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Transversals in groups. 2. Loop transversals in a group by the same subgroup

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Abstract

Connections between different loop transversals in an arbitrary group G of the same subgroup H are demonstrated. It is shown that any loop transversal in an arbitrary group G of its subgroup H can be represented through one fixed loop transversal of H in G by the determined way. The case of a group transversal of H in G is described.

1. Introduction

This article is a continuation of [6]. The connections between different loop transversals in an arbitrary group G of the same subgroup H are described. These transversals play very a important role in solving some well-known problems. For example, the problem of existence of a finite projective plane of order n is reduced to the existence of a loop transversal of $St_{ab}(S_n)$ in S_n (see [7]).

We give some necessary definitions and notations:

E is a set of indexes (E contains the distinguished element 1, left (right) cosets in a group G by its subgroup H is indexed by the elements from E);

e is the unit of a group G;

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$Core_G(H)$ is the maximal proper subgroup of G contained in H, which is normal in G;

$St_a(K)$ is the stabilizer of an element a in a permutation group K.

Definition 1. Let G be a group and H its proper subgroup. A complete system $T = \{t_i\}_{i \in E}$ of representatives of the left (right) cosets of H ($e = t_1 \in T$) is called a *left (right) transversal* of H in G (or *"to" H in G* – see [4]). (A system of representatives of left cosets of H is complete if $t, u \in T$, $u^{-1}t \in H$ implies that $t = u$.)

Let T be a left transversal of H in G. We can correctly introduce the following operation on the set E:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \quad h \in H.$$

Lemma 1. *System $< E, \overset{(T)}{\cdot}, 1 >$ is a right quasigroup with two-sided unit* 1.

Proof. See Lemma 1 in [6]. □

Definition 2. Let T be a left (right) transversal of H in G. If the system $< E, \overset{(T)}{\cdot}, 1 >$ is a loop (group), then T is called a *loop (group) transversal* of H in G.

Remark 1. As we can see in [6], Lemma 10, a loop transversal T of H in G is a two-sided transversal of H in G, i.e. it is both left and right transversal of H in G. So we can simply say "loop transversal".

According to Cayley theorem any group K can be represented as a permutation group of degree $m = card\ K$ and this representation is regular. So any group K can be represented as a group transversal of $St_1(S_m)$ in S_m..

Lemma 2. *The following conditions are equivalent for any left transversal of H in G:*

1) *T is a loop transversal of H in G;*
2) *T is a left transversal in G of $\pi H \pi^{-1}$ for any $\pi \in G$;*
3) *$\pi T \pi^{-1}$ is a left transversal of H in G for any $\pi \in G$.*

Proof. See [1] and [4]. □

In the sequel the case $Core_G(H) = \{e\}$ will be considered. According to [5], Theorem 12.2.1, in this case we have $\hat{G} \cong G$, where $\hat{G}$ is a permutation representation of the group G. If H is a subgroup of G, then

$$\hat{g}(x) = y \quad \stackrel{def}{\Longleftrightarrow} \quad gt_xH = t_yH.$$

Lemma 3. *If T is a left transversal of H in G, then*

1) $\hat{h}(1) = 1 \quad \forall h \in H,$
2) *For any* $x, y \in E$ $\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \quad \hat{t}_1(x) = \hat{t}_x(1) = x,$
   $\hat{t}_x^{-1}(y) = x \overset{(T)}{\backslash} y, \quad \hat{t}_x^{-1}(1) = x \overset{(T)}{\backslash} 1, \quad \hat{t}_x^{-1}(x) = 1,$
   where $\overset{(T)}{\backslash}$ *is a left division in the system* $< E, \overset{(T)}{\cdot}, 1 >$.
3) *The following conditions are equivalent:*
   a) *T is a loop transversal of H in G,*
   b) *$\hat{T} = \{\hat{t}_x\}_{x \in E}$ is a sharply transitive set of permutations on E.*

Proof. See Lemma 4 in [6]. □

2. Connection between loop transversals

Let T be an arbitrary fixed left transversal of a subgroup H in a group G. It is evident (see [6], equation (8)), that any other left transversal of H in G can be represented in the following form

$$s_x = t_x h_x^{(T \to S)}, \quad h_x^{(T \to S)} \in H, \quad x \in E.$$

Lemma 4. *The system $< E, \overset{(S)}{\cdot}, 1 >$ can be obtained from the system $< E, \overset{(T)}{\cdot}, 1 >$ in the following way*

$$x \overset{(S)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_x^{(T \to S)}(y). \tag{1}$$

Proof. See Lemma 13 in [6]. □

Lemma 5. *The system $< E, \overset{(S)}{\cdot}, 1 >$ is a loop iff the operations $\overset{(T)}{\cdot}$ and $B(x, y) = (\hat{h}_x^{(T \to S)})^{-1}(y)$ are orthogonal.*

Proof. (see also Theorem 2 from [3]) According to Lemma 1 the system $< E, \overset{(S)}{\cdot}, 1 >$ is a right quasigroup with the two-sided unit 1. So it is sufficient to prove the existence and uniqueness of solution of the equation

$$x \overset{(S)}{\cdot} a = b$$

for any fixed $a, b \in E$. We have

$$x \overset{(S)}{\cdot} a = b \iff x \overset{(T)}{\cdot} \hat{h}_x^{(T\to S)}(a) = b \iff \begin{cases} \hat{h}_x^{(T\to S)}(a) = z \\ x \overset{(T)}{\cdot} z = b \end{cases}$$

$$\iff \begin{cases} (\hat{h}_x^{(T\to S)})^{-1}(z) = a \\ x \overset{(T)}{\cdot} z = b \end{cases} \iff \begin{cases} B(x, z) = a \\ x \overset{(T)}{\cdot} z = b \end{cases}$$

So the existence and uniqueness of solution of the equation $x \overset{(S)}{\cdot} a = b$ is equivalent to the existence and uniqueness of solution of the last system, which gives the orthogonality of $\overset{(T)}{\cdot}$ and $B(x, z)$. □

This means that if T is a fixed left transversal of H in G, then any loop transversal S of H in G may be represented through T by formula (1) according to the orthogonality condition from Lemma 5.

V.D. Belousov proved in [2] (Lemma 3) the following criterion

Lemma 6. *An operation $A(x, y)$ defined on the set E is orthogonal to the operation $C(x, y)$ iff $C(x, y)$ can be represented in the form:*

$$C(x, y) = K(B(x, y), A(x, y)), \tag{2}$$

where $B(x, y)$ is an operation orthogonal to $A(x, y)$, and $K(x, y)$ is a left invertible operation on the set E (i.e. $K(x, a) = b$ has a unique solution in E for any fixed $a, b \in E$). □

For a given left transversal T of H in G the problem of the choice of a set $\{h_x\}_{x\in E}$ such that the operations $\overset{(T)}{\cdot}$ and $B(x, y) = \hat{h}_x^{-1}(y)$ are orthogonal is not solved. But if the transversal T of H in G is a loop transversal, then according to Lemma 2, $\pi T \pi^{-1}$ is a loop transversal for any $\pi \in G$. Fixing some $h_0 \in H \setminus \{e\}$ and choosing

$$T^{h_0} = \{r_{x'} = h_0 t_x h_0^{-1} \mid t_x \in T\},$$

we obtain a new loop transversal T^{h_0} of H in G which does not coincide with T, because $Core_G(H) = \{e\}$.

Lemma 7. *The permutation* $\hat{h}_0 : E \to E$ *is an isomorphism of the systems* $< E, \overset{(T)}{\cdot}, 1 >$ *and* $< E, \overset{(T^{h_0})}{\cdot}, 1 >$.

Proof. According to the definition of T^{h_0}, we obtain:

$$\begin{aligned} x \overset{(T)}{\cdot} y = z &\Longleftrightarrow t_x t_y = t_z h, \quad h \in H \\ &\Longleftrightarrow (h_0 t_x h_0^{-1})(h_0 t_y h_0^{-1}) = (h_0 t_z h_0^{-1})(h_0 h h_0^{-1}), \quad h \in H \\ &\Longleftrightarrow r_{x'} r_{y'} = r_{z'} h', \quad h' = (h_0 h h_0^{-1}) \in H \\ &\Longleftrightarrow x' \overset{(T^{h_0})}{\cdot} y' = z'. \end{aligned}$$

Since

$$x' = \hat{r}_{x'}(1) = \hat{h}_0 \hat{t}_x \hat{h}_0^{-1}(1) = \hat{h}_0 \hat{t}_x(1) = \hat{h}_0(x), \tag{3}$$

then we obtain

$$\hat{h}_0(x) \overset{(T^{h_0})}{\cdot} \hat{h}_0(y) = \hat{h}_0(z) = \hat{h}_0(x \overset{(T)}{\cdot} y), \tag{4}$$

i.e. permutation $\hat{h}_0$ is an isomorphism of the systems $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(T^{h_0})}{\cdot}, 1 >$. □

According to Lemma 4 there exists the set $\{h_x^{(T \to T^{h_0})}\}_{x \in E}$ such that the operation $\overset{(T^{h_0})}{\cdot}$ may be obtained from the operation $\overset{(T)}{\cdot}$ by

$$x \overset{(T^{h_0})}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_x^{(T \to T^{h_0})}(y). \tag{5}$$

Lemma 8. *The operation* $B_1(x,y) = (\hat{h}_x^{(T \to T^{h_0})})^{-1}(y)$ *has the form*

$$B_1(x,y) = x \overset{(T^{h_0})}{\backslash} (x \overset{(T)}{\cdot} y). \tag{6}$$

Proof. Let $\hat{h}_x^{(T \to T^{h_0})}(y) = z$. Then $y = (\hat{h}_x^{(T \to T^{h_0})})^{-1}(z)$. So (5) can be rewritten in the form

$$x \overset{(T^{h_0})}{\cdot} (\hat{h}_x^{(T \to T^{h_0})})^{-1}(z) = x \overset{(T)}{\cdot} z.$$

As the system $< E, \overset{(T^{h_0})}{\cdot}, 1 >$ is a loop, we obtain from the last equality

$$(\hat{h}_x^{(T\to T^{h_0})})^{-1}(z) = x \overset{(T^{h_0})}{\backslash} (x \overset{(T)}{\cdot} z).$$

Then we have

$$B_1(x,y) \leftrightharpoons (\hat{h}_x^{(T\to T^{h_0})})^{-1}(y) = x \overset{(T^{h_0})}{\backslash} (x \overset{(T)}{\cdot} y)\,, \tag{7}$$

which completes the proof of the Lemma. □

According to Lemma 5, $B_1(x,y) = (\hat{h}_x^{(T\to T^{h_0})})^{-1}(y)$ and $\overset{(T)}{\cdot}$ are orthogonal operations. So, according to Lemma 6, any operation $C(x,y)$, being orthogonal to $\overset{(T)}{\cdot}$ may be written in the form:

$$C(x,y) = K(B_1(x,y), x \overset{(T)}{\cdot} y), \tag{8}$$

where $B_1(x,y)$ is the operation from (7) and $K(x,y)$ is a left invertible operation on the set E.

Let $P = \{p_x\}_{x\in E}$ be an arbitrary left transversal of H in G. The operation $\overset{(P)}{\cdot}$ is connected with $\overset{(T)}{\cdot}$ by the the formula (1) and $< E, \overset{(P)}{\cdot}, 1 >$ is a loop iff the corresponding set $\{h_x^{(T\to P)}\}_{x\in E}$ satisfies

$$(\hat{h}_x^{(T\to P)})^{-1}(y) = C(x,y) = K(B_1(x,y), x \overset{(T)}{\cdot} y), \tag{9}$$

where $B_1(x,y)$ is the operation from (7) and $K(x,y)$ is a some left invertible operation on the set E.

Because $K(x,y)$ is left invertible on the set E, we can write it as

$$K(x,y) = \varphi_y(x),$$

where φ_y is a permutation on E (for any $y \in E$). Using (7), we can rewrite (9) in the form

$$(\hat{h}_x^{(T\to P)})^{-1}(y) = \varphi_{x \overset{(T)}{\cdot} y}(x \overset{(T^{h_0})}{\backslash} (x \overset{(T)}{\cdot} y))\,. \tag{10}$$

But by (1)

$$x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_x^{(T\to P)}(y),$$

where set $\{h_x^{(T\to P)}\}_{x\in E}$ satisfies (10).

Let $\hat{h}_x^{(T\to P)}(y)=z$. Then $y=(h_x^{(T\to P)})^{-1}(z)$ and

$$x\overset{(P)}{\cdot}(h_x^{(T\to P)})^{-1}(z)=x\overset{(T)}{\cdot}z,$$

$$(h_x^{(T\to P)})^{-1}(z)=x\overset{(P)}{\backslash}(x\overset{(T)}{\cdot}z).$$

According to (10), we have

$$x\overset{(P)}{\backslash}(x\overset{(T)}{\cdot}z)=\varphi_{x\overset{(T)}{\cdot}z}(x\overset{(T^{h_0})}{\backslash}(x\overset{(T)}{\cdot}z)),$$

which for $u=x\overset{(T)}{\cdot}z$ gives

$$x\overset{(P)}{\backslash}u=\varphi_u(x\overset{(T^{h_0})}{\backslash}u). \tag{11}$$

So for the loop transversal $P=\{p_x\}_{x\in E}$ and any $x\in E$ we have

$$\hat{p}_x^{-1}(y)=\varphi_y(x\overset{(T^{h_0})}{\backslash}y)\,. \tag{12}$$

Lemma 9. *The the following conditions hold for all $x\in E$:*

1) $\varphi_x(1)=1$,

2) $\varphi_x(x)=x$,

3) $\alpha_x(y)=\varphi_y(x\overset{(T^{h_0})}{\backslash}y)$ *is a permutation from the group* $\hat{G}$.

Proof. 1) Because $\hat{p}_x^{-1}(x)=1$ for any $x\in E$, we obtain from (12)

$$1=\hat{p}_x^{-1}(x)=\varphi_x(x\overset{(T^{h_0})}{\backslash}x)=\varphi_x(1).$$

2) As $\hat{p}_1^{-1}(x)=x$ for any $x\in E$, then

$$x=\hat{p}_1^{-1}(x)=\varphi_x(1\overset{(T^{h_0})}{\backslash}x)=\varphi_x(x).$$

3) Since for any $x\in E$ the reflection $\hat{p}_x$ is a permutation from the group $\hat{G}$, then according to (12), the reflection $\alpha_x(y)=\varphi_y(x\overset{(T^{h_0})}{\backslash}y)$ is a permutation from the group $\hat{G}$. $\square$

Now we can prove

Theorem 1. *Let $T = \{t_x\}_{x\in E}$ be a loop transversal of H in G. If a left transversal $P = \{p_x\}_{x\in E}$ of H in G is connected with T by* (1), *then the following statements are equivalent:*

1) *P is a loop transversal,*
2) *P is connected with T by* (12), *where φ_x is as in Lemma* 9 *and $\overset{(T^{h_0})}{\setminus}$ is as in Lemma* 7. *Operations $\overset{(P)}{\cdot}$ and $\overset{(T^{h_0})}{\cdot}$ are connected by* (11).

Proof. 1) $\Longrightarrow$ 2) If P is a loop transversal of H in G, then (by Lemma 5) operations $\overset{(T)}{\cdot}$ and $B(x,y) = (\hat{h}_x^{(T\to P)})^{-1}(y)$ are orthogonal and (according to Lemma 6)

$$(\hat{h}_x^{(T\to P)})^{-1}(y) = K(B_1(x,y), x \overset{(T)}{\cdot} y),$$

where $B_1(x,y)$ is the operation from (7) and $K(x,y)$ is left invertible on the set E.

Because $K(x,y)$ is left invertible on E, we can write it in the form

$$K(x,y) = \varphi_y(x),$$

where φ_y is a permutation on E (for any $y \in E$). The rest follows Lemma 9.

2) $\Longrightarrow$ 1) If the conditions of the statement 2 hold, then there exists a set $\{h_x^{(T\to P)}\}_{x\in E}$ such that

$$p_x = t_x h_x^{(T\to P)}, \qquad h_x^{(T\to P)} \in H,$$
$$x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_x^{(T\to P)}(y).$$

So we have

$$p_x^{-1} = (h_x^{(T\to P)})^{-1} t_x^{-1},$$

which by Lemma 3 implies

$$\varphi_y(x \overset{(T^{h_0})}{\setminus} y) = \hat{p}_x^{-1}(y) = (\hat{h}_x^{(T\to P)})^{-1}\hat{t}_x^{-1}(y) = (\hat{h}_x^{(T\to P)})^{-1}(x \overset{(T)}{\setminus} y)\,.$$

This for $y = x \overset{(T)}{\cdot} z$ gives

$$\varphi_{x \overset{(T)}{\cdot} z}(x \overset{(T^{h_0})}{\backslash} (x \overset{(T)}{\cdot} z)) = (\hat{h}_x^{(T \to P)})^{-1}(z)\,.$$

Since operations $\overset{(T)}{\cdot}$ and $B_1(x,z) = x \overset{(T^{h_0})}{\backslash} (x \overset{(T)}{\cdot} z) = (\hat{h}_x^{(T \to T^{h_0})})^{-1}(z)$ are orthogonal (see Lemma 8), the last equality may be written as

$$(\hat{h}_x^{(T \to P)})^{-1}(z) = K(B_1(x,z), x \overset{(T)}{\cdot} z),$$

where $K(x,y) = \varphi_y(x)$ is a left invertible operation E.

But by Lemma 6 operations $\overset{(T)}{\cdot}$ and $B_2(x,z) = (\hat{h}_x^{(T \to P)})^{-1}(z)$ are orthogonal. Thus by Lemma 5 the system $< E, \overset{(P)}{\cdot}, 1 >$ is a loop, i.e. P is a loop transversal of H in G. □

3. A group transversal

As a simple consequence of our Theorem 1 we obtain

Theorem 2. *Let $T = \{t_x\}_{x \in E}$ be a group transversal of H in G. If a left transversal $P = \{p_x\}_{x \in E}$ of H in G is connected with T by* (1), *then the following statements are equivalent:*

1) *P is a loop transversal,*
2) *P is connected with T by the formula*

$$\hat{p}_x^{-1}(y) = \varphi_y(x^{-1} \overset{(T^{h_0})}{\cdot} y)\,, \tag{13}$$

where φ_x is as in Lemma 9 and x^{-1} is the inverse of x in the group $< E, \overset{(T^{h_0})}{\cdot}, 1 >$, which is isomorphic to $< E, \overset{(T)}{\cdot}, 1 >$. Corresponding operations $\overset{(P)}{\cdot}$ and $\overset{(T^{h_0})}{\cdot}$ are connected by

$$x \overset{(P)}{\backslash} y = \varphi_y(x^{-1} \overset{(T^{h_0})}{\cdot} y)\,. \tag{14}$$

From this Theorem we obtain the criterion of the existence of a loop transversal of H in G.

Theorem 3. *If* $Core_G(H) = \{e\}$, $d = (G : H) = card\ E$, *then the following statements are equivalent:*

1) *There exists a loop transversal of* H *in* G.
2) *There exists a set* $\{\varphi_x\}_{x\in E}$ *of permutations on* E *such that*

   a) $\varphi_x \in St_{1,x}(S_d) \quad \forall x \in E$,

   b) For any $x \in E$ *the reflection* $\alpha_x(y) = \varphi_y(y \overset{(T^{h_0})}{-} x)$ *(where the operation* $\overset{(T^{h_0})}{-}$ *is the inverse operation in the fixed group* $< Z_d, \overset{(T^{h_0})}{+}, 1 >$, *which is isomorphic to the group* $< Z_d, +, 0 >$) *is a permutation from the group* $\hat{G}$.

Proof. 1) $\Longrightarrow$ 2) Let $P = \{p_x\}_{x\in E}$ be a loop transversal of H in G. Using a permutation representation $\hat{G}$ of the group G we see that $\hat{P} = \{\hat{p}_x\}_{x\in E}$ is a loop transversal of $\hat{H}$ in $\hat{G}$. According to Lemma 3, the set $\hat{P}$ is a sharply transitive set of permutations on the set E; so $\hat{P} = \{\hat{p}_x\}_{x\in E}$ is a loop transversal of $H^* = St_1(S_d)$ in the symmetric group S_d (see [6]).

By the help of the regular representation of left translations the abelian group $< Z_d, +, 0 >$ may be represented as a group transversal T of $H^* = St_1(S_d)$ in S_d (see Remark 1). According to Theorem 2, the loop transversal $\hat{P} = \{\hat{p}_x\}_{x\in E}$ may be represented as the group transversal T^{h_0} by the formula

$$\hat{p}_x^{-1}(y) = \varphi_y(-x \overset{(T^{h_0})}{+} y) = \varphi_y(y \overset{(T^{h_0})}{-} x)\,, \tag{15}$$

where permutations $\{\varphi_x\}_{x\in E}$ are as in Lemma 9.

By Lemma 7 operations $\overset{(T)}{+}$ and $\overset{(T^{h_0})}{+}$ are isomorphic. Moreover $p_x^{-1} \in G$ implies $\hat{p}_x^{-1} \in \hat{G}$. Thus putting $\alpha_x(y) = \hat{p}_x^{-1}(y)$, we see that the conditions a and b from statement 2 hold.

2 $\Longrightarrow$ 1) Let $P = \{p_x\}_{x\in E}$ be a set of permutations defined by the formula:

$$\hat{p}_x^{-1}(y) \overset{def}{=} \varphi_y(-x \overset{(T^{h_0})}{+} y).$$

Then we have for any $x \in E$

$$\hat{p}_x^{-1}(x) = \varphi_x(-x \overset{(T^{h_0})}{+} x) = \varphi_x(1) = 1 \implies p_x(1) = x,$$

$$\hat{p}_1^{-1}(x) = \varphi_x(-1 \overset{(T^{h_0})}{+} x) = \varphi_x(x) = x \implies p_1(x) = x.$$

This means that $P = \{p_x\}_{x \in E}$ is a left transversal of H in G.

Using the analogous method as in the proof of sufficiency of Theorem 1 we can prove the existence of a loop transversal of H in G. □

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