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On the associativity of multiplace operations

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Abstract

In previous paper the study of (i,j)-associative $(n+1)$-ary groupoids is reduced to the study of groupoids, which are (k,m)-associative for all k, m satisfying $k \equiv m \equiv 0 \pmod r$, $k \equiv m \pmod s$, where $r|s|n$. The last groupoids are called associates of the sort (r,s,n) and of the sort (s,n), when $r=1$. Here, first order balanced identities are described in the associates. In other words, a bracketing rule is given.

1. Introduction

It is well known that the investigation of unary transformations of a set leads to semigroups; of multiplace transformations of a set leads to superassociative groupoids, i.e. Menger algebras (see [4]); and of unary transformations of a sequence of sets reduce to multiary semigroups, alternatives (translated from Russian) and others [3, 5, 6].

The defining properties of associativity for binary and multiary operations are the same: the result of a repeated operation use doesn't depend on brackets. However, although for the binary case one identity is enough, for a multiary operation we have to demand a family of (i,j)-associative identities asserting that the result will not change if we "move" brackets from the i-th place to the j-th (we begin the

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numeration of the indexes by 0). The problem of dependence between the (i,j)-associative identities with different i,j has been under consideration by many algebraists up through the present time. In [8] the author showed that in a surjective injective $(n+1)$-ary groupoid the family of (i,j)-associative identities is equivalent to such and to a family of all identities of (i,j)-associativity, for which $s \equiv i \equiv j \equiv 0 \pmod r$, $i \equiv j \pmod s$, $n \equiv 0 \pmod s$ for some integers r, s. Such groupoids are called associates of the sort (r,s,n). An associate of the sort $(1,1,n)$ is an $(n+1)$-ary semigroup.

Here, we continue an investigation in this direction. Namely, we give an answer for the question: how to "move" brackets in a word with the repeated operation use? More exactly, what first kind balanced identities hold in associates?

2. Necessary informations

Let $(Q;f)$ be an $(n+1)$-ary groupoid, i.e. a nonempty set Q with $(n+1)$-ary operation f defined on Q. The operation f and a groupoid $(Q;f)$ are called (i,j)-*associative*, if in $(Q;f)$ the following identity holds:

$$f(x_0,\dots,x_{i-1},f(x_i,\dots,x_{i+n}),x_{i+n+1},\dots,x_{2n}) =$$
$$= f(x_0,\dots,x_{j-1},f(x_j,\dots,x_{j+n}),x_{j+n+1},\dots,x_{2n}).$$

Let M be a nonempty set of pairs of nonnegative integers, such that every of them is not greater than n. Then a groupoid $(Q;f)$ is called M-*associative*, if it is (i,j)-associative for all pairs (i,j) from the set M. If the set M is empty or consists of the pair (i,i) only, then every groupoid is M-associative. Therefore from here on we admit, that $i<j$ if $(i,j)\in M$ and $M\neq\emptyset$.

If an operation f is (i,j)-associative for all $i<j$, then it is called *associative*, respective groupoid is called $(n+1)$-*semigroup* or *semigroup of the rank* n.

One of the most important examples of binary semigroups is a semigroup of all transformations of a set. Among the examples of $(n+1)$-ary semigroups there also exist $(n+1)$-ary transformation semigroups of set sequences [5, 6].

For groupoids $(Q; f)$ of the arity $n+1$ we use the following notations:

$$\begin{aligned} &M := \{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\}, \quad n := ||f||, \\ &s := \text{g.c.d.}(j_1 - i_1, \dots, j_m - i_m, n), \quad r := \text{g.c.d.}(i_1, \dots, i_m, s). \end{aligned} \tag{1}$$

Definition 1. A groupoid $(Q; f)$ of the arity $n+1$ will be called an *associate of the sort* (r, s, n), where r divides s, and s divides n, if it is (i, j)-associative for all (i, j) such that $i \equiv j \equiv 0 \,(\operatorname{mod} r)$, and $i \equiv j \,(\operatorname{mod} s)$. In an associate of the sort (s, n) (i.e. in an associate of the sort $(1, s, n)$) the number s will be called the *degree of associativity*. The operation f will be called *s-associative*. The least of the associativity degree will be called the *period of associativity*.

Note. If the sort of an associate is equal to (r, n, n), i.e. if $s = n$, then the (i, j)-associativity and the conditions

$$0 \le i \le n, \qquad 0 \le j \le n, \qquad i \equiv j \pmod n$$

imply $i = 0$, $j = n$ or $i = j$. Hence, such associate is $(0, n)$-associative only. Therefore $r = n$, and any associate of the sort (r, n, n) is an associate of the sort (n, n, n).

It is clear, that any associate of the sort $(1, 1, n)$ is $(n+1)$-ary semigroup, but there exist associates, which are not semigroups. For example, alternatives (described in [3]), i.e. groupoids $(Q; h)$ defined by the identities of $(0, j)$-associativity, when j is even and identities of the type

$$\begin{aligned} &h(h(x_0, \dots, x_n), x_{n+1}, \dots, x_{2n}) = \\ &= h(x_0, \dots, x_{j-1}, h(x_{j+n}, x_{j+n-1}, \dots, x_j), x_{j+n+1}, \dots, x_{2n}) \end{aligned}$$

when j is odd.

To formulate the next result we need the following definition.

Definition 2. An groupoid $(Q; f)$ of the arity $n+1$ is called:

a) *surjective*, if the function f is a mapping Q^{n+1} onto Q,

b) *j-injective*, if for any different elements $a, b \in Q$ there exist elements $a_1, \dots, a_n \in Q$ such that

$$f(a_1, \dots, a_j, a, a_{j+1}, \dots, a_n) \neq f(a_1, \dots, a_j, b, a_{j+1}, \dots, a_n),$$

c) *r-multiinjective*, if it is j-injective for all j which are multiples of r,

d) *injective*, if it is 1-multiinjective, i.e. it is j-injective for all $j = 1, 2, \dots, n$.

The theorem 3 from [8] implies the following statement.

Theorem 1. *In surjective r-multiple injective groupoid $(Q; f)$ of the arity $n+1$ the following conditions are equivalent:*

1) *$(Q; f)$ is (i, j)-associative for all (i, j) from of the set M, where M is as in* (1),

2) *$(Q; f)$ is $(r, r+s)$-associative,*

3) *$(Q; f)$ is an associate of the sort (r, s, n).* □

Here we deal with balanced identities of first kind. Recall, that a word w of some signature is called *repetition-free*, if every individual variable appears in it at most one time. If the repetition-free words w_1, w_2 consist of the same variables and have no individual constant, then the formula $w_1 = w_2$ is said to be *balanced*. If in addition their variables occur in the same order, then we says that it is a *formula of the first kind*.

To write down repetition-free words and balanced formulas of the first kind, it is convenient to exploit a variable-free notation using the following convention.

$$\left(f \overset{i}{+} g\right)(x_0, \dots, x_{m+n}) = \\ = f(x_0, \dots, x_{i-1}, g(x_i, \dots, x_{i+m}), x_{i+m+1}, \dots, x_{m+n}). \tag{2}$$

The relation (2) defines on the set $\Gamma(Q)$ of all operations on Q the collection of partial compositions, which are called *positional* or *Čupona's superposition* (see [2]). The algebra $(\Phi;\Sigma)$, where

$$\Phi \subseteq \Gamma(Q) \qquad \text{and} \qquad \Sigma = \left\{\overset{0}{+},\overset{1}{+},\overset{2}{+},\dots\right\},$$

is called a *position algebra* (see [1]). In [7] it is proved that the abstract class of all position algebras is exactly the class of algebras $(G;\Sigma)$ having a mapping $\rho(f) := ||f||$ from the set G into $N\cup\{-1;0\}$ satisfying the conditions

$$f \overset{i}{+} g \ \text{ is determined} \Longleftrightarrow 0 \le i \le ||f||, \tag{3}$$

$$||f \overset{i}{+} g|| = ||f|| + ||g||, \tag{4}$$

$$f \overset{i}{+} \left(g \overset{j}{+} h\right) = \left(f \overset{i}{+} g\right) \overset{i+j}{+} h, \tag{5}$$

$$\left(f \overset{i}{+} g\right) \overset{j}{+} h = \left(f \overset{j}{+} h\right) \overset{i+||h||}{+} g, \quad \text{if} \quad i > j. \tag{6}$$

For position algebras of operations, the mapping ρ *(rank)* is defined as $\rho(f) = |f| - 1$, where $|f|$ is the arity of f.

It is easy to verify that:

$$\left(f \overset{i}{+} g\right) \overset{j}{+} h = f \overset{i}{+} \left(g \overset{j-i}{+} h\right), \qquad \text{if} \quad i \le j \le i + \|g\|; \tag{7}$$

$$\left(f \overset{i}{+} g\right) \overset{j}{+} h = \left(f \overset{j-\|g\|}{+} h\right) \overset{i}{+} g, \qquad \text{if} \quad j > i + \|g\| \tag{8}$$

where f, g, h are elements of $(G;\Sigma)$.

Hence, a groupoid $(Q;f)$ is an associate of the sort (r,s,n), if

$$f \overset{i}{+} f = f \overset{j}{+} f \tag{9}$$

for all i, j such that $i \equiv j \,(\text{mod } s)\,,\ \ i \equiv j \equiv 0\,(\text{mod } r)$.

3. The main result and corollaries

Theorem 1 describes in (i, j)-associative groupoids with the noted properties a family of the balanced identities of the first kind with twofold operation use. The following question seems to be natural: what identities are true with repeated operation use ? Here we will give an answer to this question. But first we have to introduce the notion of a functional symbol coordinate in a word.

The sequence of individual variables and constants appearing in a word w will be denoted by $\mathrm{seq}_0 w$. The sequence of the functional symbols will be denoted with $\mathrm{seq}_1 w$.

Hence, a repetition-free word can be defined as word w with the conditions: the sequence $\mathrm{seq}_0 w$ has no individual constants and the members of $\mathrm{seq}_0 w$ are different pairwise. There is no restriction on the functional symbols of the word. The appearances of functional symbols are numerated from left to right.

Definition 3. Let $(f_0, f_1, \ldots, f_k) := w$, then a number of the individual variables and constants appearing in the word w from the beginning of the word till the functional symbol f_i, $i = 0, 1, \ldots, k$ is called a *coordinate of the symbol f_i in word w.*

Hence, the coordinate of the i-th appearance of symbol f in a word w of the signature $\{f\}$ is the coordinate of symbol f_i in the word w with a new notation $(f_0, f_1, \ldots, f_k) := \mathrm{seq}_1 w$.

Lemma 2. *Let w be an arbitrary repetition-free word and let*

$$(x_0, x_1, \ldots, x_p) := \mathrm{seq}_0 w, \qquad (f_0, f_1, \ldots, f_k) := \mathrm{seq}_1 w. \tag{10}$$

Then the following relation

$$w = \left(\left(\ldots\left(\left(f_0 \overset{j_1}{+} f_1\right) \overset{j_2}{+} f_2\right) \overset{j_3}{+} \ldots\right) \overset{j_k}{+} f_k\right)(x_0, x_1, \ldots, x_p) \tag{11}$$

holds, where $j_1 \le j_2 \le \ldots \le j_k$ and j_i is a coordinate of the symbol f_i in the word w, for all $i = 0, 1, \ldots, k$.

We shall agree to make the bracketting from left to right, i.e. like the equality (11).

Proof. The proof shall be given by induction in k, that is in the number of elements in the sequence $\mathrm{seq}_1 w$. When $k = 1$, the lemma follows from the definition of the Čupona superposition (see (2)).

Let us assume that the lemma is true for all words having the length of functional symbols equal to $k-1$ and we shall consider a word w (see (10)). Let us prove the relation (11).

Since there is no functional symbol on the right from f_k in word w and, under the condition of the lemma, j_k is the coordinate of the symbol f_k, that is there are exactly j_k individual variables on the left from f_k, then there is a subword

$$f_k(x_{j_k}, \ldots, x_{j_k+m}),$$

in the word w, where $m = ||f||$. We replace this subword by the new individual variable z. As the result we obtain a word w' such that the length of its functional variable sequence is equal to $k-1$. By the inductive assumption the lemma holds for the word w', therefore

$$w' = \left(\left(\ldots\left(\left(f_0 \overset{j_1}{+} f_1\right) \overset{j_2}{+} f_2\right) \overset{j_3}{+} \ldots\right) \overset{j_{k-1}}{+} f_{k-1}\right)(x_0^{j_k-1}, z, x_{j_k+m+1}^p)$$

where $j_1 \leq j_2 \leq \ldots \leq j_{k-1}$. Note that x_i^j is the sequence x_i, x_{i+1}, $\ldots$, x_j when $i \leq j$ and the empty sequence otherwise. Since this equality holds for all values of the individual variable z, then it will fulfill when

$$z = f_k(x_{j_k}, \ldots, x_{j_k+m}).$$

Therefore

$$w = \left(\left(\ldots\left(\left(f_0 \overset{j_1}{+} f_1\right) \overset{j_2}{+} f_2\right) \overset{j_3}{+} \ldots\right) \overset{j_{k-1}}{+} f_{k-1}\right)(x_0^{j_k-1}, f_k(x_{j_k}^{j_k+m}), x_{j_k+m+1}^p).$$

Using the definition of Čupona superposition we have

$$w = \left(\left(\left(\ldots\left(\left(f_0 \overset{j_1}{+} f_1\right) \overset{j_2}{+} f_2\right) \overset{j_3}{+} \ldots\right) \overset{j_{k-1}}{+} f_{k-1}\right) \overset{j_k}{+} f_k\right)(x_0, \ldots, x_p).$$

Since the symbol f_{k-1} is more left than f_k in word w, then number of the propositional variables being more than f_{k-1} is not greater than the number of propositional variables in the word w, i.e. $j_{k-1} \leq j_k$. So the lemma is proved for k.

Thus, by induction, the lemma is true for every natural number. $\square$

The main result of this article is the following theorem.

Theorem 3. *Let $(Q; f)$ be an associate of a sort (r, s, n). If the words w_1 and w_2 differ from each other by the bracket arrangements only; the coordinate of every f's appearance in the words w_1 and w_2 is divisible by r and there exists a one-to-one correspondence between f's appearances in the word w_1 and those in the word w_2 such that the corresponding coordinates are congruent modulo s, then the formula $w_1 = w_2$ is an identity in $(Q; f)$.*

Proof. First, we shall establish that for all divisible by r nonnegative integers i, j, k with

$$k \leq i \leq k+s, \qquad k \leq j \leq k+s,$$

and for every operation g of the arity more than k, the following equality

$$\left(\left(g \overset{k}{+} f\right) \overset{j}{+} f\right) \overset{i}{+} f = \left(\left(g \overset{k}{+} f\right) \overset{i}{+} f\right) \overset{j}{+} f \tag{12}$$

is obtained. Without loss of generality we put $j > i$ and denote $\ell = j - k$, $p = i - k$, then

$$\begin{aligned}
\left(f \overset{\ell}{+} f\right) \overset{p}{+} f &\overset{(6)}{=} \left(f \overset{p}{+} f\right) \overset{\ell+n}{+} f \overset{(9)}{=} \left(f \overset{p+s}{+} f\right) \overset{\ell+n}{+} f \\
&\overset{(7)}{=} f \overset{p+s}{+} \left(f \overset{\ell-p+n-s}{+} f\right) \overset{(9)}{=} f \overset{p+s}{+} \left(f \overset{\ell-p}{+} f\right) \\
&\overset{(5)}{=} \left(f \overset{p+s}{+} f\right) \overset{\ell+s}{+} f \overset{(9)}{=} \left(f \overset{p}{+} f\right) \overset{\ell+s}{+} f \overset{(7)}{=} f \overset{p}{+} \left(f \overset{\ell-p+s}{+} f\right) \\
&\overset{(9)}{=} f \overset{p}{+} \left(f \overset{\ell-p}{+} f\right) \overset{(5)}{=} \left(f \overset{p}{+} f\right) \overset{\ell}{+} f.
\end{aligned}$$

Hence, for arbitrary ℓ, p such that $0 \le \ell \le s,\ 0 \le p \le s$, the equality

$$\left(f \overset{\ell}{+} f\right) \overset{p}{+} f = \left(f \overset{p}{+} f\right) \overset{\ell}{+} f \tag{13}$$

holds. This implies the equality (12):

$$\begin{aligned}\left(\left(g \overset{k}{+} f\right) \overset{j}{+} f\right) \overset{i}{+} f &\overset{(7)}{=} \left(g \overset{k}{+} \left(f \overset{j-k}{+} f\right)\right) \overset{i}{+} f \\ &\overset{(7)}{=} g \overset{k}{+} \left(\left(f \overset{j-k}{+} f\right) \overset{i-k}{+} f\right) \overset{(13)}{=} g \overset{k}{+} \left(\left(f \overset{i-k}{+} f\right) \overset{j-k}{+} f\right) \\ &\overset{(5)}{=} \left(g \overset{k}{+} \left(f \overset{i-k}{+} f\right)\right) \overset{j}{+} f \overset{(5)}{=} \left(\left(g \overset{k}{+} f\right) \overset{i}{+} f\right) \overset{j}{+} f.\end{aligned}$$

Lemma 2 implies that every balanced identity $w = v$ is equivalent to the equality

$$f \overset{j_1}{+} f \overset{j_2}{+} f \overset{j_3}{+} \cdots \overset{j_k}{+} f = f \overset{\ell_1}{+} f \overset{\ell_2}{+} f \overset{\ell_3}{+} \cdots \overset{\ell_k}{+} f,$$

where sequences $j_1, j_2, \ldots, j_k$; $\ell_1, \ell_2, \ldots, \ell_k$ are nondecreasing, every numbers j_m, ℓ_m are the coordinates of the m-th appearances of the symbol f in the words w, v respectively, $m = 1, 2, \ldots, k$. Hence, to prove the theorem it is enough to prove the statement $P(k)$:

"For every positive integer k every relation

$$f \overset{j_1}{+} f \overset{j_2}{+} f \overset{j_3}{+} \cdots \overset{j_k}{+} f = f \overset{i_1}{+} f \overset{i_2}{+} f \overset{i_3}{+} \cdots \overset{i_k}{+} f$$

holds, where the sequences j_1, $j_2, \ldots, j_k$; i_1, $i_2, \ldots, i_k$ are nondecreasing, and members of the sequence i_1, $i_2, \ldots, i_k$ are the remainders of the numbers j_1, $j_2, \ldots, j_k$ on division by s (not necessary respectively)."

We will prove the statement by the mathematical induction. We will consider two cases $n > s$ and $n = s$.

Let assume that $n > s$. As the first step, we will prove the assertions: $P(1)$ and $P(2)$. The assertion $P(1)$ follows from the definition of an associate.

Let us consider the assertion $P(2)$. Let $j_1 \le j_2$ and let i_1, i_2 be the remainders of the numbers j_1, j_2 on division by s respectively, then

$$g := \left(f \overset{j_1}{+} f\right) \overset{j_2}{+} f \overset{(9)}{=} \left(f \overset{i_1}{+} f\right) \overset{j_2}{+} f. \tag{14}$$

We will consider two possible cases $j_2 \le i_1 + n$ and $j_2 > i_1 + n$.

Let us assume that $j_2 \le i_1 + n$, then taking into account that $i_1 \le j_2$, we have

$$g \overset{(7)}{=} f \overset{i_1}{+} \left(f \overset{j_2 - i_1}{+} f\right).$$

If $i_2 \ge i_1$, then

$$g \overset{(9)}{=} f \overset{i_1}{+} \left(f \overset{i_2 - i_1}{+} f\right) \overset{(5)}{=} \left(f \overset{i_1}{+} f\right) \overset{i_2}{+} f.$$

Thus, the statement $P(2)$ holds. Let $i_2 < i_1$, then

$$g \overset{(9)}{=} f \overset{i_1}{+} \left(f \overset{s+i_2-i_1}{+} f\right) \overset{(5)}{=} \left(f \overset{i_1}{+} f\right) \overset{s+i_2}{+} f \overset{(9)}{=} \left(f \overset{n-s+i_1}{+} f\right) \overset{s+i_2}{+} f.$$

The inequalities $i_1 > i_2$ and $n - s \ge s$ imply the inequalities $n - s + i_1 > s + i_2$, therefore

$$g \overset{(6)}{=} \left(f \overset{s+i_2}{+} f\right) \overset{2n-s+i_1}{+} f \overset{(9)}{=} \left(f \overset{i_2}{+} f\right) \overset{2n-s+i_1}{+} f.$$

Since $i_2 + ||f|| = i_2 + n < 2n - s + i_1$, then

$$g \overset{(7)}{=} \left(f \overset{n-s+i_1}{+} f\right) \overset{i_2}{+} f \overset{(9)}{=} \left(f \overset{i_1}{+} f\right) \overset{i_2}{+} f \overset{(13)}{=} \left(f \overset{i_2}{+} f\right) \overset{i_1}{+} f.$$

Since $i_2 < i_1$, then the statement $P(2)$ is proved in this case. Hence, $P(2)$ is proved when $j_2 \le i_1 + n$.

We assume, that $j_2 > i_1 + n$, then (14) can be rewritten as follows

$$g \overset{(7)}{=} \left(f \overset{j_2-n}{+} f\right) \overset{i_1}{+} f \overset{(9)}{=} \left(f \overset{i_2}{+} f\right) \overset{i_1}{+} f.$$

If $i_2 \le i_1$, then statement $P(2)$ is proved, and if $i_2 > i_1$, then

$$g \overset{(13)}{=} \left(f \overset{i_1}{+} f\right) \overset{i_2}{+} f.$$

So the statement $P(2)$ is proved.

Now we shall establish, that for all natural numbers $k > 2$ the statement $P(k)$ implies $P(k+1)$. Let

$$g = \left(f \overset{j_1}{+} f \overset{j_2}{+} f \overset{j_3}{+} \ldots \overset{j_k}{+} f\right) \overset{j}{+} f.$$

According to the inductive assumption

$$g = \left(f \overset{i_1}{+} f \overset{i_2}{+} f \overset{i_3}{+} \ldots \overset{i_k}{+} f\right) \overset{j}{+} f. \tag{15}$$

Since $0 \le i_1 \le i_2 \le \ldots \le i_k < s \le n$, then $i_1 \le i_m < i_1 + n$ for all $m = 2, \ldots, k$, therefore we can apply ($k-1$ times) the relation (7) to the expression singled out by brackets in (15):

$$\begin{aligned}
g &= \left(\left(f \overset{i_1}{+} f\right) \overset{i_2}{+} f\right) \overset{i_3}{+} f \overset{i_4}{+} \ldots \overset{i_k}{+} f \overset{j}{+} f \\
&\overset{(7)}{=} \left(\left(f \overset{i_1}{+} \left(f \overset{i_2-i_1}{+} f\right)\right) \overset{i_3}{+} f\right) \overset{i_4}{+} \ldots \overset{i_k}{+} f \overset{j}{+} f \\
&\overset{(7)}{=} \left(f \overset{i_1}{+} \left(\left(f \overset{i_2-i_1}{+} f\right) \overset{i_3-i_1}{+} f\right)\right) \overset{i_4}{+} \ldots \overset{i_k}{+} f \overset{j}{+} f \\
&\overset{(7)}{=} \ldots = \\
&\overset{(7)}{=} \left(f \overset{i_1}{+} \left(f \overset{i_2-i_1}{+} f \overset{i_3-i_1}{+} \ldots \overset{i_k-i_1}{+} f\right)\right) \overset{j}{+} f.
\end{aligned} \tag{16}$$

We consider three cases:

1) $i_1 \le j \le i_1 + kn$, 2) $i_1 > j$, 3) $j > i_1 + kn$.

1) Let $i_1 \le j \le i_1 + kn$, then

$$g \overset{(7)}{=} f \overset{i_1}{+} \left(f \overset{i_2-i_1}{+} f \overset{i_3-i_1}{+} \ldots \overset{i_k-i_1}{+} f \overset{j-i_1}{+} f\right).$$

We apply the inductive assumption to the expression in brackets again. After the redesignation of the numbers $i_2 - i_1, \ldots, i_k - i_1$ and the remainder on division by s of the number $j - i_1$ (not necessary respectively) by ℓ_1, ℓ_2, …, ℓ_k ($\ell_1 \le \ell_2 \le \ldots \le \ell_k$) we shall get

$$g = f \overset{i_1}{+} \left(f \overset{\ell_1}{+} f \overset{\ell_2}{+} \ldots \overset{\ell_k}{+} f\right) \overset{(5)}{=} f \overset{i_1}{+} f \overset{\ell_1+i_1}{+} f \overset{\ell_2+i_1}{+} \ldots \overset{\ell_k+i_1}{+} f.$$

Since $\{i_2, \ldots, i_k\} \subset \{\ell_1 + i_1, \ldots, \ell_k + i_1\}$, then the statement $P(k+1)$ is true, if $\ell_k + i_1 < s$. Therefore we shall assume, that $\ell_k + i_1 \geq s$. We shall analyse this case, using the definition of the number ℓ_k.

If $\ell_k = i_m - i_1$ for some number $m = 2, 3, \ldots, k$, then $i_m = \ell_k + i_1$, this, by the assumption, is not less than s. This is impossible, because i_m is a remainder on division by s.

Hence, the number ℓ_k is a remainder on division of the number $j - i_1$ by s, then $\ell_1 = i_2 - i_1$, …, $\ell_{k-1} = i_k - i_1$, because

$$i_1 \leq \ldots \leq i_k, \qquad \ell_1 \leq \ldots \leq \ell_k.$$

Since i_1 is also a remainder on division by s, then $\ell_k < s$ and $i_1 < s$. The sum of these inequalities gives $\ell_k + i_1 < 2s$. But $\ell_k + i_1 \geq s$, therefore $\ell_k + i_1 = i + s$ for some number i, which satisfies the condition $0 \leq i < s$. Since $\ell_k \equiv j - i_1 \pmod{s}$, then

$$j \equiv \ell_k + i_1 = i + s \equiv i \pmod{s}.$$

This means, that the number i is a remainder on division the number j by s. Hence, the last expression for g can be rewritten as

$$g = f \overset{i_1}{+} f \overset{i_2}{+} \cdots f \overset{i_{k-1}}{+} f \overset{i_k}{+} f \overset{i+s}{+} f. \tag{17}$$

Since $i + s - i_1 = \ell_k < s$, then $i < i_1$. But $i_1 \leq i_{k-1}$, therefore $i < i_{k-1}$. Hence,

$$i_{k-1} \leq i_k < i_{k-1} + s, \qquad i_{k-1} < i + s < i_{k-1} + s.$$

This permits us to apply the relationship (12) to (17):

$$g = \left(f \overset{i_1}{+} f \overset{i_2}{+} \cdots f \overset{i_{k-1}}{+} f \overset{i+s}{+} f \right) \overset{i_k}{+} f.$$

Applying the inductive assumption to the expression in brackets, as a result, taking into account that $i < i_1$, we get

$$g = \left(f \overset{i}{+} f \overset{i_1}{+} f \overset{i_2}{+} \cdots \overset{i_{k-1}}{+} f \right) \overset{i_k}{+} f.$$

Hence, in this case the statement $P(k+1)$ is proved.

2) Let $i_1 > j$. We apply k times (6) to the relation (15):

$$g \overset{(6)}{=} \left(f \overset{j}{+} f \overset{i_1+n}{+} f \overset{i_2+n}{+} \ldots \overset{i_{k-1}+n}{+} f\right) \overset{i_k+n}{+} f.$$

Then, we apply the inductive assumption to the expression in brackets. Taking into account the inequalities $i_k \geq \ldots \geq i_1 > j$, we shall get

$$g = \left(f \overset{j}{+} f \overset{i_1}{+} f \overset{i_2}{+} \ldots \overset{i_{k-1}}{+} f\right) \overset{i_k+n}{+} f.$$

Since $k > 2$ and $j < i_k + n < j + kn$, then just being proved p.1) implies $P(k+1)$.

Then, just being proved p.1) and p.2) imply that $P(k+1)$ is proved for the case $j \leq i_1 + kn$.

3) Let $j > i_1 + kn$, then the relation (16) can be rewritten as follows

$$g \overset{(7)}{=} \left(f \overset{j-kn}{+} \left(f \overset{i_2-i_1}{+} f \overset{i_3-i_1}{+} \ldots \overset{i_k-i_1}{+} f\right)\right) \overset{i_1}{+} f$$
$$\overset{(5)}{=} \left(f \overset{j-kn}{+} f \overset{q_1}{+} f \overset{q_2}{+} \ldots \overset{q_k}{+} f\right) \overset{i_1}{+} f,$$

where $q_u = i_u - i_1 + j - kn$, for all $u = 2, \ldots, k$. We apply the inductive assumption to the expression singled out by brackets:

$$g = \left(f \overset{p_1}{+} f \overset{p_2}{+} f \ldots \overset{p_k}{+} f\right) \overset{i_1}{+} f,$$

where the numbers $p_1, \ldots, p_k$ are remainders on division (not necessary respectively) of the numbers $j - kn, q_2, \ldots, q_k$ by s. Since

$$i_1 < s < n \leq kn \leq p_1 + kn,$$

then $i_1 \leq p_1 + kn$, and just being proved 1) and 2) imply $P(k+1)$.

According to the mathematical induction, the statement $P(k)$ is true for an arbitrary positive integer k.

To prove the theorem it remains to consider the case $n = s$. From the note given to the definition of an associate, the equalities mean that $n = s = r$. Hence, the coordinate of every appearance of the symbol f in the given balanced formula is divided by n, therefore all the remainders equal to 0.

The truth of the statement $P(1)$ follows from the definition of an associate. Assume, that the statement $P(k)$ is true and consider $P(k+1)$.

Analogously to the above one we can show that the equality (15), where $i_1 = i_2 = \ldots = i_k = 0$ fulfills. Since $j = pn$, then, using the relation (7) p times, we get

$$
\begin{aligned}
g &\overset{(15)}{=} \Big(\underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{k \ \ times}\Big) \overset{pn}{+} f \overset{(7)}{=} \Big(\underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{k-1 \ \ times} \overset{(p-1)n}{+} f\Big) \overset{0}{+} f \\
&\overset{(7)}{=} \Big(\underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{k-2 \ \ times} \overset{(p-2)n}{+} f\Big) \overset{0}{+} f \overset{0}{+} f \overset{(7)}{=} \ldots \\
&\overset{(7)}{=} \Big(\underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{k-p+1 \ \ times} \overset{n}{+} f\Big) \overset{0}{+} \underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{p-1 \ \ times} \\
&\overset{(6)}{=} \Big(\underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{k-p \ \ times} \overset{0}{+} \Big(f \overset{n}{+} f\Big)\Big) \overset{0}{+} \underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{p-1 \ \ times} \\
&\overset{(9)}{=} \Big(\underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{k-p \ \ times} \overset{0}{+} \Big(f \overset{0}{+} f\Big)\Big) \overset{0}{+} \underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{p-1 \ \ times} \\
&\overset{(5)}{=} \underbrace{f \overset{0}{+} f \overset{0}{+} \ldots \overset{0}{+} f}_{k+1 \ \ times}.
\end{aligned}
$$

According to the mathematical induction, the statement $P(k)$ is proved for all k, when $n = s$. This completes our proof. □

In the associate of the sort (r, s, n) the structure of its operation is determined by theorem 4 from [8] as soon as there exists at least one r-multiple invertible element in it. In particular, this theorem reduces the study of a groupoid to the study of an associate of the sort $(1, s, n)$, that is (s, n), with invertible elements. The description of balanced identities of the first kind is in the following corollary.

Corollary 4. *Let $(Q; f)$ be an associate of a sort (r, s). The equality $w_1 = w_2$ is an identity in the groupoid $(Q; f)$, if the words w_1 and w_2 differ from each other by the bracketting only and there exists an one-to-one correspondence between f's appearances in the word w_1 and those in the word w_2 such that the corresponding coordinates are congruent modulo s.*

Proof. It is enough to put $r = 1$ in Theorem 3. □

For example, in any associate $(Q; f)$ of the sort $(2, 4, 8)$ the following identity

$$\begin{aligned}
&\overset{0}{f}(\overset{0}{f}(x_0, x_1, \overset{2}{f}(x_2, x_3, \overset{4}{f}(x_4, \dots, x_{12}), x_{13}, \dots, x_{18}), x_{19}, \dots, x_{24}),\\
&\qquad\qquad x_{25}, \dots, x_{29}, \overset{30}{f}(x_{30}, \dots, x_{38}), x_{39}, x_{40}) =\\
=&\overset{0}{f}(\overset{0}{f}(x_0, \dots, x_8), x_9, \overset{10}{f}(\overset{10}{f}(x_{10}, \dots, x_{18}), x_{19}, \dots, x_{28}),\\
&\qquad\qquad x_{29}, \dots, x_{31}, \overset{32}{f}(x_{32}, \dots, x_{40})),
\end{aligned}$$

holds, in which the number of the functional symbol is the coordinate of its appearance.

Corollary 5. *In any semigroup (of any arity) any balanced identities of the first kind holds.*

By other words, in a semigroup of an arbitrary arity the result of repeated operation use does not depend on a bracketting.

Proof. The period of the associativity of a semigroup of an arbitrary arity is equal to 1. Therefore it is enough to put $s = 1$ in Corollary 4, as the result we shall obtain the truth of the given statement. □

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