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A note on comaximal graph and maximal topology on multiplication le-modules

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Abstract. In this article, the co-maximal graph $\Gamma(M)$ on le-modules M has been introduced and studied. The graph $\Gamma(M)$ consists of vertices as elements of ${}_RM$ and two distinct elements n, m of $\Gamma(M)$ are adjacent if and only if $Rn + Rm = e$. We have established a connection between the co-maximal graph and the maximal topology on $Max(M)$ in the case of multiplication le-modules. Also, the Beck's conjecture is settled for $\Gamma(M)$ which does not contain an infinite clique.

1. Introduction

An algebraic structure known as a le-module was introduced and explored by A.K. Bhuniya and M. Kumbhakar [3, 4, 5]. They were inspired to study abstract submodule theory, in particular le-module by the study of abstract ideal theory, particularly multiplicative lattices and lattice modules.

Sharma and Bhatwadekar [10] introduced a graph on elements of commutative ring R with unity by taking vertices as elements of R with two distinct vertices x and y are adjacent if and only if the addition of ideals generated by x and y is the whole ring R. They have shown that a commutative ring R is finite if and only if the graph associated with it is finitely colorable. Also, it is proved that the chromatic number of the graph is the sum of the number of maximal ideals and the number of units of R.

H.R. Maimani and others [6] studied a subgraph of a graph introduced in [10]. They studied the connectedness and diameter of the subgraph.

K. Samai [9] studied a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ introduced in [10] with non-unit elements of R as a vertex set and obtained ring, graph as well

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as the topological properties. Also, investigated the diameter, girth, cycles and dominating sets of a subgraph $\Gamma_2(R)$.

In [8], Puranik and others studied an associated graph $\Gamma(M)$ of a le-module ${}_RM$ with all non-zero proper submodule elements of M as vertices. Any two distinct vertices n and m are adjacent if and only if their sum is equal to e, the largest element of ${}_RM$. Also, the Beck's conjecture for $\Gamma(M)$ is established for coatomic le-modules.

In Section 1 we have recalled the definition of le-module and many concepts from le-modules as well as graph theory. In Section 2, we have settled Beck's conjecture for $\Gamma(M)$ which does not contain an infinite clique. Characterized the subgraph $\Gamma_3(M)$ to be complete bipartite if the number of maximal elements is exactly 2 and shown that it is n-partite if the number of maximal elements of M is exactly n. Also, prove that the subgraph $\Gamma_3(M)$ of $\Gamma(M)$ is connected with diameter is at most 3. In Section 3, we have proven that the existence of disjoint closed sets in the maximal spectrum ensures the existence of adjacent elements in the co-maximal graph and vice-versa. Also, it is shown that if the maximal spectrum of multiplication le-modules is Hausdorff, then the diameter of the subgraphs $\Gamma_2(M)$ and $\Gamma_3(M)$ are at least 3.

Definition 1.1. An *le-semigroup* $(M, +, \leqslant, e)$ is a commutative monoid with the zero element 0_M and is a complete lattice with the greatest element e, that satisfies $m + (\vee_{i\in I} m_i) = \vee_{i\in I}(m + m_i)$. Then M is called an *le-module* over a commutative ring R with unity 1_R if there is a mapping $: R \times M \to M$ satisfying:

1. $r(m_1 + m_2) = rm_1 + rm_2$
2. $(r_1 + r_2)m \leqslant r_1m + r_2m$
3. $(r_1r_2)m = r_1(r_2m)$
4. $1_Rm = m$; $0_Rm = r0_M = 0_M$
5. $r(\vee_{i\in I} m_i) = \vee_{i\in I}(rm_i)$ holds for all $r, r_i \in R$, $m, m_i \in M$ and $i \in I$ (I is an indexed set).

An element $n \in M$ is said to be a *submodule element* if $n+n, rn \leqslant n$ for all $r \in R$. The set of all submodule elements of M is denoted by $Sub(M)$.

Observe that if $n, m \in Sub(M)$ then $n + m \in Sub(M)$, $rn \in Sub(M)$, $n \wedge m \in Sub(M)$ and $n + n = n$. Let M be an le-module, $n \in M$ and

I be an ideal in R. Then $In = \vee\{\sum_{i=0}^{k} r_i n : k \in \mathbb{N}; r_i \in I\}$. If for each $n \in Sub(M)$, $n = Ie$ for some ideal I of R, then the le-module M is known as a *multiplication le-module*. An element $m \in Sub(M)$ is said to be *maximal* if $m < n$ for some $n \in Sub(M)$ implies $n = e$. The set of all maximal elements of M is denoted by $Max(M)$. If $l \in Sub(M)$ and $n \in M$, then $(l : n) = \{r \in R : rn \leqslant l\}$ is an ideal in R. If $t \in Sub(M)$ then $Ann(t) = \{r \in R : rt = 0\}$. Note that $Ann(t)$ is an ideal in R. We define radical of an le-module M as $Rad(M) = \wedge_{m \in Max(M)} m$.

A *graph* G is the pair $(V(G); E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The *degree* of a vertex n is denoted by $deg(n)$ and is equal to the number of edges incident on n. In G, the *distance* between two distinct vertices n and m, denoted by $d(n; m)$ is the length of the shortest path between n and m. The *diameter* of a graph G is given by $diam(G) = sup\{d(n; m) | n, m \in V(G)\}$. Graph G is called *connected*, if there is a path between any two vertices of G. The length of the shortest cycle in G is called the *girth* of G. A graph is called *complete* if each pair of vertices in G is adjacent. A *complete* $r-partite$ graph is one in which each vertex is joined to every other vertex not in the same subset. A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$, is called the *clique number of* G. The minimum n for which a graph G is n-colorable is called the *chromatic number* of G, and is denoted by $\chi(G)$.

Proposition 1.2. (cf. [5]) *Let M be an le-module and I be an ideal of R. Then $In \in Sub(M)$ for all $n \in M$ and Rn is the smallest element of $Sub(M)$ covering n i.e. if $l \in Sub(M)$ and $n \leqslant l$, then $n \leqslant Rn \leqslant l$.*

In particular, $Rn = n$ for all $n \in Sub(M)$.

Proposition 1.3. *Let M be a multiplication le-module. If $m \in Max(M)$ and $n_1, n_2, \ldots, n_m \in Sub(M)$ such that $(\wedge_\lambda n_\lambda) \leqslant m$, then there exist some λ such that $n_\lambda \leqslant m$.*

2. Comaximal graph of multiplication le-modules

Let M be an le-module and let $\Gamma(M)$ consist of vertices as elements of M and two distinct elements n, m of $\Gamma(M)$ are adjacent if and only if $Rn+Rm = e$. We denote $U(M) = \{n \in M | Rn = e\}$.

The following theorem shows that the Beck's conjecture is true for $\Gamma(M)$ which does not contain infinite clique.

Theorem 2.4. *Let M be an le-module. If $\Gamma(M)$ does not contain infinite clique, then $\chi(\Gamma(M)) = \omega(\Gamma(M)) = t + s$, where $t = |U(M)|$ and $s = |Max(M)|$.*

Proof. Note that $|U(M)|$ and $|Max(M)|$ are finite, otherwise $\Gamma(M)$ contains infinite clique. Suppose that $U(M) = \{n_1, n_2, \ldots, n_t\}$ and $Max(M) = \{m_1, m_2, \ldots, m_s\}$. Then $C = U(M) \cup Max(M)$ is a clique in $\Gamma(M)$. Then $\chi(\Gamma(M)) \geq t + s$. Let $V_1 = \{m \in M | m \leqslant m_1\}$ and for $i = 1, 2, \ldots, s; V_i = \{m \in M | m \leqslant m_i$ but $m \nleqslant m_j$ for $j = 1, 2, \ldots, i-1\}$. Then $M = U(M) \cup V_1 \cup V_2 \cup \ldots \cup V_s$ is a disjoint union of sets. Define $f : M \to \{1, 2, \ldots, t+s\}$ as $f(n_i) = i$ where $n_i \in U(M)$ and $f(v_j) = t + j$ where $v_j \in V_j$ for $j = 1, 2, \ldots, s$. If $k_1, k_2 \in M$ with $k_1 \neq k_2$ and $Rk_1 + Rk_2 = e$ implies $f(k_1) \neq f(k_2)$. Thus the map f gives colouring implies $\chi(\Gamma(M)) = t + s$. □

In [10] Sharma and Bhatwadekar have shown that, every ring without infinite clique is finite. But the following example illustrates that even an infinite le-module can have a finite clique.

Example 2.5. Let $M = \{a_i | i \in \mathbb{N}\} \cup \{b_i | i \in \mathbb{N}\} \cup \{0, e\}$ is a le-module over $\mathbb{Z}_2$ with $+$ as $a_i + a_j = a_1, b_i + b_j = b_1$ and $a_i + b_j = e$ and scalar multiplication is $0x = 0$ and $1x = x$ for all $x \in M$. By Proposition 1.2, each a_i is adjacent to each b_j, because $Ra_i + Rb_j = a_1 + b_1 = e$.

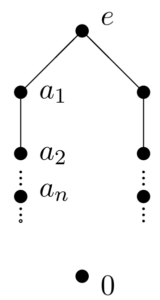

Figure 1 : Lattice of M.

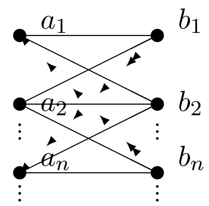

Figure 2 : $\Gamma(M)-$ Comaximal graph of M.

Here $Sub(M) = \{a_1, b_1\}$ and we have only 2 vertices clique because a_i is not adjacent to a_j and b_i is not adjacent to b_j for any $i, j \in \mathbb{N}$.

We consider subgraph $\Gamma_2(M)$ with the vertex set $\{n \in M | n \notin U(M)\}$.

Theorem 2.6. *The graph with the vertex set $U(M)$ is complete. Moreover, $m \leqslant Rad(M)$ if and only if $deg_{\Gamma_2}(m) = 0$, where $deg_{\Gamma_2}(m)$ is a degree of M in a subgraph $\Gamma_2(M)$.*

Proof. 1. Let $m_1, m_2 \in U(M)$. Then $Rm_1 = e$ and $Rm_2 = e$. Consequently, $Rm_1 + Rm_2 = e$ and hence every pair of elements of $U(M)$ are adjacent.

2. Let $m \leqslant Rad(M)$, which implies $m \leqslant m_i$ for all $m_i \in Max(M)$. If $deg_{\Gamma_2}(m) \neq 0$, then there exists $n \in \Gamma_2(M)$ such that $Rn + Rm = e$. Now, there exists $m_j \in Max(M)$ such that $n \leqslant m_j$. Therefore by Proposition 1.2, we have $Rn + Rm \leqslant Rm_j + Rm_j = m_j + m_j = m_j \neq e$, a contradiction. Hence $deg_{\Gamma_2}(m) = 0$.

Conversely, suppose that $deg_{\Gamma_2}(m) = 0$. If $m \nleqslant Rad(M)$, then there exists $m_j \in Max(M)$ such that $m \nleqslant m_j$. Thus $Rm + m_j = Rm + Rm_j = e$, a contradiction to $deg_{\Gamma_2}(m) = 0$. □

We consider subgraph $\Gamma_3(M)$ with the vertex set

$$\{n \in M | n \notin U(M) \text{ and } n \nleqslant Rad(M)\}.$$

Theorem 2.7. *Let M be an le-module. Then $\Gamma_3(M)$ is a complete bipartite if and only if $|Max(M)| = 2$.*

Proof. Let $Max(M) = \{m_1, m_2\}$. Then the vertex set of $\Gamma_3(M) = V_1 \cup V_2$, where

$$V_1 = \{m | m \leqslant m_1 \text{ and } m \nleqslant m_2\} \text{ and } V_2 = \{m | m \leqslant m_2 \text{ and } m \nleqslant m_1\}.$$

Now for $n_1 \in V_1$ and $n_2 \in V_2$ we have $Rn_1 \nleqslant m_2$ and $Rn_2 \nleqslant m_1$. Hence $Rn_i \leqslant Rn_1 + Rn_2 \nleqslant m_i$ for $i = 1, 2$. But $Rn_1 + Rn_2 \in Sub(M)$ and which implies $Rn_1 + Rn_2 = e$. Therefore $\Gamma_3(M)$ is a complete bipartite.

Conversely, suppose that $\Gamma_3(M)$ is a complete bipartite with V_1 and V_2 are two parts. Let $m_1 = \vee\{v_{i_1} | v_{i_1} \in V_1\}$ and $m_2 = \vee\{v_{i_2} | v_{i_2} \in V_2\}$. We first prove that $m_1 \in V_1$. Otherwise, we have following two cases: Let $v_{i_1}, v_{j_1} \in V_1$.

1. If $v_{i_1} \vee v_{j_1} \in U(M)$, then $R(v_{i_1} \vee v_{j_1}) = e$. Now $v_{i_1} \vee v_{j_1} \leqslant v_{i_1} + v_{j_1}$ implies $R(v_{i_1} \vee v_{j_1}) \leqslant R(v_{i_1} + v_{j_1}) = R(v_{i_1}) + R(v_{j_1})$. Therefore $R(v_{i_1} \vee v_{j_1}) = e$ implies $R(v_{i_1}) + R(v_{j_1}) = e$, a contradiction.

2. If $v_{i_1} \vee v_{j_1} \in V_2$, then $R(v_{i_1}) + R(v_{i_1} \vee v_{j_1}) = e$. Now $v_{i_1} \vee v_{j_1} \leqslant v_{i_1} + v_{j_1}$ implies $R(v_{i_1} \vee v_{j_1}) \leqslant R(v_{i_1} + v_{j_1}) = R(v_{i_1}) + R(v_{j_1})$. Therefore $R(v_{i_1}) + R(v_{i_1} \vee v_{j_1}) = e$ implies $R(v_{i_1}) + R(v_{i_1}) + R(v_{j_1}) = e$. Therefore, $R(v_{i_1}) + R(v_{j_1}) = e$, a contradiction.

Hence $m_1 \in V_1$ and similarly we have $m_2 \in V_2$. Since $m_1 \in V_1$, we have $Rm_1 \neq e$ and also $Rm_1 + Rv_{i_1} = Rm_1 \neq e$ implies $Rm_1 \notin V_2$. Similarly we have $Rm_2 \notin V_1$. If $n \in Max(M)$ then $n \leqslant m_1$ or $n \leqslant m_2$. Otherwise $Rn + Rm_1 = e$ and $Rn + Rm_2 = e$, which is a contradiction to $\Gamma_3(M)$ is a complete bipartite. □

Proposition 2.8. *Let M be an le-module and $n > 1$.*

1. *If $|Max(M)| = n < \infty$, then $\Gamma_3(M)$ is an n-partite.*
2. *If $\Gamma_3(M)$ is an n-partite, then $|Max(M)| \leqslant n$ and if $\Gamma_3(M)$ is not an $(n-1)$-partite, then $|Max(M)| = n$.*

Proof. 1. Let $Max(M) = \{m_1, m_2, \dots, m_n\}$. Take $V_1 = \{m \in \Gamma_3(M) | m \leqslant m_1\}$ and $V_i = \{m \in \Gamma_3(M) | m \leqslant m_i$ and $m \nleqslant m_j$ for $j = 1, 2, \dots, i-1\}$ for $i = 2, 3, \dots, n$. If $m_{i_1}, m_{i_2} \in V_i$, then $Rm_{i_1} + Rm_{i_2} \leqslant Rm_i + Rm_i = m_i + m_i = m_i < e$. Thus m_{i_1} and m_{i_2} are not adjacent. Similarly no two elements of V_1 are adjacent. Therefore, $\Gamma_3(M)$ is n-partite.

2. Suppose that $\Gamma_3(M)$ is $n-partite$ graph. Let $V_1, V_2, \dots, V_n$ be the n parts of vertices of $\Gamma_3(M)$. Suppose that $|Max(M)| > n$. Let $\{m_1, m_2, \dots, m_{n+1}\} \subseteq Max(M)$. Let $t_i \leqslant m_i$ but $t_i \nleqslant m_j$ for $i \neq j$. Note that $Rt_i + Rt_j \geqslant t_i, t_j$. If $Rt_i + Rt_j \neq e$ then $Rt_i + Rt_j \leqslant m_k$ for some $m_k \in Max(M)$. Therefore $t_i, t_j \leqslant m_k$, a contradiction. Hence $Rt_i + Rt_j = e$. Therefore $\{t_1, t_2, \dots, t_{n+1}\}$ is a clique in $\Gamma_3(M)$. As we have $V_1, V_2, \dots V_n$ are n parts of vertices of $\Gamma_3(M)$ and $\{t_1, t_2, \dots, t_{n+1}\}$ is a clique in $\Gamma_3(M)$, by the Pigeonhole principle two $t_i \in V_i$ for some i, a contradiction. Therefore $|Max(M)| \leqslant n$.

Now, if $\Gamma_3(M)$ is not $(n-1)$-partite and if $|Max(M)| = s < n$, then by part (1), $\Gamma_3(M)$ is s-partite, a contradiction. Hence $|Max(M)| = n$. □

Theorem 2.9. *Let M be a multiplication le-module and $|Max(M)| \geqslant 2$. If $\Gamma_3(M)$ is a complete n-partite, then $n = 2$.*

Proof. Suppose that $\Gamma_3(M)$ is a complete n-partite. For $m_1, m_2 \in Max(M)$, let $V_1 = \{m \in \Gamma_3(M) | m \leqslant m_1$ and $m \nleqslant m_2\}$ and $V_2 = \{m \in \Gamma_3(M) | m \leqslant m_2$ and $m \nleqslant m_1\}$. Observe that the elements of V_i are not adjacent for $i = 1, 2$ and every element of V_1 is adjacent to each element of V_2. Since $\Gamma_3(M)$ is a complete n-partite graph implies V_1 and V_2 are two parts of $\Gamma_3(M)$. Now, we claim that $Rad(M) = m_1 \wedge m_2$. Suppose that $Rad(M) < m \leqslant m_1 \wedge m_2$ for some $m \in M$. This implies m is not adjacent to any element of V_1 and of V_2. This is contradiction to $\Gamma_3(M)$ is complete n-partite. Therefore $Rad(M) = m_1 \wedge m_2$ and for any $m_3 \in Max(M)$, we have $m_1 \wedge m_2 \wedge m_3 = m_1 \wedge m_2$. Which implies $m_1 \wedge m_2 \leqslant m_3$. Then by Propostion 1.3, we have $m_1 \leqslant m_3$ or $m_2 \leqslant m_3$. As $m_1, m_2, m_3 \in Max(M)$, implies $m_1 = m_3$ or $m_2 = m_3$ and therefore $|Max(M)| = 2$. Hence by Theorem 2.7, $\Gamma_3(M)$ is a complete bipartite. □

Theorem 2.10. *If M is a multiplication le-module, then $\Gamma_3(M)$ is connected and $diam(\Gamma_3(M)) \leqslant 3$.*

Proof. Let $n, l \in \Gamma_3(M)$. Then we consider the following two cases:

1. Suppose that $n \wedge l \nleqslant Rad(M)$. Then $n \wedge l \nleqslant m$ for some $m \in Max(M)$. Hence, $R(n \wedge l) + Rm = e$ and which implies $Rn + Rm = e$ and $Rl + Rm = e$. Therefore $n - m - l$ is a path and so $d(n, m) \leqslant 2$.

2. Suppose that $n \wedge l \leqslant Rad(M)$. Let $S_n = \{m \in Max(M) | n \leqslant m\}$ and $S_l = \{m \in Max(M) | l \leqslant m\}$ implies $Max(M) = S_n \cup S_l$. Because if there exist $m_0 \in Max(M)$ such that $m_0 \notin S_m$ and $m_0 \notin S_n$, then $n \wedge l \leqslant m_0$ implies $Rn \wedge Rl \leqslant m_0$. Suppose n is adjacent to t in $\Gamma_2(M)$. Then $t \nleqslant Rad(M)$. If $n \leqslant m_1$, then $t \nleqslant m_1$ and so $t \leqslant m_2$ for some $m_2 \in S_l - S_n$. If $Rt \wedge Rl \leqslant Rad(M)$ then by Proposition 1.3, $Rt \leqslant Rad(M)$ or $Rl \leqslant Rad(M)$. But $l \nleqslant m$ for some $m \in S_n$ implies $Rl \nleqslant m$ for some $m \in S_n$ and therefore$Rl \nleqslant Rad(M)$. Similarly $Rt \nleqslant Rad(M)$. Hence $Rt \wedge Rl \nleqslant Rad(M)$. Therefore by Case(i), there exists a path between Rt and Rl and $d(Rt, Rl) \leqslant 2$. Suppose $Rt - m - Rl$ is a path for some $m \in M$ and hence $n - Rt - m - l$ is a path between n and l. Consequently, $d(n, l) \leqslant 3$. □

3. Maximal spectrum and comaximal graph

In [5], Kumbhakar and Bhuniya, studied the Zariski topology on le-modules. They have defined $V(n) = \{p \in Spec(M) | n \leqslant p\}$ and $V^*(n) = \{p \in Spec(M) | (p : e) \subseteq (n : e)\}$ for $n \in Sub(M)$. If M is a multiplication le-module, then $\{V(n) | n \in Sub(M)\}$ forms the Zarisky topology of closed sets on the prime spectrum $Spec(M)$.

Throughout this section, M denotes a multiplication le-module unless otherwise stated.

Here, we consider $Max(M) = \{m \in Sub(M) | m \text{ is maximal element}\}$ as a subset of $Spec(M) = \{p \in Sub(M) | p \text{ is prime element}\}$ with the subspace topology.

Thus, if $M(t) = \{m \in Max(M) | t \leqslant m\}$, then $T = \{M(t) | t \in Sub(M)\}$ forms a basis of closed subsets on $Max(M)$.

Lemma 3.11. *Let M be a multiplication le-module. If A and B are disjoint closed subsets of $Max(M)$, then there exist $t_1, t_2 \in Sub(M)$ such that $A = M(t_1), B = M(t_2)$ and $Rt_1 + Rt_2 = e$. Also if A is closed and open set, then there exist $t_1, t_2 \in Sub(M)$ such that $Rt_1 + Rt_2 = e$ and $t_1 \wedge t_2 \leqslant Rad(M)$.*

Proof. If A and B are closed sets implies there exist $t_1, t_2 \in Sub(M)$ such that $A = M(t_1), B = M(t_2)$. We have $t_1 \leqslant Rt_1, t_2 \leqslant Rt_2$ and therefore $t_1 \leqslant Rt_1 + Rt_2$ and $t_2 \leqslant Rt_1 + Rt_2$. If $Rt_1 + Rt_2 \neq e$, then such that $Rt_1 + Rt_2 \leqslant m$ for some $m \in Max(M)$. But $t_1, t_2 \leqslant Rt_1 + Rt_2 \leqslant m$ and this implies $m \in M(t_1) \cap M(t_2) = A \cap B$, a contradiction. Consequently $Rt_1 + Rt_2 = e$.

Now, if A is both closed and open, then A and A^c are closed sets. Therefore by above argument there exist $t_1, t_2 \in Sub(M)$ such that $A = M(t_1), A^c = M(t_2)$ and $Rt_1 + Rt_2 = e$. Now we have $t_1 \leqslant m_1$ for all $m_1 \in A$ and $t_2 \leqslant m_2$ for all $m_2 \in A^c$. This implies $t_1 \wedge t_2 \leqslant m_1$ for all $m_1 \in A$ and $t_1 \wedge t_2 \leqslant m_2$ for all $m_2 \in A^c$. Therefore $t_1 \wedge t_2 \leqslant m$ for all $m \in Max(M)$. This implies $t_1 \wedge t_2 \leqslant Rad(M)$. □

Remark 3.12. The existence of disjoint closed subsets in the maximal spectrum gives the existence of adjacent elements in the comaximal graph.

Proposition 3.13. *Let $n_1, n_2, n_3 \in \Gamma_3(M)$ be distinct elements and let $D(t) = Max(M)/M(t)$. Then*

(1) *n_1 is adjacent to n_2 and n_3 if and only if $M(Rn_1) \subseteq D(Rn_2 \wedge Rn_3)$.*

(2) *$d(n_1, n_2) = 1$ if and only if $M(Rn_1) \cap M(Rn_2) = \emptyset$.*

(3) *$d(n_1, n_2) = 2$ if and only if $M(Rn_1) \cap M(Rn_2) \neq \emptyset$ and $Rn_1 \wedge Rn_2 \nleqslant Rad(M)$.*

(4) *$d(n_1, n_2) = 3$ if and only if $M(Rn_1) \cap M(Rn_2) \neq \emptyset$ and $Rn_1 \wedge Rn_2 \leqslant Rad(M)$.*

Proof. (1). Suppose that $M(Rn_1) \subseteq D(Rn_2 \wedge Rn_3)$. This implies $Rn_1 + (Rn_2 \wedge Rn_3) = e$. Therefore, $Rn_1 + Rn_2 = e$ and $Rn_1 + Rn_3 = e$. Thus n_1 is adjacent to both n_2 and n_3.

Conversely, suppose that n_1 is adjacent to both n_2 and n_3. Therefore $Rn_1 + Rn_2 = e$ and $Rn_1 + Rn_3 = e$, which implies $M(Rn_1) \cap M(Rn_2) = \emptyset$ and $M(Rn_1) \cap M(Rn_3) = \emptyset$. On contrary, if there exist $m \in M(Rn_1)$ and $m \notin D(Rn_2 \wedge Rn_3)$, then $Rn_2 \wedge Rn_3 \leqslant m$, and by Proposition 1.3, we have $Rn_2 \leqslant m$ or $Rn_3 \leqslant m$. Hence we have $m \in M(Rn_2)$ or $m \in M(Rn_3)$ and consequently $m \notin M(Rn_1)$, a contradiction.

(2). $d(n_1, n_2) = 1$ if and only if $Rn_1 + Rn_2 = e$ if and only if $M(Rn_1) \cap M(Rn_2) = \emptyset$.

(3). Suppose that,$d(n_1, n_2) = 2$. Which implies $Rn_1 + Rt = e$ and $Rn_2 + Rt = e$ for some $t \in M$. Note that t is adjacent to both n_1 and n_2 and hence by (i) above we have $M(Rt) \subseteq D(Rn_1 \wedge Rn_2)$. Thus $m \in M(Rt)$ implies $m \notin M(Rn_1 \wedge Rn_2)$. Hence $Rn_1 \wedge Rn_2 \nleqslant Rad(M)$. Conversely, suppose that $M(Rn_1) \cap M(Rn_2) \neq \emptyset$ and $Rn_1 \wedge Rn_2 \nleqslant Rad(M)$. Thus there exists $m \in Max(M)$ such that $Rn_1 \wedge Rn_2 \nleqslant m$ implies $Rn_1 + m = Rn_1 + Rm = e$ and $Rn_2 + m = Rn_2 + Rm = e$. Therefore $n_1 - m - n_2$ is a shortest path and which implies $d(n_1, n_2) = 2$.

(4) Follows from $(2), (3)$ and Theorem 2.10. □

Theorem 3.14. *Let M be a multiplication le-module with $Max(M)$ is Hausdorff. Then $diam(\Gamma_3(M)) = min\{|Max(M)|, 3\}$. If $|Max(M)| = 2$, then $gr(\Gamma_3(M)) = 4$ or ∞ otherwise $gr(\Gamma_3(M)) = 3$.*

Proof. First we prove that $|Max(M)| \geqslant 3$ if and only if $diam(\Gamma_3(M)) = 3$. Suppose that $|Max(M)| \geqslant 3$ and m_1, m_2, m_3 are distinct maximal elements in M. Since $Max(M)$ is Hausdorff, there are $t_i \in Sub(M)$ such that $m_i \in D(t_i)$ and $D(t_i) \cap D(t_j) = \emptyset$ for $i \neq j$. Thus $D(t_i) \subseteq M(t_j)$ for $i \neq j$. Now $D(t_i) \cup M(t_i) = Max(M)$ implies $M(t_i) \cup M(t_j) = Max(M)$. Hence $t_i \wedge t_j \leqslant m$ for all $m \in Max(M)$ implies $t_i \wedge t_j \leqslant Rad(M)$. Now $m_3 \in M(t_1) \cap M(t_2)$ implies $M(t_1) \cap M(t_2) \neq \emptyset$. Therefore by the Proposition 3.13, $d(t_1, t_2) = 3$ implies $diam(\Gamma_3(M)) = 3$.

Conversely, suppose that $diam(\Gamma_3(M)) = 3$. On contrary if $|Max(M)| < 3$, then either $|Max(M)| = 1$ or 2. The case $|Max(M)| = 1$ is not possible, because then $\Gamma_3(M)$ will contain only one vertex, a contradiction to $diam(\Gamma_3(M)) = 3$. Now suppose that $Max(M) = \{m_1, m_2\}$ and for vertices n_1, n_2 we have $d(n_1, n_2) = 3$. Hence there are vertices t_1, t_2 such that $n_1 - t_1 - t_2 - n_2$ is a shortest path between n_1 and n_2. If $n_1 \leqslant m_1$ then $t_1 \leqslant m_2$ implies $t_2 \leqslant m_1$ and hence $n_2 \leqslant m_2$. This gives a contradiction, because n_1 and n_2 are not adjacent. Similarly $n_2 \leqslant m_2$ is not possible. Therefore $|Max(M)| \geqslant 3$.

Now let $|Max(M)| = 2$. Then $Max(M) = \{m_1, m_2\}$ and $Max(M)$ is Hausdorff implies there exist $t_1, t_2 \in Sub(M)$ with $M(t_1) = \{m_1\}$ and $M(t_2) = \{m_2\}$. Therefore, we have $t_1 + t_2 = Rt_1 + Rt_2 = m_1 + m_2 = e$ and we have shortest cycle of length 4 namely $t_1 - t_2 - m_1 - m_2 - t_1$. If $t_1, t_2, t_3 \in \Gamma_3(M)$, then by the Pigeonhole Principle at least two of them $\leqslant m_1$ or m_2. Therefore there is no triangle in $\Gamma_3(M)$. If $|M_{m_1}| = 2$ or $|M_{m_2}| = 2$ then $t \leqslant m_1$ implies $t = 0$ or $t = m_1$ for $|M_{m_1}| = 2$. Hence in this case we have no cycle implies $gr(\Gamma_3(M)) = \infty$. □

Corollary 3.15. *Let M be a multiplication le-module with $Max(M)$ is*

Hausdorff. Then $diam(\Gamma_2(M)) = min\{|Max(M)|, 3\}$. *If* $|Max(M)| = 2$, *then* $gr(\Gamma_2(M)) = 4$ *or* ∞ *otherwise* $gr(\Gamma_2(M)) = 3$.

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