_1^{n-2},B(B(z,c_1^{n-2},z),c_1^{n-2},a)))=b$$

hold in (Q,B), then there is an n-group (Q,A) such that ${}^{-1}\!A=B$. (see Theorem 2.2 in [7]).

2° There exists the n-ary operation ${}^{-1}B$ in Q such that $(Q,{}^{-1}B)$ is an n-group and ${}^{-1}B=B$.

Indeed, by 1°, we conclude that there is an n-group (Q,A) such that ${}^{-1}\!A=B$. Hence

$${}^{-1}({}^{-1}\!A)(x,a_1^{n-2},y)=z\Leftrightarrow {}^{-1}\!A(z,a_1^{n-2},y)=x\Leftrightarrow A(x,a_1^{n-2},y)=z.$$

Moreover for all $x,y\in Q$ and $a_1^{n-2}\in Q$ we have

$$B(x,a_1^{n-2},y)=B(x,a_1^{n-2},B(B(y,a_1^{n-2},y),a_1^{n-2},y)),$$

and

$${}^{-1}B(x,a_1^{n-2},y)=B(x,a_1^{n-2},B(B(y,a_1^{n-2},y),a_1^{n-2},y)),$$

which proves that ${}^{-1}B=B$.

3° For all $x\in Q$ and for every sequence a_1^{n-2} over Q we have $(a_1^{n-2},x)^{-1}=x$ (see Proposition 1.2 and Remark 1.3 in [7]). Thus $B^{-1}=B$, because by [7] we have

$$B^{-1}(x,a_1^{n-2},y)=B((a_1^{n-2},x)^{-1},a_1^{n-2},y).$$

4° For all $x,y\in Q$ and for every sequence a_1^{n-2} over Q the following equality holds $B(x,a_1^{n-2},y)=B(y,a_1^{n-2},x)$. Indeed,

$$B(x, a_1^{n-2}, y) = z \Longleftrightarrow {}^{-1}B(x, a_1^{n-2}, y) = z \Longleftrightarrow B(z, a_1^{n-2}, y) = x$$
$$\Longleftrightarrow B^{-1}(z, a_1^{n-2}, y) = x \Longleftrightarrow B(z, a_1^{n-2}, x) = y$$
$$\Longleftrightarrow {}^{-1}B(y, a_1^{n-2}, x) = z \Longleftrightarrow B(y, a_1^{n-2}, x) = z.$$

5° Let $n \geq 3$ and $\mathbf{e}$ be a $\{1, n\}$-neutral operation of the n-group (Q, B). Then for all $x, y \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$B(\mathbf{e}(a_1^{n-2}), x, a_1^{n-2}) = x.$$

To prove it we consider the new operation F defined by

$$F(x, a_1^{n-2}) \overset{def}{=} B(x, \mathbf{e}(a_1^{n-2}), a_1^{n-2}).$$

Then

$$B(F(x, a_1^{n-2}), \mathbf{e}(a_1^{n-2}), a_1^{n-2}) = B(B(x, \mathbf{e}(a_1^{n-2}), a_1^{n-2}), \mathbf{e}(a_1^{n-2}), a_1^{n-2})$$

and

$$B(F(x, a_1^{n-2}), \mathbf{e}(a_1^{n-2}), a_1^{n-2}) = B(x, B(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, \mathbf{e}(a_1^{n-2})), a_1^{n-2}).$$

This implies

$$B(F(x, a_1^{n-2}), \mathbf{e}(a_1^{n-2}), a_1^{n-2}) = B(x, \mathbf{e}(a_1^{n-2}), a_1^{n-2}).$$

Thus

$$F(x, a_1^{n-2}) = x \Longleftrightarrow B(x, \mathbf{e}(a_1^{n-2}), a_1^{n-2}) = x.$$

But by (iv) we have

$$B(\mathbf{e}(a_1^{n-2}), x, a_1^{n-2}) = B(x, \mathbf{e}(a_1^{n-2}), a_1^{n-2}) = x,$$

which completes the proof of 5°.

6° Let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary nHG-algebra associated to the n-group (Q, B) (see [8]). Then, by Proposition 1.6 from [8], there is at least one sequence $a_1^{n-2} \in Q$ such that

$$x \cdot y = B(x, a_1^{n-2}, y) \quad \text{and} \quad \varphi(x) = B(\mathbf{e}(a_1^{n-2}), x, a_1^{n-2})$$

for all $x, y \in Q$. Whence, by 4° and 5°, we conclude that

$$x \cdot y = y \cdot x \quad \text{and} \quad \varphi(x) = x.$$

Thus

$$\mathbf{e}(a_1^{n-2}) \cdot x = x \cdot \mathbf{e}(a_1^{n-2}) = B(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$$

and

$$(a_1^{n-2},x)^{-1}\cdot x = x\cdot (a_1^{n-2},x)^{-1} = B(x,a_1^{n-2},(a_1^{n-2},x)^{-1}) = \mathbf{e}(a_1^{n-2})$$

by [7]. Hence $x^{-\mathbf{1}} \stackrel{def}{=} (a_1^{n-2},x)^{-1} = x$, which by our Proposition 1.3 completes the proof. □

Remark 2.3. Let $(K,\cdot)$, where $K=\{1,2,3,4\}$, be the Klein's group with the multiplication defined by the following table:

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Then the permutation φ of K defined by

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

is an automorphism of $(K,\cdot)$ and $(K,\{\cdot,\varphi,2\})$ is a 3HG-algebra associated to a 3-group (K,A), where

$$A(x,y,z) = x\cdot\varphi(y)\cdot z\cdot 2.$$

Moreover, $\mathbf{e}(x) = 2\cdot\varphi(x)$, $(a,x)^{-1} = x$, and ${}^{-1}\!A = A = A^{-1}$.

It is not difficult to see that the laws $(i)-(iii)$ hold in this 3-group, but $A(2,4,2) = 4 \neq 3 = A(4,2,2)$.

References

[1] **V. D. Belousov**: *n-Ary quasigroups*, (Russian), Ştiinţa, Chişinău 1972.

[2] **W. A. Dudek and J. Michalski**: *On a generalization of Hosszú theorem*, Demonstratio Math. **15** (1982), 783 – 805.

[3] **W. A. Dudek and J. Michalski**: *On retracts of polyadic groups*, Demonstratio Math. **17** (1984), 281 – 301.

[4] **W. A. Dudek and J. Michalski**: *On a generalization of a theorem of Timm*, Demonstratio Math. **18** (1985), 869 − 883.

[5] **M. Polonijo**: *Abelian totally symmetric n-quasigroups*, Proc. Symp. n-ary Structures, Skopje 1982, 185 − 193.

[6] **J. Ušan**: *On Hosszú-Gluskin algebras corresponding to the same n-group*, Rev. of Research, Fac. of Sci. Univ. of Novi Sad, Math. Ser. **25.1** (1995), 101 − 119.

[7] **J. Ušan**: *n−groups as n-groupoids with laws*, Quasigroups and Related Systems **4** (1997), 67 − 76.

[8] **J. Ušan and M. Žižović**: *On ordered n-groups*, Quasigroups and Related Systems **4** (1997), 77 − 87.

Institute of Mathematics
University of Novi Sad
Trg D. Obradovića 4
21000 Novi Sad
Yugoslavia

Received December 29, 1998