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Injective and projective poset acts

Leila Shahbaz

Dedicated to my mother Malak

Abstract. In this paper, after recalling the category **PosAct**-S of all poset acts over a pomonoid S; an S-act in the category **Pos** of all posets, with action preserving monotone maps between them, some categorical properties of the category **PosAct**-S are considered. In particular, we describe limits and colimits such as products, coproducts, equalizers, coequalizers and etc. in this category. Also, several kinds of epimorphisms and monomorphisms are characterized in **PosAct**-S. Finally, we study injectivity and projectivity in **PosAct**-S with respect to (regular) monomorphisms and (regular) epimorphisms, respectively, and see that although there is no non-trivial injective poset act with respect to monomorphisms, **PosAct**-S has enough regular injectives with respect to regular monomorphisms. Also, it is proved that regular injective poset acts are exactly retracts of cofree poset acts over complete posets.

1. Introduction

General ordered algebraic structures play an important role in mathematics and other mathematical areas such as analysis, logic, and theoretical computer science. Combining the notions of a poset and an act, we get a special kind of these structures, namely S-poset, poset on which the actions of a pomonoid S preserve the order. Fakhruddin in the 1980s (see [10] and [11]), has done the preliminary work on properties of S-posets and many researchers continued in recent papers [4, 6, 7, 8, 13, 14, 15, 17, 18].

The category of poset acts with action preserving monotone maps between them, first has been introduced and studied by Skornyakov in [19], [20] and continued by Shahbaz in [16], where congruences in this category are introduced and characterized and adjoint relations between this category and the categories **Pos** of posets, **Act**-S of S-acts, and **Set** of sets, are discussed.

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The main objective of this paper, is to study some categorical ingredients of the category of poset acts. In particular, limits and colimits such as products and coproducts in this category are described. Several kinds of epimorphisms and monomorphisms are characterized in **PosAct**-S. Also, it is proved that regular injective poset acts are exactly retracts of cofree poset acts over complete posets.

First note that the category of all partially ordered sets (posets) with order preserving (monotone) maps between them is denoted by **Pos**. A poset is said to be *complete* if each of its subsets has an infimum and a supremum.

Let S be a monoid with 1 as its identity. A *right S-act* is a set A equipped with an action $\lambda : A \times S \to A, (\lambda(a, s)$ is denoted by $as)$ such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. An *S-map* $f : A \to B$ between S-acts is an action preserving map, that is $f(as) = f(a)s$ for each $a \in A, s \in S$. The category of all S-acts and S-maps between them is denoted by **Act**-S.

Recall that a monoid (semigroup) S is said to be a *pomonoid* (*posemigroup*) if it is also a poset whose partial order $\leqslant$ is compatible with its binary operation (that is, $s \leqslant t$, $s' \leqslant t'$ imply $ss' \leqslant tt'$).

A *right S-poset* over a pomonoid S is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. An *S-poset map* (or *morphism*) is an action preserving monotone map between S-posets. Moreover, regular monomorphisms (equalizers) are exactly *order embeddings*; that is, morphisms $f : A \to B$ for which $f(a) \leqslant f(a')$ if and only if $a \leqslant a'$, for all $a, a' \in A$. The category of all S-posets and S-poset maps between them is denoted by **Pos**-S.

A *poset act* over a pomonoid S is a poset A together with a mapping $A \times S \to A, (a, s) \mapsto as$ such that

1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leqslant a'$ implies $as \leqslant a's$ for every $a, a' \in A$ and $s, t \in S$.

This makes a poset act an ordered algebra in the sense of [3], where all operations R_s are unary.

By a *poset act map* between poset acts, we mean an order preserving map which is also an S-map.

The category of all poset acts with action-preserving monotone maps between them is denoted by **PosAct**-S. It is easily seen that the category **Pos**-S is a full subcategory of **PosAct**-S.

Note that each S-poset is a poset act but the converse is not true generally. For example, let $G = \{0, 1\}, 00 = 11 = 1, 01 = 10 = 0, 0 < 1$ be the two element pogroup and $A = \{a, b, c\}$ with the order $b < c$ be a poset. Then with the action $a0 = b0 = c0 = a, a1 = a, b1 = b, c1 = c$, A becomes a poset act which is not an S-poset. This example shows that even when S is a pogroup the notions of S-poset and poset act are not the same.

If A is a poset act, a *congruence* θ on A is an equivalence relation on A that is compatible with the S-action, and has the further property that A/θ can be equipped with a partial order so that A/θ is a poset act and the natural map $A \to A/\theta$ is a poset act morphism.

Recall that if θ is any binary relation on A, we write $a \leqslant_\theta a'$ if a so-called θ-chain

$$a \leqslant a_1 \theta a'_1 \leqslant a_2 \theta a'_2 \leqslant \cdots \leqslant \theta a'_m \leqslant a'$$

from a to a' exists in A. Congruences on poset acts are characterized the same as congruences on S-posets.

We recall the following results from [16].

Theorem 1.1. *The functor K' : $\mathbf{PosAct} - S \longrightarrow \mathbf{Pos} - S$, given by $K'(A) = A^{(S)}$ of all monotone maps from S to A, with pointwise order and action given by $(fs)(t) = f(st)$ for $s, t \in S$ and $f \in A^{(S)}$, is a right adjoint to the inclusion functor $i : \mathbf{Pos} - S \longrightarrow \mathbf{PosAct} - S$.*

Proposition 1.2. *The (free) functor F'_1 : $\mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$ given by $F'_1(P) = P \times S$ is a left adjoint to the forgetful functor $U'_1 : \mathbf{PosAct} - S \longrightarrow$ $\mathbf{Pos}$.*

Proposition 1.3. *The (cofree) functor K'_1 : $\mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$, given by $K'_1(P) = P^S$, is the right adjoint to the forgetful functor $U'_1 : \mathbf{PosAct} - S \longrightarrow \mathbf{Pos}$.*

Theorem 1.4. *The functor H' : $\mathbf{PosAct} - S \longrightarrow \mathbf{Pos}$ given by $H'(A) = A/\nu(W)$ where $\nu(W)$ is the poset congruence induced on the poset act A by the set $W = \{(a, as) : a \in A, s \in S\}$ is the left adjoint of the functor $G' : \mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$ (that equips a poset with the trivial action).*

2. Limits and colimits in PosAct-S

The category **PosAct**-S is complete and cocomplete. In fact, having the adjunctions given in Section 1, and the fact that right adjoints preserve limits and left adjoints preserve colimits, we get that limits and colimits such as products, coproducts, (sub)equalizers, (sub)coequalizers, (sub)pullbacks, and (sub)pushouts in **PosAct**-S exist and are computed the same as the category of S-posets.

The product of a family of poset acts is their cartesian product, with componentwise action and order. The coproduct is their disjoint union, with natural action and componentwise order. In particular, the terminal poset act is the singleton poset act, and the initial poset act is empty.

The equalizer of a pair $f, g : A \to B$ of poset act maps is given by $E = \{a \in A : f(a) = g(a)\}$ with action and order inherited from A. The coequalizer of the pair above is the quotient of B by the congruence $\theta(H)$ generated by $H = \{(f(a), g(a)) : a \in A\}$.

The pullback of poset act maps $f : A \to C$ and $g : B \to C$ is the sub poset act $P = \{(a, b) : f(a) = g(b)\}$ of $A \times B$, together with the restricted projection maps. The pushout of poset act maps $f : A \to B$ and $g : A \to C$ is the quotient of the coproduct

$$B \sqcup C = (\{1\} \times B) \cup (\{2\} \times C)$$

by the congruence $\theta(H)$ generated by $H = \{((1, f(a)), (2, g(a))) : a \in A\}$.

By substituting "$=$" in the definition of the equalizer E above (pullback P above) by "$\leqslant$", the subequalizer (subpullback) of f and g is obtained. (To ensure that all (sub)equalizers and (sub)pullbacks exist, it is assumed that poset acts may be empty.) Also, by substituting "$\theta(H)$" in the definition of coequalizer (pushout) by "$\nu(H)$", the subcoequalizer (subpushout) is obtained.

3. Injectivity and regular injectivity in PosAct-S

In this section, we first study monomorphisms and regular monomorphisms and show that monomorphisms in **PosAct**-S are exactly one-one morphisms and regular monomorphisms in **PosAct**-S are exactly order embeddings. Then recalling the fact that the category **Pos** does not have any non-trivial (non-singleton) injective object with respect to monomorphisms, we see that

PosAct-S has no non-trivial injective object, too. Then we study regular injectivity, that is, injectivity with respect to regular monomorphisms.

3.1 Monomorphisms and regular monomorphisms

First, we show that the monomorphisms in **PosAct**-S are just the injective poset act maps. Note that a *monomorphism* in **PosAct**-S is a morphism that is left cancelable under composition.

Theorem 3.1. *Monomorphisms in* **PosAct**-S *are exactly one-one monotone S-maps.*

Proof. Having the adjunction given in Proposition 1.2, and the fact that right adjoints preserve limits, and in particular monomorphisms, we get that monomorphisms in **PosAct**-S are exactly monotone S-maps which are monomorphisms in **Pos**. Then the result follows by the fact that monomorphisms in **Pos** are exactly one-one morphisms by [5], Lemma 1. $\square$

Notice that poset act order embeddings are injective, but the converse is not true. For example, the identity map from the discrete two element set $\mathbf{1} \sqcup \mathbf{1} = \{0, 1\}$ onto the two element chain $\mathbf{2} = \{0, 1\}$ with $0 < 1$, both considered as poset acts over a one-element pomonoid, is a monomorphism but it is not an order embedding.

Recall (see [1] for example) that a monomorphism f is called *regular* if it is the equalizer of a pair of morphisms, and f is *extremal* if, whenever $f = h \circ g$ and g is an epimorphism, then g is an isomorphism. Also, a poset act map $f : A \to B$ is called an order embedding if $f(a) \leqslant f(a')$ implies $a \leqslant a'$, for all $a, a' \in A$.

Similar to the case for S-posets, one can show that the classes of regular and extremal monomorphisms coincide with each other, and in fact are exactly poset act order embeddings (see [8]).

Theorem 3.2. *For a monomorphism $h : A \to B$ in* **PosAct**-S*, the following are equivalent:*

(i) *h is regular,*

(ii) *h is extremal,*

(iii) *h is an order embedding.*

Proof. (i) $\Rightarrow$ (ii) This is a general category theoretic result.

(ii) $\Rightarrow$ (iii) Suppose h is extremal. Let A' denote the sub poset act with universe A, equipped with the order $\ll$ given by

$$a \ll a' \Leftrightarrow h(a) \leqslant h(a').$$

The identity map $\mathrm{id}_A : A \to A'$, and the map $h' = h : A' \to B$ are poset act maps and $h = h' \circ \mathrm{id}_A$. Since id_A is an epimorphism and h is extremal, it follows that id_A is an isomorphism and therefore, h is an order embedding.

(iii) $\Rightarrow$ (i) Since h is an order embedding, the image of h is a poset act map isomorphic to A. So, it is enough to prove that in the case where h is the inclusion map from a sub poset act A into B, h is a regular monomorphism. Consider the amalgamated coproduct $B \sqcup_A B$ and the morphisms $g_1, g_2 : B \to B \sqcup_A B$. Then h is obviously the equalizer of g_1 and g_2. □

Definition 3.3. A poset act monomorphism is called *subregular* if it is the subequalizer of a pair of poset act maps.

As in the case of equalizers, one can prove that a subequalizer is always a monomorphism. Also, it is shown that in **PosAct**-S, not all monomorphisms are subregular. In fact, by showing that subregular monomorphisms are exactly order embedding morphisms and then applying Theorem 3.2, it is shown that, in **PosAct**-S, the regular monomorphisms coincide with the subregular monomorphisms.

Theorem 3.4. *A poset act monomorphism $h : A \to B$ is subregular if and only if it is an order embedding.*

Proof. Let h be the subequalizer of $f, g : B \to C$. Define a new order $\ll$ on A by

$$a \ll a' \Leftrightarrow h(a) \leqslant h(a').$$

One can show that $\ll$ is a partial order on A, by using the fact that h is injective. Denoting the corresponding poset act by A', then $h' = h : A' \to B$ is clearly a poset act map. Also, $f \circ h' \leqslant g \circ h'$. So, by the universal property of subequalizers, there exists a unique poset act map $k : A' \to A$ such that $h \circ k = h'$. Using the fact that h is injective, it is concluded that k is the identity map on A. Then, k being monotone means that h is an order embedding, as required.

Conversely, suppose that h is an order embedding. Without loss of generality, assume that A is a sub poset act of B. Consider the poset act

$\{x, y\} \times B$, with action on the right-hand component, and partial order $\ll$ given by

$$(i, b) \ll (j, b') \Leftrightarrow (i = j, b \leqslant b') \ \text{ or } (i < j, b \leqslant a \leqslant b', \ \text{ for some } a \in A),$$

where $\{x, y\}$ is ordered so that $x < y$. Then, the natural injection maps $g_x, g_y : B \to \{x, y\} \times B$ $(g_i(b) = (i, b))$ are poset act maps, and the natural embedding of A into B is their subequalizer: if $g_x(b) \ll g_y(b)$, then $(x, b) \ll (y, b)$, and so there exists $a \in A$ such that $b \leqslant a \leqslant b$, i.e. b belongs to A. $\square$

3.2 Injective and regular injective poset acts

First recall the following lemma from [9] which will be useful in the sequel.

Lemma 3.5. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors such that $F \dashv G$. Also, let $\mathcal{M}, \mathcal{M}'$ be certain subclasses of morphisms of $\mathcal{C}, \mathcal{D}$, respectively. If for all $f \in \mathcal{M}$, $Ff \in \mathcal{M}'$, then for any $\mathcal{M}'$-injective object $D \in \mathcal{D}$, GD is an $\mathcal{M}$-injective object of $\mathcal{C}$.*

As a consequence of Proposition 1.2, Lemma 3.5, and the fact that the free functor $F_1' : \mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$ preserves monomorphisms, we get the following result.

Theorem 3.6. **PosAct**-*S has no non-trivial injective object.*

We now study regular injectivity of poset acts. Recall that a regular injective object in **Pos**-S has zero bottom and top elements. As for poset acts, we have

Proposition 3.7. *Every non-trivial (non-singleton) regular injective poset act A is bounded by two zero elements.*

Theorem 3.8. *If A_S is a regular injective poset act then the S-poset $A^{(S)}$ is regular injective.*

Proof. Having the adjunction given in Theorem 1.1, Lemma 3.5, and the fact that the inclusion functor $i : \mathbf{Pos} - S \longrightarrow \mathbf{PosAct} - S$ preserves regular monomorphisms, we get the result. $\square$

In the following we give answer to the question that does **PosAct**-S have enough regular injectives? That is for any $A \in \mathbf{PosAct} - S$, does there exist a regular injective $E \in \mathbf{PosAct} - S$ with a regular monomorphisms $A \to E$? First recall the following:

Proposition 3.9 ([5]). *Regular injective posets are exactly complete posets.*

Now applying Lemma 3.5 to the adjoint pairs in Proposition 1.3, we have the following two results.

Proposition 3.10. *If A is regular injective in* **Pos** *then the cofree poset act A^S is regular injective in* **PosAct**-*S.*

Theorem 3.11. *Each poset act can be regularly embedded into a regular injective poset act.*

Proof. Let A be a poset act. Recall that A, as a poset, can be regularly embedded into a complete poset $\overline{A}$ (see [5]). Also, by Propositions 3.9 and 3.10, $\overline{A}^S$ is a regular injective poset act. Now, it is straightforward to see that the map $\varphi : A \to \overline{A}^S$, given by $a \mapsto \varphi_a$ with $\varphi_a : S \to \overline{A}$ defined by $\varphi_a(s) = \downarrow(as)$, is the required order embedding poset act map. □

Theorem 3.12. *A poset act A is regular injective if and only if every regular embedding $A \to B$ has a left inverse.*

Proof. ($\Leftarrow$) Take a regular embedding $h : B \to C$ and a poset act map $f : B \to A$. By Theorem 3.11, A can be regularly embedded into the regular injective poset act $\overline{A}^S$ via φ. Then, since $\overline{A}^S$ is regular injective, there exists a morphism $k : C \to \overline{A}^S$ such that $kh = \varphi f$. Also, by hypothesis, there exists a retraction $l : \overline{A}^S \to A$. Now, $g = lk$ is a poset act map with $gh = lkh = l\varphi f = \mathrm{id}_A f = f$. The other direction is clear. □

As a corollary of Proposition 3.10 we give an explicit characterization of regular injective poset acts.

Theorem 3.13. *A poset act A is regular injective if and only if it is a retract of a cofree poset act over a complete poset.*

Proof. ($\Rightarrow$) Follows from Theorem 3.12.

($\Leftarrow$) Note that, by Propositions 3.9 and 3.10, a cofree poset act over a regular injective poset is regular injective. Also, it is straightforward that a retract of a regular injective poset act is regular injective and so we get the result. □

The following proves the converse of Proposition 3.10. A poset act is called *complete* if it is complete as a poset.

Theorem 3.14. *A poset act A is complete if and only if A^S is a regular injective poset act.*

Proof. One part was proved in Proposition 3.10. To prove the only if part, let A^S be a regular injective poset act. But, A is a retract of A^S. Now, A being a retract of a regular injective poset act is clearly regular injective and so it is complete. □

4. Projectivity and regular projectivity in PosAct-S

First, we show that the epimorphisms in **PosAct**-S are just the surjective poset act maps.

Theorem 4.1. *Epimorphisms in* **PosAct**-S *are exactly surjective monotone S-maps.*

Proof. Having the adjunction given in Proposition 1.3, and the fact that left adjoints preserve colimits, and in particular epimorphisms, we get that epimorphisms in **PosAct**-S are exactly monotone S-maps which are epimorphisms in **Pos**. Then the result follows by the fact that epimorphisms in **Pos** are exactly surjective morphisms by [5], Lemma 1. □

Recall from [1] that a poset act epimorphism g is called *regular* if it is the coequalizer of a pair of poset act maps, and it is called *extremal* if $g = h \circ f$, where h, f are poset act maps and h is a monomorphism, implies that h is an isomorphism. It is proved that not all epimorphisms in **PosAct**-S are regular. By a similar proof for S-posets it can be shown that the classes of regular and extremal epimorphisms coincide with each other.

Theorem 4.2. *For an epimorphism $g : A \to B$ in* **PosAct**-S*, the following assertions are equivalent:*

(i) *g is regular,*

(ii) *g is extremal,*

(iii) *if $b \leqslant b'$ in B, then there exist $a_1, a'_1, \cdots, a_n, a'_n \in A$ such that*

$$\begin{array}{cccc} b = g(a_1) & g(a'_1) = g(a_2) & \cdots & g(a'_n) = b'; \\ & a_1 \leqslant a'_1 \quad a_2 \leqslant a'_2 & \cdots & a_n \leqslant a'_n. \end{array}$$

An object P in a category is called projective (regular projective) if given any epimorphism (regular epimorphism) $g : A \to B$ and any morphism $f : P \to B$ there exists a morphism $h : P \to A$ such that $gh = f$.

Now, recall the following proposition from [8] which is needed in the sequel.

Proposition 4.3. *Let P be a poset. The following assertions are equivalent:*
(i) *P is projective,*
(ii) *P is regular projective,*
(iii) *P has discrete order.*

Also, we recall the following fact from category theory (see [2], in which the dual result appears).

Lemma 4.4. *A left adjoint preserves (regular) projectivity if its right adjoint preserves (regular) epimorphisms.*

Lemma 4.5. *The functor $K_1' : \mathbf{Pos} \longrightarrow \mathbf{PosAct} - S$, given in Proposition 1.3, preserves (regular) epimorphisms.*

Proof. Let $f : P \to Q$ be a regular epimorphism. It is clear that $K_1'(f) : P^S \to Q^S$ given by $\bar{f}(h) = K_1'(f)(h) = fh, h \in P^S$ is a surjective monotone map. Now, let $h_1 \leqslant h_2$. Then for all $s \in S, h_1(s) \leqslant h_2(s) \in Q$. Since f is a regular epimorphism, there exist $a_{1s}, a_{2s}, \cdots, a_{ns}, a'_{1s}, \cdots, a'_{ns} \in A$ such that

$$\begin{array}{cccc} h_1(s) = f(a_{1s}) & f(a'_{1s}) = f(a_{2s}) & \dots & f(a'_{ns}) = h_2(s); \\ a_{1s} \leqslant a'_{1s} & a_{2s} \leqslant a'_{2s} & \dots & a_{ns} \leqslant a'_{ns}. \end{array}$$

Define the mappings $g_i : S \to P$ by $g_i(s) = a_{is}$ and $g'_i : S \to P$ by $g'_i(s) = a'_{is}$ for $i = 1, \cdots, n$. Then

$$\begin{array}{cccc} h_1(s) = fg_1(s) & fg'_1(s) = fg'_2(s) & \dots & fg'_n(s) = h_2(s); \\ g_1(s) \leqslant g'_1(s) & g_2(s) \leqslant g'_2(s) & \dots & g_n(s) \leqslant g'_n(s), \end{array}$$

and so one has the following,

$$\begin{array}{cccc} h_1 = \bar{f}(g_1) & \bar{f}(g'_1) = \bar{f}(g'_2) & \dots & \bar{f}(g'_n) = h_2; \\ g_1 \leqslant g'_1 & g_2 \leqslant g'_2 & \dots & g_n \leqslant g'_n, \end{array}$$

as required. $\square$

Now, applying the above lemmas to the adjoint pair $U'_1 \dashv K'_1$ given in Corollary 1.3, and using Proposition 4.3, we get the following.

Theorem 4.6. *For a (regular) projective poset act A, A is (regular) projective as a poset, and so it has discrete order.*

Applying Lemma 4.4 and the fact that U_1' preserves (regular) epimorphisms, and using Proposition 4.3, we get the following.

Theorem 4.7. *If P is a* (*regular*) *projective poset, then the free poset act over a poset P, $P \times S$, is* (*regular*) *projective, and it has discrete order.*

Also, applying Lemma 4.4 to the adjoint pair $H' \dashv G'$ given in Theorem 1.4, and using Proposition 4.3, we obtain:

Theorem 4.8. *For a* (*regular*) *projective poset act A, the poset $H'(A) = A/\nu(W)$ is a* (*regular*) *projective poset, and so has discrete order.*

Corollary 4.9. *If A is a projective poset act then, for every $a \leqslant a'$ in A, there exist $c_1, \ldots, c_n, d_1 \ldots, d_n \in A, s_1, \ldots, s_n \in S$ such that*

$$a' \leqslant c_1 s_1, \quad d_1 s_1 \leqslant c_2 s_2, \quad d_2 s_2 \leqslant c_3 s_3, \quad \ldots, \quad d_n s_n \leqslant a,$$

where $(c_i, d_i) \in W$ for $i = 1, 2, \ldots, n$.

Proof. By the preceding theorem, for a projective poset act A the poset $A/\nu(W)$ has discrete order. If $a \leqslant a'$ then $a\nu(W)a'$ and then $a' \leqslant_W a$ which means
$c_1, \ldots, c_n, d_1 \ldots, d_n \in A, s_1, \ldots, s_n \in S$ exist as required. □

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Department of Mathematics, University of Maragheh, Maragheh, 55181-83111 Iran
E-mail: leilashahbaz@yahoo.com, l_shahbaz@maragheh.ac.ir