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A new quasigroup isomorphism invariant arising from fractal image patterns

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Abstract. The analysis and recognition of fractal image patterns derived from Cayley tables has turned out to play a relevant role for distributing distinct types of algebraic and combinatorial structures into isomorphism classes. In this regard, Dimitrova and Markovski described in 2007 a graphical representation of quasigroups by means of fractal image patterns. It is based on the construction of pseudo-random sequences arising from a given quasigroup. In particular, isomorphic quasigroups give rise to the same fractal image pattern, up to permutation of underlying colors. This possible difference may be avoided by homogenizing the standard sets related to these patterns. Based on the differential box-counting method, the mean fractal dimension of homogenized standard sets constitutes a quasigroup isomorphism invariant which is analyzed in this paper in order to distribute quasigroups of the same order into isomorphism classes.

1. Introduction

A *Latin square* of order n is an $n \times n$ array whose entries are chosen from a set of n distinct symbols so that each symbol appears precisely once in each row and precisely once in each column. From here on, let $\mathcal{L}_n$ denote the set of Latin squares of order n whose entries are chosen from the set of symbols $[n] := \{1, \ldots, n\}$. Every Latin square $L = (l_{i,j}) \in \mathcal{L}_n$ is uniquely identified with its set of entries

$$\mathrm{Ent}(L) := \{(i, j, l_{i,j})\colon\ 1 \leqslant i, j \leqslant n\}.$$

The set $\mathcal{L}_n$ is preserved by the action of the symmetric group S_n on the set $[n]$. It is so that every permutation $\pi \in S_n$ acts on a Latin square $L = (l_{i,j}) \in \mathcal{L}_n$ by giving rise to its *isomorphic* Latin square $L^\pi \in \mathcal{L}_n$, where $\mathrm{Ent}(L^\pi) = \{(\pi(i), \pi(j), \pi(l_{i,j}))\colon\ 1 \leqslant i, j \leqslant n\}$. The permutation π constitutes a Latin square *isomorphism*. To be isomorphic is an equivalence relation among Latin squares. Currently, it is only known [7] the distribution into isomorphism classes of Latin squares of order $n \leqslant 11$. In order to deal with higher orders, new Latin square isomorphism invariants are being introduced in the recent literature [2, 6, 14]. This paper delves into this topic by focusing on the fractal dimension of the standard set of image patterns associated to any given Latin square.

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Every Latin square in $\mathcal{L}_n$ is the Cayley table of a *quasigroup* $([n], \cdot)$, where the set $[n]$ is endowed with a binary operation so that both left and right divisions are feasible. Two quasigroups $([n], \cdot)$ and $([n], \circ)$ are *isomorphic* if and only if there exists a bijection f on the set $[n]$ such that $f(i \cdot j) = f(i) \circ f(j)$, for all $i, j \in [n]$. Equivalently, if and only if their associated Latin squares are isomorphic.

In 2007, Dimitrova and Markovski [3] proposed a graphical representation of quasigroups, which was based on the construction of pseudo-random sequences arising from the Cayley table of these algebraic structures [9, 10, 11]. Some of the resulting images revealed fractal patterns that enable one to distribute quasigroups into fractal and non-fractal classes. Particularly, fractal quasigroups are recommended for designing error detecting codes [8], whereas non-fractal quasigroups are recommended for designing cryptographic primitives [1, 12]. An interesting aspect to take into account is the fact that isomorphic quasigroups give rise to the same image pattern, up to permutation of underlying colors. Due to it, the recognition and analysis of image patterns arising from quasigroups have recently arisen as an efficient new approach for classifying quasigroups and related structures into isomorphism classes [4, 5]. In this paper, the standard set of image patterns is homogenized in order to determine its mean fractal dimension, which turns out to be a relevant quasigroup isomorphism invariant.

The paper is organized as follows. In Section 2, it is reminded the construction of standard sets of image patterns arising from any given quasigroup. Based on this construction, it is introduced the concept of homogenized standard set, which constitutes itself an invariant of quasigroup isomorphisms. Then, based on the differential box-counting method, and in order to distinguish homogenized standard sets of image patterns arising from non-isomorphic quasigroups, it is introduced in Section 3 the notion of mean fractal dimension of any given homogenized standard set. In addition, it is shown how this new invariant characterizes all the isomorphism classes of quasigroups of order $n \leqslant 4$. Finally, Section 4 illustrates the effectiveness of this new isomorphism invariant by focusing on the problem of distributing random Latin squares into isomorphism classes.

2. Preliminaries

Let $L \in \mathcal{L}_n$ be the Cayley table of a quasigroup $([n], \cdot)$ and let $T = t_1 \dots t_m$ be a plaintext, for some positive integer m, where $t_i \in [n]$, for all $i \leqslant m$. Then, for each $s \in [n]$, it is defined [3] the encrypted string $E_s(T) := e_1 \dots e_{m-1}$, where $e_1 := s \cdot t_1$ and $e_i := e_{i-1} \cdot t_i$, whenever $1 < i \leqslant m$. An iterative implementation of this encryption gives rise to arrays describing fractal patterns. In order to see it, let $r \geqslant 2$ be a positive integer and let $S = (s_1, \dots, s_{r-1})$ be an $(r-1)$-tuple of positive integers such that $s_i \in [n]$, for all $i < r$. Then, the $r \times m$ *image pattern* based on L is the $r \times m$ array $(p_{i,j})$ such that, for each pair of positive integers $i \leqslant r$ and $j \leqslant m$,

$$p_{i,j} := \begin{cases} t_j, & \text{if } i = 1, \\ s_{i-1} \cdot p_{i-1,1}, & \text{if } i > 1 \text{ and } j = 1, \\ p_{i,j-1} \cdot p_{i-1,j}, & \text{otherwise.} \end{cases} \tag{1}$$

Each one of its cells constitutes a *pixel* of the image pattern under consideration. In addition, the symbol within each pixel may uniquely be identified with a color of a given palette of n colors. Further, the image pattern is called s-*standard* [4], with $s \in [n]$, if S is the constant $(r-1)$-tuple $(s, \dots, s)$ and T is the constant plaintext $s \dots s$ of length m. From here on, let $\mathcal{P}_{r,m;s}(L)$ denote this array. The *standard set* of $r \times m$ image patterns associated to the Latin square L is the set $\{\mathcal{P}_{r,m;s}(L)\colon\ s \in [n]\}$.

Isomorphic Latin squares give rise, up to permutation of colors, to the same standard set of $r \times m$ image patterns (see [4]). In order to avoid this possible difference, let us fix a palette $\mathfrak{P} = \{c_1, \dots, c_n\}$ of n distinct colors so that they are ordered in natural way by means of their corresponding intensity. Then, we say that an s-standard $r \times m$ image pattern $\mathcal{P}_{r,m;s}(L)$ is *homogenized* with respect to the palette $\mathfrak{P}$ if its colors appear in natural order (according to their intensity) when the image pixels are read row by row then column by column.

In addition, we say that the *homogenized standard set* of $r \times m$ image patterns associated to the Latin square L, with respect to the palette $\mathfrak{P}$, is that one in which all its $r \times m$ image patterns are homogenized with respect to $\mathfrak{P}$. In this way, isomorphic Latin squares give rise to exactly the same homogenized standard set of $r \times m$ image patterns with respect to a given palette.

Example 1. Let us consider the quasigroups having as respective Cayley tables the following four Latin squares in $\mathcal{L}_4$.

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

L_1

3	1	4	2
4	3	2	1
1	2	3	4
2	4	1	3

L_2

1	2	4	3
2	1	3	4
3	4	1	2
4	3	2	1

L_3

1	2	3	4
3	1	4	2
4	3	2	1
2	4	1	3

L_4

L_1 and L_2 are isomorphic by means of the Latin square isomorphism $(13)(2)(4) \in S_4$. Visual representations of their respective standard sets of 90×90 image patterns are shown in Figure 1. It consists of a collage in form of 2×4 array, whose cell (i, j) represents the j-standard 90×90 image pattern of the Latin square L_i, for all $i \leqslant 2$ and $j \leqslant 4$. Each standard image pattern is represented as a 90×90 pixel array so that each symbol is uniquely replaced by a colour within the same grayscale palette of four colours. Notice that both standard sets coincide, up to permutation of colors. Similarly, the homogenized standard sets of 90×90 image patterns of the Latin squares L_1, L_3 and L_4, with respect to a grayscale palette of four colors, with respective gray-level intensities 0.25, 0.5, 0.75 and 1, are shown in the collage of form of 3×4 array in Figure 2.

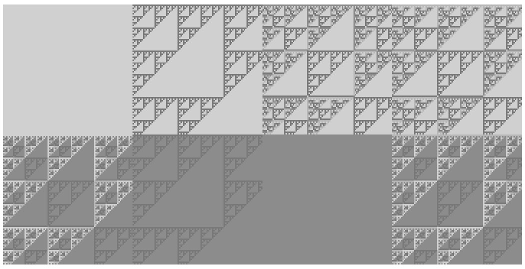

Figure 1. Standard sets of 90×90 image patterns of the Latin squares L_1 (upper row) and L_2 (lower row).

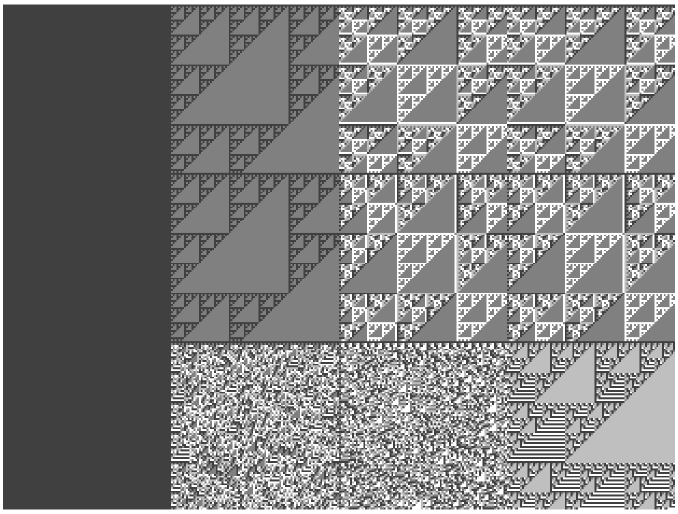

Figure 2. Homogenized standard sets of 90×90 image patterns of the Latin squares L_1 (upper row), L_3 (second row) and L_4 (lower row).

From Figure 2, it is readily verified, even visually, that L_4 is isomorphic neither to L_1 nor to L_3. Nevertheless, it is not so evident that the homogenized standard sets of L_1 and L_3 are distinct. In order to make easier the distinction of homogenized standard sets, the next section delves into the study of the fractal dimension of their image patterns.

3. Mean fractal dimension of homogenized standard sets

Let n be a positive integer and let $\mathfrak{P}_n = \{c_1, \ldots, c_n\}$ be the grayscale palette such that, for each positive integer $i \leqslant n$, the gray-level intensity of the color c_i is i/n. (In this way, the color c_n is always white.) Then, let $\mathcal{H}_{r,m}(L)$ denote the homogenized standard set of $r \times m$ image patterns of a Latin square $L \in \mathcal{L}_n$, with respect to the palette $\mathfrak{P}_n$. There is no loss of generality in assuming a grayscale palette, because the distribution of Latin squares into isomorphism classes is currently only known for all $n \leqslant 11$.

Further, let $\mathrm{Div}(r,m)$ denote the set of common divisors of both parameters r and m. Since each positive integer $k \in \mathrm{Div}(r,m)$ is in compliance with the image dimensions of every image pattern $\mathcal{P}_{r,m;s}(L) \in \mathcal{H}_{r,m}(L)$, with $s \in [n]$, it is always possible to cover the latter by an $\frac{r}{k} \times \frac{m}{k}$ grid formed by two-dimensional boxes of side length k. Let $I_{i,j,k}(\mathcal{P}_{r,m;s}(L))$ denote the range of gray-level intensities within the region of $\mathcal{P}_{r,m;s}(L)$ that is bounded by the cell (i,j) of the mentioned grid. Then, let us consider the value

$$I_k(\mathcal{P}_{r,m;s}(L)) := \sum_{(i,j)\in\left[\frac{r}{k}\right]\times\left[\frac{m}{k}\right]} (1 + I_{i,j,k}(\mathcal{P}_{r,m;s}(L))).$$

Now, based on the known method of *differential box-counting* [13] for determining the fractal dimension of a given grayscale image, we define the *differential box-counting fractal dimension* $D_B(\mathcal{P}_{r,m;s}(L))$ of the image pattern $\mathcal{P}_{r,m;s}(L)$ as the slope of the linear regression line of the set of points

$$\{(\ln(I_k(\mathcal{P}_{r,m;s}(L))),\, \ln(1/k)) :\ k \in \mathrm{Div}(r,m)\}\,.$$

We call the mean value of this fractal dimension, averaged over all the positive integers $k \in \mathrm{Div}(r,m)$, the *mean fractal dimension* $D_B(\mathcal{H}_{r,m}(L))$ of the homogenized standard set $\mathcal{H}_{r,m}(L)$. The following result follows readily.

Proposition 1. *Let r, m and n be three positive integers, and let L_1 and L_2 be two Latin squares in $\mathcal{L}_n$. If $D_B(\mathcal{H}_{r,m}(L_1)) \neq D_B(\mathcal{H}_{r,m}(L_2))$, then L_1 and L_2 are not isomorphic.*

In order to illustrate Proposition 1, Table 1 enumerates both the differential box-counting dimension and the mean fractal dimension of each one of the three homogenized standard sets described in Example 1. Notice in particular that their mean fractal dimensions are pairwise distinct, which enables one to ensure that the Latin squares L_1, L_2 and L_3 correspond to different isomorphism classes.

		L	
	L_1	L_3	L_4
$D_B(\mathcal{P}_{90,90;1}(L))$	2.00000	2.00000	2.00000
$D_B(\mathcal{P}_{90,90;2}(L))$	1.95165	1.95165	1.92136
$D_B(\mathcal{P}_{90,90;3}(L))$	1.8877	1.88873	1.92331
$D_B(\mathcal{P}_{90,90;4}(L))$	1.8877	1.88873	1.90088
$D_B(\mathcal{H}_{90,90}(L))$	1.9317625	1.9322775	1.9363875

Table 1. Differential box-counting and mean fractal dimensions of the homogenized standard sets of 90×90 image patterns in Example 1.

Furthermore, Table 2 enables one to ensure that the mean fractal dimension of homogenized standard sets of 90×90 image patterns characterizes the five isomorphism classes of the set $\mathcal{L}_3$ and the 35 isomorphism classes of the set $\mathcal{L}_4$. (Notice the existence of only one isomorphism class for all $n \in \{1, 2\}$.) In the table, isomorphism classes of each order $n \in \{3, 4\}$ are arranged according to the increasing value of their associated mean fractal dimension (see also Figure 3). In order to identify each isomorphism class, it has been used the notation introduced in [4, 5]. It is so that each class is identified with the corresponding representative $L_{n.i}$ appearing in the mentioned references. The mean fractal dimension $D_B(\mathcal{H}_{90,90}(L_{n.i}))$ is shown in the column $D_B(n.i)$.

n	i	$D_B(n,i)$	n	i	$D_B(n,i)$	n	i	$D_B(n,i)$	n	i	$D_B(n,i)$
3	1	1.9285267	4	30	1.9212575	4	35	1.9338325	4	23	1.9428575
	5	1.9335900		29	1.9213650		19	1.9357400		15	1.9472450
	2	1.9524867		27	1.9215325		21	1.9359200		12	1.9476600
	3	1.9527467		25	1.9216950		4	1.9363875		22	1.9495350
	4	2.0000000		7	1.9230125		5	1.9366250		18	1.9504400
4	32	1.9072150		8	1.9274475		20	1.9411800		14	1.9511850
	28	1.9099250		9	1.9285825		13	1.9411950		11	1.9606500
	33	1.9137400		2	1.9296225		16	1.9413250		34	1.9637375
	31	1.9139600		1	1.9317625		10	1.9413500		17	1.9807250
	26	1.9210725		6	1.9322775		3	1.9413775		24	2.0000000

Table 2. Mean fractal dimensions of the homogenized standard sets of 90×90 image patterns of each isomorphism class of Latin squares of order $n \in \{3, 4\}$.

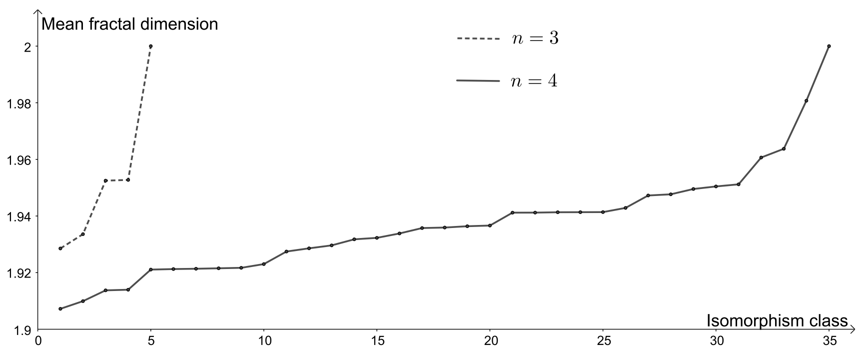

Figure 3. Mean fractal dimensions of the homogenized standard sets of 90×90 image patterns of each isomorphism class of Latin squares of order $n \in \{3, 4\}$.

Notice in particular the existence of exactly one isomorphism class associated to the maximum mean fractal dimension 2 in each one of the sets $\mathcal{L}_3$ and $\mathcal{L}_4$. Their representatives are the Latin squares

1	3	2
3	2	1
2	1	3

and

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

.

Both of them are multiplication tables of *idempotent* quasigroups. That is, the cell (i, i) contains the symbol i, for all $i \in [n]$. In fact, the following result is readily verified.

Proposition 2. *The mean fractal dimension of the homogenized standard set of $r \times m$ image patterns based on the multiplication table of an idempotent quasigroup is 2, for every pair of positive integers r and m.*

The previous result, together with the existence of non-isomorphic idempotent quasigroups even from order $n \geqslant 5$, implies that the mean fractal dimension is not definitive for characterizing isomorphism classes of Latin squares of higher orders. It is the case, for instance of the five non-isomorphic idempotent quasigroups of order five, which are represented by the following Latin squares in $\mathcal{L}_5$.

1	3	2	5	4
4	2	5	1	3
5	4	3	2	1
3	5	1	4	2
2	1	4	3	5

1	3	2	5	4
5	2	4	1	3
4	5	3	2	1
3	1	5	4	2
2	4	1	3	5

1	3	4	5	2
3	2	5	4	1
2	5	3	1	4
5	1	2	4	3
4	3	1	2	5

1	3	4	5	2
4	2	5	3	1
5	1	3	2	4
2	5	1	4	3
3	4	2	1	5

1	3	4	5	2
3	2	5	1	4
4	5	3	2	1
5	1	2	4	3
2	4	1	3	5

4. Patterns arising from random Latin squares

Let us illustrate the effectiveness of the new isomorphism invariant that we have just introduced by focusing on the problem of distributing random Latin squares into isomorphism classes. To this end, it has been chosen the randomization method described in [2], which consists of adding sequentially a set of feasible random entries to an empty $n \times n$ array until a Latin square is reached. The computation of the mean fractal dimension of the homogenized set of 90×90 image patterns of each one of these random Latin squares enables one to distinguish non-isomorphic classes among them.

By means of this procedure, the five isomorphism classes of $\mathcal{L}_3$ have been obtained after six attempts. Furthermore, the 35 isomorphism classes of $\mathcal{L}_4$ have been obtained after 326 attempts. Figure 4 illustrates the computational progression for obtaining such classes in this last case.

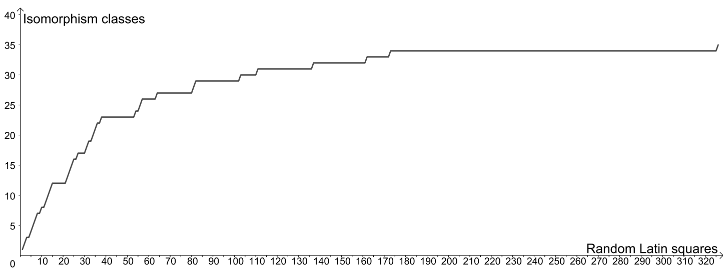

Figure 4. Computational progression for obtaining the 35 isomorphism classes of $\mathcal{L}_4$.

In a similar way, Figure 5 illustrates the case $n = 5$, for which, after $20,000$ attempts, it has been distinguished $1,404$ isomorphism classes of the $1,411$ ones. Figure 6 illustrates all their mean fractal dimensions in increasing order.

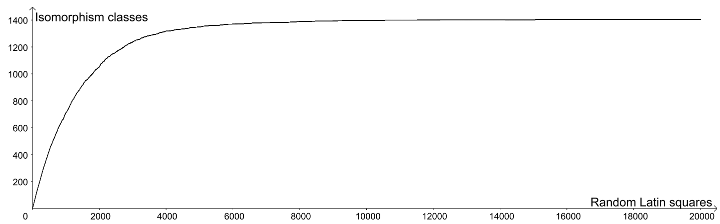

Figure 5. Computational progression for obtaining isomorphism classes of $\mathcal{L}_5$.

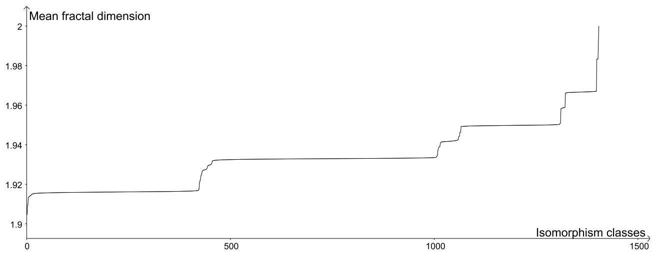

Figure 6. Mean fractal dimensions of the homogenized standard sets of 90×90 image patterns associated to each distinct isomorphism class of $20,000$ random Latin squares of order $n = 5$.

5. Conclusion and further work

The analysis of standard sets of image patterns related to a given quasigroup has recently arisen as an efficient way for distinguishing, even visually, distinct isomorphism classes of quasigroups. In order to avoid possible discrepancies concerning the underlying colors, the concept of homogenized standard set has been introduced in this paper. Based on the differential box-counting method, the mean fractal dimension of these homogenized standard sets has also been introduced as an efficient invariant for dealing with the open problem of distributing quasigroups of order $n \geqslant 12$ into isomorphism classes. Concerning the computational efficiency of this invariant, notice that the maximum running time that is required to compute any of the mean fractal dimensions appearing in this paper has been less than one second in an *Intel Core i7-8750H CPU (6 cores), with a 2.2 GHz processor and 8 GB of RAM.* In the same computer system, the mean fractal dimension of the homogenized standard sets of 90×90 image patterns associated to the quasigroup of order 256 described in [4] is obtained in 81.63 seconds. Its mean fractal dimension is 1.88926. It is, therefore, computationally feasible to make use of this new invariant even to deal with the possible characterization of isomorphism classes of quasigroups of order $n = 256$, which are the most commonly used in the literature for designing codes and cryptographic primitives. It is established as further work. As a preliminary stage, it has been shown the computational progression for obtaining the isomorphism classes of quasigroups of order $n \leqslant 5$ by focusing to this end on the mean fractal dimension of a given set of random Latin squares.

The existence of non-isomorphic idempotent quasigroups even from order $n \geqslant 5$ implies that the mean fractal dimension is not definitive for characterizing quasigroup isomorphism classes. In this regard, it is also established as further work the study of algebraic and combinatorial properties of those isomorphism classes of Latin square whose homogenized standard sets of image patterns are associated to the same mean fractal dimension.

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