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About some computer investigation of the endomorphisms of the linear isotopes of small order non-cyclic groups

Oleg U. Kirnasovsky and Sergej Sevastianov

Abstract

The order of the automorphism group and the endomorphism monoid of linear isotopes for non-cyclic groups are found up to 15-th order.

1. Introduction

A groupoid $(Q;\cdot)$ is called a *linear isotope of a group* $(Q;+)$, or a *linear group isotope*, iff there exist an element a and automorphisms φ, ψ of the group such that the equality

$$x \cdot y = \varphi x + \psi y + a \tag{1}$$

holds. It is easy to see that a group isotope is a quasigroup.

The group isotopes were studied in [3], [4] and [5]. Isomorphisms between two group isotopes are described in [1]. The list of all pairwise non-isomorphic linear group isotopes up to 15-th order is printed in [2]. This list contains 1554 quasigroups. Exactly 975 of them are linear isotopes of non-cyclic groups.

Combining the results from [2] and [3] we obtain the following

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Lemma. *A permutation α of a linear isotope $(Q;\cdot)$ of a group $(Q;+)$ defined by the equality* (1) *is an endomorphism of the isotope iff*

$$\alpha = R_c\theta, \quad \theta\varphi = \varphi\theta, \quad I_{\varphi c}\theta\psi = \psi\theta, \quad \theta a = \varphi c + \psi c + a - c$$

for some element c and some endomorphism θ of the group, where R_c denotes the right translation of the group with an element c, and $I_{\varphi c}$ denotes the inner automorphism $I_{\varphi c}x = -\varphi c + x + \varphi c$.

2. Description of the main algorithm

An algorithm, which describes all linear isotopes of an arbitrary fixed group defined by its Cayley table and its generator system, is given in [2]. Let us alter the algorithm supplementing it with some part.

Note that for a real computer employment it is useful to execute two almost the same algorithms. In the first one we limit ourself to the search of the list of all linear isotopes of the given group saving on magnetic information carriers the respective ordinal numbers of the automorphisms and of the free members from the canonical decompositions. It gives us the possibility in the second algorithm to avoid the preservation of the Cayley table (which is important in the first algorithm) of the automorphism group of the given group which economizes the necessary operative memory. There is also no necessity to use blocks, which were needed in the initial algorithm from [2].

To construct the second algorithm we add to the algorithm from [2] new blocks:

1. we compose information on the endomorphisms of the given group as we did it in [2] for automorphisms, and, in addition, we construct the table where every square corresponds to each next endomorphism and contains the information about its bijectivity;

2. in the certain place of the algorithm we read the parameters of each next linear group isotope;

3. looking over all pairs $\langle\theta, c\rangle$, where θ is an endomorphism of the given group and c is an element of the group, we verify fulfillment of the equalities of criterion, given in the Lemma; if $R_c\theta$

is an endomorphism of the investigated isotope, we add the unit to the score of number of the endomorphisms of the isotope (endomorphism doesn't appear twice, because different pairs define different endomorphisms);

4. taking into account that the transformation $R_c\theta$ is an automorphism of this isotope iff the transformation θ is bijective, we calculate the number of automorphisms of the isotope remembering the respective pairs in an individual table (in fact, we can remember the ordinal numbers of θ and c);

5. if the number of the automorphisms is not greater than 15, we create the Cayley table of the automorphism group of the isotope; for this purpose, we make the search of all triples $\langle\alpha,\beta,\gamma\rangle$ of the automorphisms of this isotope, and we put $\gamma=\alpha\beta$ iff γ and $\alpha\beta$ define the same act on the basis set;

6. we determine commutativity of the automorphism group and the number of its subgroups in the same way as we did for the main group in [2]; these two characteristics together with the order of this automorphism group synonymously define this group up to isomorphism, since its order is not greater than 15.

3. Main results

This algorithm was applied to all 13 non-cyclic groups up to 15-th order inclusively using IBM PC.

If $(abcd, efgh, ij)$ is the representation from [2] for a linear isotope of the group D_3, then for the linear isotope of the 12-th order group

$$G_{12}=\langle\, a,b\,|\,a^4=b^3=1,\ ba=ab^2\,\rangle$$

with the representation $(d^{'}cba,\ h^{'}gfe,\ ji)$, where

$$d\equiv d^{'}(\bmod 2),\quad h\equiv h^{'}(\bmod 2),$$

the number of all endomorphisms is twice greater than the number of all endomorphisms of the respective isotope of the group D_3. The automorphism group is isomorphic to the direct product of the group

Z_2 on the automorphism group of the respective isotope of the group D_3 (recall, that $Z_2 \times D_3 \simeq D_6$).

The numbers of all automorphisms and all endomorphisms are given across the symbol / . With that, the automorphism groups having the order up to 15 are discerned with the help of the letter placed after the number of the automorphisms. If such group is cyclic, then this letter is omitted. We use also the following symbols:

- the group $Z_2 \times Z_2$ is denoted as 4a ,
- the group $Z_6 \times Z_2$ is denoted as 12a ,
- the group $Z_3 \times Z_3$ is denoted as 9a ,
- the group $Z_2 \times Z_2 \times Z_2$ is denoted as 8a ,
- the group D_3 is denoted as 6a ,
- the group D_4 is denoted as 8b ,
- the group D_6 is denoted as 12b ,
- the group A_4 is denoted as 12c .

The symbol * denotes the automorphism group of the isotope which is isomorphic to the respective group.

The group $Z_2 \times Z_2$.
2/4. 2/4. 2/2. 2/4. 2/4. 2/2. 2/4. 2/2. 3/4. 3/4. 12c/16. 4a/4. 2/4. 3/4. 6a/16*.

The group $Z_4 \times Z_2$.
8b/32*. 4a/8. 4a/8. 4/8. 8b/32. 4a/8. 4a/8. 2/4. 2/4. 4a/8. 4a/8. 4a/8. 2/4. 4a/8. 2/4. 4a/8. 4a/8. 4/8. 2/4. 2/4. 4/8. 4/8. 4/8. 8b/32. 4a/8. 4a/8. 4/8. 8b/32.

The group $Z_6 \times Z_2$.
12b/48*. 4a/12. 6/12. 4a/12. 6/12. 12b/48. 4a/12. 4a/12. 4a/12. 4a/6. 4a/12. 4a/6. 4a/12. 4a/6. 4a/12. 4a/6. 4a/12. 4a/12. 6/12. 4a/12. 4a/6. 6/12. 4a/12. 4a/6. 24/48. 8a/12. 24/48. 8a/12. 6/12. 6/12. 4a/12.

4a/12. 4a/6. 4a/12. 4a/6. 12b/36. 6/12. 12b/36. 12b/18. 6/12. 6/6. 4a/12. 12b/36. 6/12. 12b/36. 6/6. 6/12. 12b/18. 6/12. 4a/12. 4a/6. 24/48. 8a/12. 12b/36. 12b/18. 6/12. 6/6. 18/36. 9a/12. 6/12. 18/36. 9a/12. 72/144. 24/36. 12a/12. 36/48. 12b/48. 4a/12. 6/12. 12b/36. 6/12. 18/36. 9a/12. 36/144. 18/48.

The group $Z_3 \times Z_3$.
12b/27. 6/9. 2/3. 2/3. 2/3. 6a/9. 3/3. 2/3. 6a/9. 3/3. 2/3. 4a/9. 6a/9. 3/3. 2/3. 4a/9. 2/3. 2/3. 2/3. 6a/9. 3/3. 6a/9. 3/3. 2/3. 2/3. 6a/9. 3/3. 6a/9. 3/3. 6a/9. 3/3. 12b/27. 6/9. 2/3. 2/3. 2/3. 2/3. 8/9. 6a/9. 3/3. 6a/9. 3/3. 2/3. 2/3. 6a/9. 3/3. 2/3. 8/9. 8/9. 2/3. 6a/9. 3/3. 8/9. 6a/9. 3/3. 72/81. 9a/9. 8/9. 8/9. 8/9. 2/3. 6a/9. 3/3. 8/9. 2/3. 2/3. 6a/9. 3/3. 2/3. 8/9. 6a/9. 3/3. 8/9. 2/3. 6a/9. 3/3. 8/9. 6a/9. 3/3. 72/81. 9a/9. 8/9. 8/9. 8/9. 2/3. 6a/9. 3/3. 2/3. 8/9. 2/3. 6a/9. 3/3. 8/9. 6a/9. 3/3. 2/3. 8/9. 6a/9. 3/3. 2/3. 6a/9. 3/3. 8/9. 72/81. 9a/9. 8/9. 8/9. 8/9. 6a/9. 3/3. 2/3. 2/3. 2/3. 18/27. 9a/9. 2/3. 6a/9. 3/3. 2/3. 6/9. 6a/9. 3/3. 6/9. 2/3. 6a/9. 3/3. 2/3. 6a/9. 3/3. 2/3. 54/81. 9a/9. 27/27. 6a/9. 3/3. 18/27. 9a/9. 6/9. 2/3. 2/3. 6a/9. 3/3. 6a/9. 3/3. 2/3. 6/9. 2/3. 6/9. 6a/9. 3/3. 2/3. 6/9. 2/3. 6a/9. 3/3. 2/3. 6a/9. 3/3. 6/9. 2/3. 6a/9. 3/3. 6/9. 6/9. 4a/9. 8/9. 8/9. 8/9. 6/9. 6/9. 48/81*. 48/81. 12b/27. 6/9. 8/9. 8/9. 8/9. 18/27. 9a/9. 6/9. 48/81. 432/729. 54/81.

The group $Z_2 \times Z_2 \times Z_2$.
8b/32. 2/8. 2/4. 2/4. 2/4. 2/4. 2/2. 1/2. 2/4. 2/2. 2/4. 2/2. 1/2. 1/2. 2/4. 2/2. 4a/8. 2/4. 2/2. 1/2. 2/8. 2/4. 1/2. 2/4. 2/2. 2/4. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 4a/8. 2/4. 4/8. 8b/32. 2/8. 2/4. 3/8. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 1/2. 1/2. 1/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 1/2. 2/4. 2/2. 2/4. 2/2. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 4a/8. 4a/4. 4a/4. 4a/4. 2/4. 2/2. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 1/2. 2/4. 2/2. 3/8. 1/2. 1/2. 1/2. 2/4. 2/2. 2/4. 2/2. 12c/32. 4a/8. 4a/8. 4a/4. 4a/4. 4a/4. 4a/8. 4a/4. 4a/4. 4a/4. 2/4. 2/2. 2/4. 2/2. 2/4. 2/4. 2/2. 4/8. 2/4. 2/2. 1/2. 1/2. 2/4. 2/2. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 4a/8. 4a/4. 4a/4. 4a/4. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2.

1/2. 2/4. 2/2. 1/2. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 4/8. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 4a/8. 4a/4. 4a/4. 4a/4. 4/8. 2/4. 2/4. 2/2. 2/4. 2/2. 4/8. 2/4. 2/2. 1/2. 1/2. 2/4. 2/2. 7/8. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 4a/8. 4a/4. 4a/4. 4a/4. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 7/8. 7/8. 2/4. 2/2. 7/8. 56/64. 8a/8. 7/8. 7/8. 1/2. 2/4. 2/2. 2/4. 2/2. 1/2. 2/4. 2/2. 7/8. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 2/4. 2/2. 4a/8. 4a/4. 4a/4. 4a/4. 1/2. 1/2. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 1/2. 2/4. 2/2. 1/2. 7/8. 2/4. 2/2. 2/4. 2/2. 2/4. 2/2. 7/8. 4a/8. 4a/4. 4a/4. 4a/4. 7/8. 56/64. 8a/8. 7/8. 7/8. 8b/32. 3/8. 4/8. 7/8. 7/8. 168/512*.

The group D_3.
6a/10*. 2/4. 3/4. 3/4. 1/2. 3/4. 3/4. 2/4. 2/6. 1/2. 1/1.

The group D_4.
8b/36*. 4a/20. 8b/12. 4/6. 4a/6. 4/6. 2/4. 4/6. 4/6. 2/4. 4/6. 8b/12. 4a/8. 8b/12. 4/6. 4a/6. 4a/20. 4a/20. 4a/8. 4a/8. 2/4. 2/4. 4a/6. 2/4. 4a/6. 2/4. 4a/6. 4a/6.

The group D_5.
20/26*. 4/6. 5/6. 4/6. 4/6. 5/6. 1/2. 5/6. 5/6. 5/6. 5/6. 1/2. 1/2. 4/6. 4/10. 1/2. 1/1. 4/6. 4/6. 1/2. 1/2. 4/6. 4/6. 1/2. 1/2. 4/6. 4/10. 1/2. 1/1. 4/6. 4/6. 1/2. 1/2. 4/10. 4/6. 1/1. 1/2.

The group D_6.
12b/64*. 4a/24. 6/16. 6/8. 4a/8. 12b/20. 6/8. 2/4. 6/8. 6/8. 6/8. 2/4. 6/8. 6/8. 6/16. 2/8. 6/16. 6/16. 6/8. 2/4. 6/8. 6/8. 12b/20. 4a/8. 6/8. 6/8. 4a/8. 12b/20. 4a/24. 4a/32. 2/8. 2/4. 2/4. 2/2. 4a/12. 4a/8. 4a/8. 2/2. 2/4. 4a/12. 2/4. 4a/12. 2/2. 4a/8.

The group D_7.
42/50*. 6/8. 7/8. 6/8. 6/8. 6/8. 6/8. 7/8. 1/2. 7/8. 7/8. 7/8. 7/8. 7/8. 7/8. 1/2. 1/2. 1/2. 1/2. 6/8. 6/14. 1/2. 1/1. 6/8. 6/8. 1/2. 1/2. 6/8. 6/8. 1/2. 1/2. 6/8. 6/8. 1/2. 1/2. 6/8. 6/14. 1/2. 1/1. 6/8. 6/8. 1/2. 1/2. 6/8. 6/8. 1/2. 1/2. 6/8. 6/8. 1/2. 1/2. 6/8. 6/14. 1/2. 1/1. 6/8. 6/8. 1/2. 1/2. 6/8. 6/8. 1/2. 1/2. 6/14. 6/8. 1/1. 1/2. 6/8. 6/8. 1/2.

1/2. 6/14. 6/8. 1/1. 1/2. 6/8. 6/8. 1/2. 1/2.

The group Q_8.
4a/6. 2/4. 4a/6. 1/3. 1/1. 4a/6. 2/4. 4a/6. 2/4. 2/4. 2/2. 2/2. 1/3. 1/1. 1/3. 1/1. 3/4. 1/2. 3/4. 1/2. 3/7. 1/1. 2/4. 4/6. 4/6. 1/3. 1/1. 4/6. 2/4. 4/6. 2/4. 2/4. 2/2. 2/2. 4a/6. 4/6. 3/4. 24/28*. 8b/12. 2/4. 2/4. 1/2. 8b/12. 4a/8. 8b/12. 4/6. 4a/6.

The group A_4.
24/33*. 3/6. 8b/9. 4a/9. 4/5. 3/6. 3/9. 3/6. 1/2. 1/2. 1/1. 1/5. 1/1. 1/1. 1/3. 4a/9. 1/5. 4a/5. 1/1. 2/3. 4a/21. 1/4. 4a/5. 1/1. 2/5. 4/5. 1/1. 1/3. 4/5. 2/3. 2/5. 1/1. 1/1. 4/5. 4/9. 8b/9. 1/2. 4a/5. 8b/9. 2/3. 4a/5. 4/5. 2/3.

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References

[1] **V. I. Izbash**: *Isomorphisms of quasigroups isotopic to groups*, Quasigroups and Related Systems **2** (1995), 34 – 50.

[2] **O. U. Kirnasovsky**: *Linear isotopes of small order groups*, Quasigroups and Related Systems **2** (1995), 51 – 82.

[3] **F. Sokhatsky**: *On isotopes of groups. I*, (Ukrainian), Ukrain. Math. Zh. **47** (1995), 1387 – 1398.

[4] **F. Sokhatsky**: *On isotopes of groups. II*, (Ukrainian), Ukrain. Math. Zh. **47** (1995), 1692 – 1703. (English translation in Ukrainian Math. J. **47** (1995), 1935 – 1948.)

[5] **F. Sokhatsky**: *On isotopes of groups. III*, (Ukrainian), Ukrain. Math. Zh. **48** (1996), 251 – 259.

Department of Algebra
Vinnytsia State Pedagogical University
Vinnytsia 287100
Ukraine

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