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On the simple Suzuki group $Sz(2^9)$ by the average order

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Abstract. We prove that simple Suzuki group $Sz(2^9)$, can be uniquely determined by its average order.

1. Introduction

Throughout this paper G is a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of elements order in G. Next, the sum of element orders in a group G denoted by $\psi(G) = \sum_{g\in G} o(g)$ where $o(g)$ denotes the order of $g \in G$ and also we define average order of G as $o(G) = \frac{\psi(G)}{|G|}$. In the way, the function $\psi(G)$ was introduced by Amiri, Jafarian and Isaacs [3]. They proved that if G is a non-cyclic group of order n, then $\psi(G) < \psi(Z_n)$. In fact Z_n is characterized by $\psi(Z_n)$ and $|Z_n|$.

In this paper we goal discuss about average order of group. In fact, we say the group G is characterized by $o(G)$, when ever there exist the group H, so that if $o(G) = o(H)$, then $G \cong H$. In the way, for example the authors in [1, 4, 7, 10, 11] proved that $PSL(2;5)$ and $PSL(2;7)$ are uniquely determined by their orders and the sum of the element orders. But only some of groups is characterized by average order. For example, the authors in ([6],[12],[14]) proved the alternating group A_5 and symetric group S_4, Suzuki group $Sz(8)$ can be determined by average order.

In this paper, we prove that the simple Suzuki group $Sz(2^9)$, can be uniquely determined by its average order. In fact, we prove the following main theorem.

Main Theorem. *Let G be a non-soluble group such and the simple Suzuki group $Sz(2^9)$ so that $o(G) = o(Sz(2^9))$ then, $G \cong Sz(2^9)$.*

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2. Notation and Preliminaries

In this section, we bring some lemmas and theorems where that in after section we used them. Next, Let $m_i(G)$ be the number of elements of order i and $\pi_e(G)$ denotes the set of element orders of G. We define $\psi(G) = \sum_{i\in\pi_e(G)} im_i(G)$ and $|G| = \sum_{i\in\pi_e(G)} m_i(G)$ and $o(G) = \frac{\psi(G)}{|G|}$.

Lemma 0.1. [8] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*
4. *There exists a subgroup H_0 of H with maximum index $[H : H_0] \leq 2$, where $H_0 \cong SL_2(5) \times Z$, $\pi(Z) \cup \{2, 3, 5\} = 1$, Sylow subgroups of Z are cyclic.*

Definition 0.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 0.3. [5] *Let G be a 2-Frobenius group of even order. Then*

1. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$. In particular, every 2-Frobenius group is soluble group.*

Lemma 0.4. [14] *Let G be a group and H a non-trivial normal subgroup of G.*

1. *For each $x \in G$ and $h \in H$, $o(xH) \mid o(xh)$,*
2. *$\frac{\psi(H)-|H|}{|G|} + o(\frac{G}{H}) \leq o(G)$. In particular $o(G/H) < o(G)$.*

Lemma 0.5. [13] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |Out(K/H)|$.*

Lemma 0.6. [2] *Let $P \in Syl_p(G)$, and assume that $P \trianglelefteq G$ and that P is cyclic. Then $\psi(G) \leq \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

Lemma 0.7. [9] *Let G_1, G_2 be finite groups, p be a prime number and let a, b be positive integers. Then the following hold:*

1. $\psi(G_1 \times G_2) = \psi(G_1) \times \psi(G_2)$ *if and only if* $(|G_1|, |G_2|) = 1$ ($i.e. \psi$ *is multiplicative*);
2. $\psi(p^a) = \frac{p^{2a+1}+1}{p+1}$,
3. $\psi(p^a) \mid \psi(p^b)$ *if and only if* $2a+1 \mid 2b+1$,
4. $(\psi(p), \psi(p^2)) = (\psi(p), \psi(p^3)) = (\psi(p^2), \psi(p^3)) = 1$.

Theorem 0.8. [12]. *Let G be a finite group. If $o(G) < \frac{211}{60}$ then G is solvable.*

Proof of the Main Theorem. We prove that the simple Suzuki groups $Sz(2^9)$ can be uniquely determined by the average order. In fact, we prove that if G is a group with $o(G) = o(Sz(2^9)$. Then $G \cong Sz(2^9)$.

First, assume $|Sz(2^9)| = 2^{18}.5.7.13.37.73.109$ and $\psi(Sz(2^9)) = 155694$ 38561270784 then $o(Sz(2^9)) = \frac{\psi(2^9)}{|Sz(2^9)|} = o(G) = 443.3$. Now, since G is not soluble group so $o(G) > \frac{211}{60}$. Therefore G is not 2-Frobenius group. Now, we prove G is not a Frobenius group. On opposite, assume G be a Frobenius group with complement H and kernel K. Now by Lemma0.1 $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and there exists subgroup H_0 of H with maximum index so that $[H : H_0] \leq 2$ where $H_0 \cong SL_2(5) \times Z$, $\pi(Z) \cup \{2, 3, 5\} = 1$. Since $|Sz(2^9)| = 2^{18}.5.7.13.37.73.109$ we can see $3 \notin \pi(G)$ that is a contradiction. Hence G is not a Frobenius group. Now, by lemma 0.5 and Lemma 0.4 then $o(G/H) < o(G)$. On the other hand, we know, Suzuki group $Sz(q)$ is only group where 3 not divided order of it. So $G/H \cong Sz(q')$, where $q' = 2^{2m+1}$ and $o(G/H) < 443.3$ in special case $q' = 2^9$. In other words, $G/H \cong Sz(2^9)$, on the other hand G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ as $H = 1$, so $G \cong Sz(2^9)$, the proof be completed. □

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