Quasigroups and Related Systems **5** (1998), 69 – 80

# Symmetric $n$-loops with the inverse property

*Leonid A. Ursu*


### Abstract

It is proved that the matrix $\|I_{ij}\|$, where the substitutions $I_{ij}$ are defined by the equalities $\left(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}\right) = e$ is one of the inversion matrices of the symmetric $n$-$IP$-loop with an unique unit $e$. From this result it follows that the matrix $\|I_{ij}\|$ is a unique inversion matrix of such loops of an odd arity.


A quasigroup $Q(A)$ of arity $n$ is said to be an $IP$-*quasigroup* [1] if there exist substitutions $\nu_{ij}$, $i, j = \overline{1, n}$, on $Q$ with $\nu_{ii} = \varepsilon$ ($\varepsilon$ is the identical substitition) such that the equalities (the identities with parameters)

$$A(\{\nu_{ij}x_j\}_{j=1}^{i-1}, A(x_1^n), \{\nu_{ij}x_j\}_{j=i+1}^{n}) = x_i \tag{1}$$

hold for any $x_i \in Q$, $i \in \overline{1, n}$.

The substitutions $\nu_{ij}$ are called *inversion substitutions* and the matrix $\|\nu_{ij}\|$ is called an *inversion matrix*, $i \in \overline{1, n}$, $j \in \overline{1, n+1}$, where $\nu_{i,n+1} = \varepsilon$ for all $i$. The rows of this matrix are called *inversion systems* (*rows*) of an $n$-$IP$-quasigroup.

A quasigroup $Q(A)$ of an arity $n$ is said to be an $IP$-quasigroup if the following equalities

$$A^{\pi_i} = A^{T_i} \tag{2}$$

hold for all $i \in \overline{1, n}$, where $\pi_i$ is the transposition $(i, n+1)$, $A^{\pi_i}$ is the $i$-th inverse operation for $A$ and

$$T_i = (\{\nu_{ij}\}_{j=1}^{i-1}, \varepsilon, \{\nu_{ij}\}_{j=i+1}^{n}, \varepsilon).$$

1991 Mathematics Subject Classification: 20N15
Keywords: n-ary quasigroup, IP-loop, autotopy

If an $n$-$IP$-quasigroup $Q(A)$ has a unit $e$, then it is called an $n$-$IP$-*loop*.

In [1] the substitutions $I_{ij}$ are defined by the equalities

$$A\left(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}\right) = e \tag{3}$$

in an $n$-loop $Q(A)$ with a unit $e$ for any $x \in Q$, $i, j \in \overline{1, n}$, with $I_{ii} = I_{i,n+1} = \varepsilon$. From (3) it follows that $I_{ij}^{-1} = I_{ji}$ and $I_{ij}e = e$.

The following equality (cf. [1])

$$I_{ij}x = L_i(\bar{e}_j)\nu_{ji}x \tag{4}$$

shows a relation between $I_{ij}$ and $\nu_{ij}$, where

$$\bar{e}_j = \{\nu_{jk}e\}_{k=1}^{n}, \ \nu_{jj}e = e,$$
$$L_i(\bar{e}_j)x = A(\nu_{j1}e, \nu_{j2}e, \ldots, \nu_{j,i-1}e, x, \nu_{j,i+1}e, \ldots, \nu_{jn}e).$$

It is evident that $L_i(\bar{e}_j)$ is a substitution on $Q$. From (4) we get the following equality for the corresponding matrices

$$\|I_{ij}\| = \|L_i(\bar{e}_j)\| \cdot \|\nu_{ji}\|. \tag{5}$$

An $(n+1)$-tuple $T = (\alpha_1^{n+1})$ of substitutions on $Q$ is called an *autotopy* for an $n$-quasigroup $Q(A)$ if $A^T = A$.

A quasigroup $Q(A)$ of arity $n$ is said to be *symmetric* (cf. [2]) if

$$A(x_{\alpha 1}^{\alpha n}) = A(x_1^n)$$

for all $x_1^n \in Q^n$ and any $\alpha \in S_n$ where $S_n$ is the symmetric qroup of degree $n$.

It is known that an $n$-$IP$-quasigroup $(n > 2)$ can have more than one inversion matrix [1]. In [2] some examples of nonsymmetric $n$-$IP$-loops with the inversion matrix $\|I_{ij}\|$ are constructed.

Related to this V.D. Belousov has asced the following questions.

*Is the matrix $\|I_{ij}\|$ always one of the inversion matrices of an $n$-$IP$-loop ?*

*Does an $n$-$IP$-loop exist such that the matrix $\|I_{ij}\|$ is a unique inversion matrix ?*

In this article some properties of the symmetric $n$-$IP$-loops are established and answers are given to the V.D. Belousov's questions for such loops.

Let $Q(A)$ be a symmetric $n$-$IP$-quasigroup with an inversion matrix $\|\nu_{ij}\|$. Then the following properties are true.

**1.** *A symmetric $n$-$IP$-quasigroup is defined by a unique identity.* Indeed, let the $i$-th inverse identity

$$A(\{\nu_{ik}x_k\}_{k=1}^{i-1}, A(x_1^n), \{\nu_{ik}x_k\}_{k=i+1}^{n}) = x_i$$

holds in an $n$-$IP$-quasigroup $Q(A)$. Then

$$A(\{\nu_{ik}x_k\}_{k=1}^{i-1}, A(x_1^n), \{\nu_{ik}x_k\}_{k=i+1}^{n}) =$$
$$A(\{\nu_{ik}x_k\}_{k=1}^{j-1}, A(x_1^{j-1}, x_i, x_{j+1}^n), \{\nu_{ik}x_k\}_{k=j+1}^{n}) = x_i.$$

Thus, the $j$-th inverse identity holds in $Q(A)$ for all $j = 1, 2, \ldots, i-1, i+1, \ldots n$. It means that if $(\{\nu_{ik}\}_{k=1}^{i-1}, \varepsilon, \{\nu_{ik}\}_{k=i+1}^{n}, \varepsilon)$ is the $i$-th row of the inversion matrix $\|\nu_{ij}\|$, then $(\{\nu_{ik}\}_{k=1}^{j-1}, \varepsilon, \{\nu_{ik}\}_{k=j+1}^{n}, \varepsilon)$ is the $j$-th row of this matrix, $i, j \in \overline{1,n}$, $i \neq j$.

Hence if one inversion row is known, then the inversion matrix is known.

**2.** *If $(\nu_{i1}, \nu_{i2}, \ldots, \nu_{i,i-1}, \varepsilon, \nu_{i,i+1}, \ldots, \nu_{in}, \varepsilon)$ is the $i$-th inversion row, $i \in \overline{1,n}$, of a symmetric $n$-$IP$-quasigroup, then any permutation of the substitutions $\nu_{ik}$, $k = 1, 2, \ldots, i-1, i+1, \ldots, n$, of this row is the $i$-th inverse row of the quasigroup.*

In fact, from

$$A(\{\nu_{ik}x_k\}_{k=1}^{i-1}, A(x_1^n), \{\nu_{ik}x_k\}_{k=i+1}^{n}) = x_i$$

it follows that

$$A(\{\nu_{ik}x_k\}_{k=1}^{i-1}, \nu_{it}x_j, \{\nu_{ik}x_k\}_{k=j+1}^{i-1}, A(x_1^n), \{\nu_{ik}x_k\}_{k=i+1}^{t-1}, \nu_{ij}x_t, \{\nu_{ik}x_k\}_{t+1}^{n})$$
$$= A(\{\nu_{ik}x_k\}_{k=1}^{j-1}, \nu_{ij}x_t, \{\nu_{ik}x_k\}_{k=j+1}^{i-1}, A(x_1^{j-1}, x_t, x_{j+1}^{i-1}, x_i, x_{i+1}^{t-1}, x_j, x_{t+1}^n),$$
$$\{\nu_{ik}x_k\}_{k=i+1}^{t-1}, \nu_{it}x_j, \{\nu_{ik}x_k\}_{k=t+1}^{n}) = x_i$$

for any $i, j, t \in \overline{1,n}$, $j < i < t$.

Next, for the sake of simplicity we shall take the first inversion identity, i.e. the first inversion row, as definition of an $n$-$IP$-quasigroup. The corresponding inversion matrix we shall denote by $\|\nu_1\|$.

**3.** *If $T = (\alpha_1^n, \beta)$ is an autotopy of a symmetric $n$-$IP$-quasigroup $Q(A)$, then $\tilde{T} = (\alpha_{\sigma 1}^{\sigma n}, \beta)$ is an autotopy of $Q(A)$ also for any $\sigma \in S_n$.*

In other words, any permutation of the first $n$ components of an autotopy of a symmetric $n$-$IP$-quasigroup is an autotopy of this quasigroup as well.

Indeed, the equality

$$A(\{\alpha_k x_k\}_{k=1}^{i-1}, \alpha_i x_i, \{\alpha_k x_k\}_{k=i+1}^{j-1}, \alpha_j x_j, \{\alpha_k x_k\}_{k=j+1}^{n}) = \beta A(x_1^n)$$

implies that

$$A(\{\alpha_k x_k\}_{k=1}^{i-1}, \alpha_j x_j, \{\alpha_k x_k\}_{k=i+1}^{j-1}, \alpha_i x_i, \{\alpha_k x_k\}_{k=j+1}^{n}) \\ = \beta A(x_1^{i-1}, x_j, x_{i+1}^{j-1}, x_i, x_{j+1}^n),$$

i.e. $T_{i,j} = (\alpha_1^{i-1}, \alpha_j, \alpha_{i+1}^{j-1}, \alpha_i, \alpha_{j+1}^n, \beta)$ is an autotopy of $Q(A)$ for any $i, j \in \overline{1,n}$, $i \neq j$.

**4.** *If $T = (\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_n, \beta)$ is an autotopy of a symmetric $n$-$IP$-quasigroup $Q(A)$ with an inversion system $(\varepsilon, \nu_{12}, \ldots, \nu_{1n}, \varepsilon)$ (i.e. with an inverse matrix $\|\nu_1\|$), then*

$$(\beta,\ \nu_{12}\alpha_2\nu_{12},\ \nu_{13}\alpha_3\nu_{13}, \ldots, \nu_{1,i-1}\alpha_{i-1}\nu_{1,i-1}, \\ \nu_{1i}\alpha_1\nu_{1i}, \nu_{1,i+1}\alpha_{i+1}\nu_{1,i+1}, \ldots, \nu_{1n}\alpha_n\nu_{1n}, \alpha_i)$$

*is an autotopy of this quasigroup for any $i \in \overline{1,n}$.*

In fact, if $(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_n, \beta) \in \mathfrak{A}_A$, where $\mathfrak{A}_A$ is the autotopy group of $Q(A)$, then by property 3

$$(\alpha_i, \alpha_2, \alpha_3, \ldots, \alpha_{i-1}, \alpha_1, \alpha_{i+1}, \ldots, \alpha_n, \beta) \in \mathfrak{A}_A.$$

and by the property of autotopies of $n$-$IP$-quasigroups, proved in [1],

$$(\beta,\ \nu_{12}\alpha_2\nu_{12},\ \nu_{13}\alpha_3\nu_{13}, \ldots, \nu_{1,i-1}\alpha_{i-1}\nu_{1,i-1},\ \nu_{1i}\alpha_1\nu_{1i},$$

$$\nu_{1,i+1}\alpha_{i+1}\nu_{1,i+1}, \ldots, \nu_{1n}\alpha_n\nu_{1n}, \alpha_i) \in \mathfrak{A}_A.$$

As a corollary from this result, we get that if

$$T_1 = (\varepsilon, \nu_{12}, \nu_{13}, \dots, \nu_{1n}, \varepsilon)$$

is an inversion system of a symmetric $n$-$IP$-quasigroup, then

$$T_1^2 = (\varepsilon, \nu_{12}^2, \nu_{13}^2, \dots, \nu_{1n}^2, \varepsilon)$$

is its autotopy, since $T = \binom{n+1}{\varepsilon} \in \mathfrak{A}_A$.

**5.** *Let $Q(A)$ be a symmetric $n$-$IP$-loop with a unit $e$ and with an inversion matrix $\|\nu_1\|$. Then $\nu_{1j}^2 = \varepsilon$ for any $j \in \overline{2,n}$.*

Indeed, from the equality

$$A(A(x_1^k), \{\nu_{1i}x_i\}_{i=2}^n) = x_1$$

by $x_1 = x_2 = \dots = x_{j-1} = x_{j+1} = \dots = x_n = e$ we get

$$A(x_j, \{\nu_{1i}e\}_{i=2}^{j-1}, \nu_{1j}x_j, \{\nu_{1i}e\}_{i=j+1}^n) = e.$$

Changing in this equality $x_j$ for $\nu_{1j}e$ we get

$$A(\nu_{1j}e, \{\nu_{1i}e\}_{i=2}^{j-1},\ \nu_{1j}^2 e, \{\nu_{1i}e\}_{i=j+1}^n) = e$$

or $A(\nu_{1j}^2 e, \{\nu_{1i}e\}_{i=2}^n) = e$. Thus, $A^{\pi_1}(e, \{\nu_{1i}e\}_{i=2}^n) = \nu_{1j}^2 e$ from which according to (2) and symmetry we have

$$A(e, \{\nu_{1i}^2 e\}_{i=2}^n) = \nu_{1j}^2 e.$$

But $T_1^2 \in \mathfrak{A}_A$ so $A(e, \{\nu_{1i}^2 e\}_{i=2}^n) = A\binom{n}{e} = e$ and $\nu_{1j}^2 e = e$ for all $j \in \overline{2,n}$.

Now from

$$A(x_1, \{\nu_{1i}^2 x_i\}_{i=2}^n) = A(x_1^n)$$

by $x_1 = x_2 = \dots = x_{j-1} = x_{j+1} = \dots = x_n = e$, $x_j = x$, $j > 1$, one has $\nu_{1j}^2 x = x$ for any $j \in \overline{2,n}$.

**6.** *If $(\varepsilon, \nu_{12}, \nu_{13}, \dots, \nu_{1n}, \varepsilon)$ is an inversion system of a symmetric $n$-$IP$-loop with a unique unit, then*

$$\left(\overset{i-1}{\varepsilon}, \nu_{1i}\nu_{1j}, \overset{j-i-1}{\varepsilon}, \nu_{1j}\nu_{1i}, \overset{n-j+1}{\varepsilon}\right)$$

*is an autotopy of this loop for any* $i, j \in \overline{2, n}$.

This statement follows from properties 2 and 5 since the product (in the sense of component-wise multiplication) of two $i$-th inversion systems of an $n$-$IP$-quasigroup is an autotopy of this quasigroup [1].

In fact, let $(\varepsilon, \nu_{12}, \ldots, \nu_{1i}, \ldots, \nu_{1n}, \varepsilon)$ be an inversion system of a symmetric $n$-$IP$-loop. Then by property 2

$$(\varepsilon, \nu_{12}, \ldots, \nu_{1,i-1}, \nu_{1j}, \nu_{1,i+1}, \ldots, \nu_{1,j-1}, \nu_{1i}, \nu_{1,j+1}, \ldots, \nu_{1n}, \varepsilon)$$

is an inversion system of this loop too, and their product (since $\nu_{1i}^2 = \varepsilon$)

$$\left(\overset{i-1}{\varepsilon}, \nu_{1i}\nu_{1j}, \overset{j-i-1}{e}, \nu_{1j}\nu_{1i}, \overset{n-j-1}{\varepsilon}\right)$$

is an autotopy of the loop for all $i, j \in \overline{2, n}$.

**7.** *In a symmetric* $n$-$IP$*-loop* $Q()$ *with an inversion matrix* $\|\nu_1\|$ *the following equalities are true*

$$\nu_{1i}(x_1^n) = (\nu_{1i}x_i, \nu_{12}x_2, \nu_{13}x_3, \ldots, \nu_{1,i-1}x_{i-1}, \nu_{1i}x_i, \nu_{1,i+1}x_{i+1}, \ldots, \nu_{1n}x_n)$$

*for any* $i \in \overline{2, n}$.

Indeed, from

$$((x_1^n), \nu_{12}x_2, \ldots, \nu_{1,i-1}x_{i-1}, \nu_{1i}x_i, \nu_{1,i+1}x_{i+1}, \ldots, \nu_{1n}x_n) = x_1$$

it follows that

$$(\nu_{1i}x_i, \nu_{12}x_2, \ldots, \nu_{1,i-1}x_{i-1}, (x_1^n), \nu_{1,i+1}x_{i+1}, \ldots, \nu_{1n}x_n) = x_1.$$

Using (2) and taking into account that $\nu_{1i}^2 = \varepsilon$ for all $i \in \overline{2, n}$ we get

$$(x_1^{i-1}, \nu_{1i}(x_1^n), x_{i+1}^n) = \nu_{1i}x_i$$

or

$$(\nu_{1i}(x_1^n), x_2^{i-1}, x_1, x_{i+1}^n) = \nu_{1i}x_i.$$

Using (2) again one has

$$\nu_{1i}(x_1^n) = (\nu_{1i}x_i, \nu_{12}x_2, \nu_{13}x_3, \ldots, \nu_{1,i-1}x_{i-1}, \nu_{1i}x_1, \nu_{1,i+1}x_{i+1}, \ldots, \nu_{1n}x_n)$$

for any $i \in \overline{2,n}$.

**8.** *In a symmetric $n$-$IP$-loop*
*a) all substitutions $I_{ij}$ are equal, i.e. $I_{ij}x = Ix$ for any $i,j \in \overline{1,n}$, $i \neq j$ and any $x \in Q$,*
*b) $I^2 = \varepsilon$.*

We prove these statements.

a) Let $e$ be $a$ unit of a symmetric $n$-$IP$-loop. Then from the equalities

$$\left(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}\right) = \left(\overset{k-1}{e}, x, \overset{t-k-1}{e}, I_{kt}x, \overset{n-t}{e}\right) = e$$

it follows that

$$\left(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}\right) = \left(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{kt}x, \overset{n-j}{e}\right),$$

i.e. $I_{ij}x = I_{kt}x = I_x$ for all $i,j,k,t \in \overline{1,n}$, $i \neq j$, $k \neq t$ and any $x \in Q$.

b) Changing in $\left(\overset{i-1}{e}, x, \overset{j-i-1}{e}, Ix, \overset{n-1}{e}\right) = e$ the element $x$ for $Ix$ we get

$$\left(\overset{i-1}{\to} e, Ix, \overset{j-i-1}{\to} e, I^2x, \overset{n-j}{\to} e\right) = e = \left(\overset{i-1}{e}, Ix, \overset{j-i-1}{e}, x, \overset{n-j}{e}\right)$$

from which it follows that

$$I^2x = x \quad \text{for any} \quad x \in Q.$$

It is known (cf. [1]) that
i) the product of two autotopies of an $n$-quasigroup is an autotopy,
ii) the product of two $i$-th inversion systems, $i \in \overline{1,n}$, of an $n$-$IP$-quasigroup is an autotopy,
iii) the product of an autotopy and an inversion system of an $n$-$IP$-quasigroup is an inversion system of this quasigroup.

The analogous results are true for the product of corresponding matrices.

Let $Q(A)$ be a symmetric $n$-$IP$-loop with $a$ unique unit $e$ and with an inversion matrix $\|\nu_1\|$. Then a connection between the substitute $I$ and the inversion substitutions $\nu_{1i}$ is given by the following equality (see [1])

$$Ix = (e, \nu_{12}e, \dots, \nu_{1,i-1}e, \nu_{1i}x, \nu_{1,i+1}e, \dots, \nu_{1n}e) = L_i(\bar{e})\nu_{1i}x \qquad (6)$$

where

$$L_i(\bar{e})x = (e, \nu_{12}e, \dots, \nu_{1,i-1}e, x, \nu_{1,i+1}e, \dots, \nu_{1n}e)$$

are substitutions of $Q$, $i \in \overline{1,n}$, $(\bar{e}) = (e, \nu_{12}e, \dots, \nu_{1n}e)$.

Denote by $\mathfrak{O}_A$ the set of all inversion matrices and by $\mathfrak{A}_A$ the set of all matrices of autotopies of a symmetric $n$-$IP$-loop $Q(A)$. Let $\|\mathcal{L}\| = \|L_i(\bar{e})\|$. Then the equality (6) takes the form

$$\|I\| = \|\mathcal{L}\| \cdot \|\nu_1\|, \qquad (7)$$

i.e.

$$\begin{pmatrix} \varepsilon & I & I & \cdots & I & I & \varepsilon \\ I & \varepsilon & I & \cdots & I & I & \varepsilon \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I & I & I & \cdots & I & I & \varepsilon \end{pmatrix} =$$

$$\begin{pmatrix} \varepsilon & L_2(\bar{e}) & L_3(\bar{e}) & \cdots & L_{n-1}(\bar{e}) & L_n(\bar{e}) & \varepsilon \\ L_2(\bar{e}) & \varepsilon & L_3(\bar{e}) & \cdots & L_{n-1}(\bar{e}) & L_n(\bar{e}) & \varepsilon \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_2(\bar{e}) & L_3(\bar{e}) & L_4(\bar{e}) & \cdots & L_{n-1}(\bar{e}) & \varepsilon & \varepsilon \end{pmatrix} \times$$

$$\times \begin{pmatrix} \varepsilon & \nu_{12} & \nu_{13} & \cdots & \nu_{1,n-1} & \nu_{1n} & \varepsilon \\ \nu_{12} & \varepsilon & \nu_{13} & \cdots & \nu_{1,n-1} & \nu_{1n} & \varepsilon \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \nu_{12} & \nu_{13} & \nu_{14} & \cdots & \nu_{1,n-1} & \varepsilon & \varepsilon \end{pmatrix}.$$

From (7) it follows that

$$\|I\| \in \mathfrak{O}_A \Longleftrightarrow \|\mathcal{L}\| \in \mathfrak{A}_A. \qquad (8)$$

**Theorem 1.** *The matrix $\|I\|$ is one of the inversion matrices of a symmetric $n$-$IP$-loop with a unique unit.*

*Proof.* Let $Q(A) = Q(\ )$ be a symmetric $n$-$IP$-loop with an inversion matrix $\|\nu_1\|$ and with a unique unit $e$. Then $(\varepsilon, \nu_{12}, \nu_{13}, \ldots, \nu_{1n}, \varepsilon) \in \mathfrak{O}_A$, and by property 3 any permutation of the first $n$ substitutions of this inversion system gives an inversion system of this loop. According to property 6

$$\left(\overset{i-1}{\varepsilon}, \nu_{1i}\nu_{1j}, \overset{j-i-1}{\varepsilon}, \nu_{1j}\nu_{1i}, \overset{n-j+1}{\varepsilon}\right) \in \mathfrak{A}_A$$

for any $i, j \in \overline{2, n}$. By property 3 any permutation of the first $n$ components is an autotopy of the loop. Thus, by $1 < i < j < n$, we have

$$(\nu_{1i}, \nu_{1j}, \nu_{12}, \nu_{13}, \ldots, \nu_{1,i-1}, \nu_{1,i+1}, \ldots, \nu_{1,j-1}, \nu_{1,j+1}, \ldots, \nu_{1n}, \varepsilon, \varepsilon)\times$$
$$\times(\nu_{1j}\nu_{1i}, \nu_{1i}\nu_{1j}, \overset{n-1}{\rightarrow} \varepsilon) = (\nu_{1i}\nu_{1j}\nu_{1i}, \nu_{1j}\nu_{1i}\nu_{1j}, \nu_{12}, \nu_{13}, \ldots$$
$$\ldots, \nu_{1,i-1}, \nu_{1,i+1}, \ldots, \nu_{1,j-1}, \nu_{1,j+1}, \ldots, \nu_{1n}, \varepsilon, \varepsilon) \in \mathfrak{O}_A.$$

Then by property 5

$$(\varepsilon, \nu_{1j}, \nu_{12}, \nu_{13}, \ldots, \nu_{1,i-1}, \nu_{1,i+1}, \ldots, \nu_{1,j-1}, \nu_{1i}, \nu_{1,j+1}, \ldots, \nu_{1n}, \varepsilon)\times$$
$$\times(\nu_{1i}\nu_{1j}\nu_{1i}, \nu_{1j}\nu_{1i}\nu_{1j}, \nu_{12}, \nu_{13}, \ldots, \nu_{1,i-1}, \nu_{1,i+1}, \ldots, \nu_{1,j-1}, \nu_{1,j+1}, \ldots$$
$$\ldots, \nu_{1n}, \varepsilon, \varepsilon) = (\nu_{1i}\nu_{1j}\nu_{1i}, \nu_{1i}\nu_{1j}, \overset{j-3}{\rightarrow} \varepsilon, \nu_{1i}, \overset{n-j+1}{\rightarrow} \varepsilon) \in \mathfrak{A}_A \qquad .$$

Next,

$$(\nu_{1i}\nu_{1j}\nu_{1i}, \nu_{1i}\nu_{1j}, \overset{j-3}{\rightarrow} \varepsilon, \nu_{1i}, \overset{n-j+1}{\rightarrow} \varepsilon) \cdot (\varepsilon, \nu_{1j}\nu_{1i}, \overset{j-3}{\rightarrow} \varepsilon, \nu_{1i}\nu_{1j}, \overset{n-j+1}{\rightarrow} \varepsilon)$$
$$= (\nu_{1i}\nu_{1j}\nu_{1i}, \overset{j-2}{\rightarrow} \varepsilon, \nu_{1j}, \overset{n-j+1}{\rightarrow} \varepsilon) \in \mathfrak{A}_A.$$

Now use properties 4 and 5:

$$\left(\overset{j-1}{\varepsilon}, \nu_{1j}, \overset{n-j}{\varepsilon}, \nu_{1i}\nu_{1j}\nu_{1i}\right) \in \mathfrak{A}_A,$$

i.e.

$$\nu_{1i}\nu_{1j}\nu_{1i}A(x_1^n) = A(x_1^{j-1}, \nu_{1j}x, x_{j+1}^n).$$

From these equalities by $x_1 = x_2 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_n = e$ we get that

$$\nu_{1i}\nu_{1j}\nu_{1i}x = \nu_{1j}x.$$

Replacing $x$ by $\nu_{1i}x$ and using property 5 one has

$$\nu_{1i}\nu_{1j}x = \nu_{1j}\nu_{1i}x \tag{9}$$

for all $x \in Q$ and any $i, j \in \overline{1,n}$.

Now let $Q(A)$ have an odd arity. Then by property 6 and equality (9) the equality

$$(x, \nu_{1i}\nu_{1j}e, \nu_{ij}\nu_{1i}e, \nu_{1i}\nu_{1j}e, \nu_{1j}\nu_{1i}e, \ldots, \nu_{1i}\nu_{1j}e, \nu_{1j}\nu_{1i}e) = x$$

implies

$$\left(\overset{k-1}{\nu_{1i}\nu_{1j}e}, x, \overset{n-k}{\nu_{1i}\nu_{1j}e}\right) = x$$

for any $k \in \overline{1,n}$ and $x \in Q$. Thus, $\nu_{1i}\nu_{1j}e = e$, since $n-1$ is an even number and $e$ is a unique unit. But then $\nu_{1i}e = \nu_{1j}e$ for any $i, j \in \overline{2,n}$ since the inverse substitutions have order two. Therefore,

$$\nu_{12}e = \nu_{13}e = \cdots = \nu_{1n}e.$$

Next, since

$$\left(\overset{i-1}{\varepsilon}, \nu_{1i}\nu_{1j}, \overset{j-i-1}{\varepsilon}, \nu_{1j}\nu_{1i}, \overset{n-j+1}{\varepsilon}\right) \in \mathfrak{A}_A$$

then $\left(\overset{i-1}{e}, x, \overset{n-i}{e}\right) = x$ implies

$$\left(\overset{i-1}{e}, \nu_{1i}\nu_{1j}x, \overset{j-i-1}{e}, \nu_{1j}\nu_{1i}e, \overset{n-j}{e}\right) = x,$$

from which receive $\nu_{1i}\nu_{1j}x = x$ and $\nu_{1i}x = \nu_{1j}x$ for any $i, j \in \overline{2,n}$ and any $x \in Q$. From $(x, \nu_{12}e, \nu_{13}e, \ldots, \nu_{1n}e) = x$ (see (1) by $i = 1$) it follows that $\left(\overset{i-1}{\nu_{12}e}, x, \overset{n-i}{\nu_{12}e}\right) = x$ for any $i \in \overline{1,n}$ and $x \in Q$. Thus, $\nu_{12}e = \nu_{13}e = \cdots = \nu_{1n}e = e$ and the equality

$$Ix = (e, \nu_{12}e, \nu_{13}e, \ldots, \nu_{1,i-1}e, \nu_{1i}x, \nu_{1,i+1}e, \ldots, \nu_{1n}e)$$

implies $Ix = \nu_{1i}x$ for any $i \in \overline{2,n}$, $x \in Q$.

Now from (6) we have that $L_i(\bar{e}) = \varepsilon$. Thus, $\|\mathcal{L}\| = \|E\|$, where $\|E\|$ is the identical matrix, i.e. the matrix consisting of $\varepsilon$, and so $\|\mathcal{L}\| \in \mathfrak{A}_A$. But according to (8) and (7)

$$\|I\| = \|\nu_1\|. \tag{10}$$

Now let $Q(A)$ have an even arity. In this case

$$(\varepsilon, \nu_{12}\nu_{13}\ldots\nu_{1n}, \nu_{13}\nu_{14}\ldots\nu_{1n}\nu_{12}, \nu_{14}\nu_{15}\ldots\nu_{1n}\nu_{12}\nu_{13}, \ldots$$
$$\ldots, \nu_{1n}\nu_{12}\nu_{13}\ldots\nu_{1,n-1}, \varepsilon) \in \mathfrak{A}_A$$

and according to (9)

$$\left(\varepsilon, \nu_{12}\nu_{13}\overset{n-1}{\cdots}\nu_{1n}, \varepsilon\right) \in \mathfrak{A}_A.$$

Hence, by property 3 from $\left(\overset{i-1}{e}, x, \overset{n-i}{e}\right) = x$ it follows that

$$\left(\nu_{12}\nu_{13}\overset{i-1}{\cdots}\nu_{1n}e, x, \nu_{12}\nu_{13}\overset{n-i}{\cdots}\nu_{1n}e\right) = x$$

for all $i \in Q$, i.e. $\nu_{12}\nu_{13}\ldots\nu_{1n}e = e$. On the other hand, since $n$ is an even arity, then

$$T = (\varepsilon, \nu_{13}\nu_{14}\ldots\nu_{1n}, \nu_{12}\nu_{14}\nu_{15}\ldots\nu_{1n}, \ldots, \nu_{12}\nu_{13}\ldots\nu_{1,i-1}\nu_{1,i+1}\ldots\nu_{1n}, \\ \ldots, \nu_{12}\nu_{13}\ldots\nu_{1,n-1}, \varepsilon) \in \mathfrak{A}_A.$$

Using this autotopy, equality (9) and property 5 we get

$$L_i(\bar{e})x = (e, \nu_{12}e, \nu_{13}e, \ldots, \nu_{1,i-1}e, x, \nu_{1,i+1}e, \ldots, \nu_{1n}e) = \\ (e, \nu_{12}\nu_{13}\ldots\nu_{1n}e, \nu_{12}\nu_{13}\ldots\nu_{1n}e, \ldots, \nu_{12}\nu_{13}\ldots\nu_{1,i-1}\nu_{1,i+1}\ldots\nu_{1n}x, \\ \nu_{12}\nu_{13}\ldots\nu_{1n}e, \ldots, \nu_{12}\nu_{13}\ldots\nu_{1n}e) = \nu_{12}\nu_{13}\ldots\nu_{1,i-1}\nu_{1,i+1}\ldots\nu_{1n}x$$

for any $i \in \overline{2, n}$. Thus,

$$(\varepsilon, L_2(\bar{e}), L_3(\bar{e}), \ldots, L_i(\bar{e}), \ldots, L_n(\bar{e}), \varepsilon) \in \mathfrak{A}_A.$$

It means that $\|\mathcal{L}\| \in \mathfrak{A}_A$. Then by (8) $\|I\| \in \mathfrak{A}_A$ and

$$Ix = (e, \nu_{12}e, \nu_{13}e, \ldots, \nu_{1,i-1}e, \nu_{1i}x, \nu_{1,i+1}e, \ldots, \nu_{1n}e) = \\ (e, \nu_{12}\nu_{13}\ldots\nu_{1n}e, \nu_{12}\nu_{13}\ldots\nu_{1n}e, \ldots, \nu_{12}\nu_{13}\ldots\nu_{1,i-1}\nu_{1i}\nu_{1,i+1}\ldots\nu_{1n}x, \\ \nu_{12}\nu_{13}\ldots\nu_{1n}e, \ldots, \nu_{12}\nu_{13}\ldots\nu_{1n}e) = \nu_{12}\nu_{13}\ldots\nu_{1n}x.$$

The theorem is proved. □

**Corollary 1.** *Any symmetric $n$-$IP$-loop of an odd arity with a unique unit has only one inversion matrix, namely, the matrix $\|I\|$.*

This statement follows from the proof of the first part of Theorem, since any inversion matrix of a symmetric $n$-$IP$-loop of an odd arity with a unique unit coincides with the matrix $\|I\|$.

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Department of Higher Mathematics
Technical State University
Dacia str., 41
MD-2060 Chishinau
Moldova

Received August 16, 1998