Quasigroups and Related Systems **30** (2022), 63 − 80
https://doi.org/10.56415/qrs.v30.05

Rad-supplemented property in the lattices

Shahabaddin Ebrahimi Atani, Mehdi Khoramdel
and Saboura Dolati Pish Hesari

Abstract. Let L be a lattice with the greatest element 1. Following the concept of Rad-supplemented modules, we define Rad-supplemented filters and we will make an intensive investigate the basic properties and possible structures of these filters.

1. Introduction

Lattices are natural topic in algebra to study because they now play an important role in many disciplines of mathematics such as combinatorics, number theory and group theory and, hence, ought to be in the literature. In structure, lattices lie between semigroups and rings. In this paper, we extend several concepts from module theory to lattice theory. With a careful generalization, we can cover some basic corresponding results in the former setting. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [2, 3, 5, 6, 7, 8, 9]).

The notion of a supplement submodule was introduced in [12, 14] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective cover. For submodules U and V of a module M, V is said to be a supplement of U in M or U is said to have a supplement V in M if $U+V=M$ and $U \cap V \ll V$. The module M is called supplemented if every submodule of M has a supplement in M. See [4] and [16] for results and the definitions related to supplements and supplemented modules. In a series of papers, Zöschinger has obtained detailed information about supplemented and related modules [17]. Supplemented modules are also discussed in [13]. Recently, several authors have studied different generalizations of supplemented modules. Rad-supplemented modules have been studied in [1] and [15]. See [15]; these modules are called generalized supplemented modules. For submodules U and V of a module M, V is said to be a Rad-supplement of U in M or U is said to have a Rad-supplement V in M if $U + V = M$ and $U \cap V \subseteq \text{Rad}(V)$. M is called a Rad-supplemented module if every submodule of M has a Rad-supplement in M. Recently, the study of

2010 Mathematics Subject Classification: 06B05.
Keywords: Lattice; Filter; Small; Supplemented; Rad-supplemented.

the supplemented property in the rings, modules, and lattices has become quite popular (see for example [1, 3, 8, 9, 10, 11, 12].

Let L be a distributive lattice with 1. In the present paper, we are interested in investigating (amply) Rad-supplemented filters to use other notions of (amply) Rad-supplemented, and associate which exist in the literature as laid forth in [15]. It is of interest to know how far the old theories extend to the new situation. If A is a subset of a lattice L, then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A. A subfilter G of a filter F of L is called small in F, written $G \ll F$, if, for every subfilter H of F, the equality $T(G \cup H) = F$ implies $H = F$. Here is a brief outline of the article. Among many results in this paper, in Section 2, we investigate some properties of Rad-supplemented filters, weakly Rad-supplemented filters and amply Rad-supplemented filters. We prove in Theorem 2.5 that if H is a Rad-supplement in a filter F of L, then $\mathrm{Rad}(H) = H \cap \mathrm{Rad}(F)$. As one of the main results of this section, we prove in Theorem 2.10 that if F is a Rad-supplemented filter of L, then there exist a semisimple subfilter K and a subfilter G with $\mathrm{Rad}(G) \trianglelefteq G$ such that $F = K \oplus G$. We prove that if F_1 and F_2 are Rad-supplemented filters of L and $F = T(F_1 \cup F_2)$, then F is a Rad-supplemented filter; Theorem 2.12. Also, we prove in Theorem 2.14 that a filter F of L is semisimple if and only if every subfilter of F is Rad-supplement in F. Using a similar proof like that in Theorem 2.12, we also prove that if F_1 and F_2 are weakly Rad-supplemented filters of L and $F = T(F_1 \cup F_2)$, then F is a weakly Rad-supplemented filter; Theorem 2.18. Finally, if F is an amply Rad-supplemented filter of L, then every direct summand of F (resp. every supplement of a subfilter of F) is an amply Rad-supplemented filter, Theorem 2.24. Section 3 contains some basic properties of quotient Rad-supplemented filters, quotient weakly Rad-supplemented filters and quotient amply Rad-supplemented filters. It starts by stating useful properties of quotient filters. One of the main results of this part is: If F is a Rad-supplemented filter of L (resp. a weakly Rad-supplemented filter), then every quotient filter of F is Rad-supplemented (weakly Rad-supplemented); Theorem 3.5. We prove in Theorem 3.7 that if F is a Rad-supplemented filter of L, then $\frac{F}{\mathrm{Rad}(F)}$ is a semisimple filter. Finally, using the characterization of lifting filters given in Theorem 3.11, we prove in Theorem 3.14 that if F is a filter of L with ACC on small subfilters, then F is an amply Rad-supplemented filter and every Rad-supplement is a direct summand if and only if F is a lifting filter.

Let us briefly review some definitions and tools that will be used later [3]. By a lattice we mean a poset $(L, \leqslant)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if

$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter P of L is said to be maximal if E is a filter in L with $P \subsetneqq E$, then $E = L$. If F is a filter of a lattice L, then the radical of F, denoted by $\mathrm{Rad}(F)$, is the intersection of all maximal subfilters of F.

Let L be a lattice. If A is a subset of L, then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A. A filter F is called finitely generated if there is a finite subset A of F such that $F = T(A)$. A subfilter G of a filter F of L is called *small* in F, written $G \ll F$, if, for every subfilter H of F, the equality $T(G \cup H) = F$ implies $H = F$. A subfilter G of F is called *essential* in F, written $G \trianglelefteq F$, if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of F. Let G be a subfilter of a filter F of L. A subfilter H of F is called a *supplement* of G in F if $F = T(G \cup H)$ and H is minimal with respect to this property, or equivalently, $F = T(G \cup H)$ and $G \cap H \ll H$. H is said to be a supplement subfilter of F if H is a supplement of some subfilter of F. F is called a *supplemented* filter if every subfilter of F has a supplemented in F. A subfilter G of a filter F of L has *ample supplements* in F if, for every subfilter H of F with $F = T(H \cup G)$, there is a supplement H' of G with $H' \subseteq H$. If every subfilter of a filter F has ample supplements in F, then we call F *amply supplemented.* A filter F of a lattice L is called *hollow* if $F \neq \{1\}$ and every proper subfilter G of F is small in F. F is called *local* if it has exactly one maximal subfilter that contains all proper subfilters.

Proposition 1.1. [6, 7] *A non-empty subset F of a lattice L is a filter if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

Proposition 1.2. [8, Lemma 2.4 and Theorem 2.6] *Let F be a filter of a distributive lattice L with* 1.

(1) *If $A \ll F$ and $C \subseteq A$, then $C \ll F$.*
(2) *If A, B are subfilters of F with $A \ll B$, then A is a small subfilter in subfilters of F that contains the subfilter of B. In particular, $A \ll F$.*
(3) *If $F_1, F_2, \cdots, F_n$ are small subfilters of F, then $T(F_1 \cup F_2 \cup \cdots \cup F_n)$ is also small in F.*
(4) *If A, B, C and D are subfilters of F with $A \ll B$ and $C \ll D$, then $T(A \cup C) \ll T(B \cup D)$.*
(5) *$x \in \mathrm{Rad}(F)$ if and only if $T(\{x\}) \ll F$. Moreover,* $\mathrm{rad}(F) = T(\cup_{G \ll F} G)$.

2. Basic properties of Rad-supplemented filters

Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. At this stage it is useful to make an elementary lemma about filters of L which we will use without further comment.

Lemma 2.1. [8, Lemma 2.1]

(1) *Let A be an arbitrary non-empty subset of L. Then*

$T(A) = \{x \in L: \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x$ *for some* $a_i \in A$ $(1 \leqslant i \leqslant n)\}$.

Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$,

$$T(F) = F \text{ and } T(T(A)) = T(A).$$

(2) *Let A, B and C be subfilters of a filter F of L. Then*

$$T(T(A \cup B) \cup C) \subseteq T(A \cup T(B \cup C)).$$

In particular, if $F = T(T(A \cup B) \cup C)$, then

$$F = T(T(C \cup B) \cup A) = T(T(A \cup C) \cup B).$$

(3) (*Modular law*) *If F_1, F_2, F_3 are filters of L with $F_2 \subseteq F_1$, then*

$$F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3)).$$

Definition 2.2. Let G, H be subfilters of a filter F of L. If $F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{Rad}(H)$, then H is called a *Rad-supplement* of G in F. If every subfilter of F has a Rad-supplement in F, then F is called a Rad-supplemented filter.

A filter F of L is called *radical* if $\mathrm{Rad}(F) = F$, and F is called reduced if it has no radical subfilter $G \neq \{1\}$.

Remark 2.3.

(1) Clearly, each supplement is a Rad-supplement (since $G \cap H \ll H$ gives $G \cap H \subseteq \mathrm{Rad}(H)$). So (amply) supplemented filters, hollow filters and local filters are Rad-supplemented filters.

(2) Let F be a radical filter. For every subfilter G of F, we have $F = T(F \cup G)$ and $G \cap F = G \subseteq \mathrm{Rad}(F)$. Hence F is Rad-supplemented.

(3) Let L be a lattice such that $\mathrm{Rad}(H) \ll H$ for every filter $H \neq \{1\}$ of L. Let F be a Rad-supplemented filter. We show that F is supplemented. If G is a proper subfilter of F, then there exists a subfilter H of F such that $F = T(H \cup G)$ and $G \cap H \subseteq \mathrm{Rad}(H) \ll H$; so $G \cap H \ll H$. Thus F is supplemented.

Proposition 2.4. *Let U, V be subfilters of a filter F of L. Then V is a Rad-supplement of U in F if and only if $F = T(U \cup V)$ and $T(\{x\}) \ll V$ for all $x \in U \cap V$.*

Proof. Let V be a Rad-supplement of U in F and $x \in U \cap V$; so $F = T(U \cup V)$ and $x \in U \cap V \subseteq \mathrm{Rad}(V)$. Then $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k} \leqslant x$, where $x_{i_1} \in F_{i_1} \ll V, \cdots, x_{i_k} \in F_{i_k} \ll V$ by Proposition 1.2. By Lemma 2.1, $T(\{x\}) \subseteq T(F_{i_1} \cup \cdots \cup F_{i_k}) \ll V$; so $T(\{x\}) \ll V$. Conversely, assume that $F = T(U \cup V)$ and $T(\{x\}) \ll V$ for all $x \in U \cap V$; so $x \in \mathrm{Rad}(V)$ and so $U \cap V \subseteq \mathrm{Rad}(V)$. □

Theorem 2.5. *If H is a Rad-supplement in a filter F of L, then* $\mathrm{Rad}(H) = H \cap \mathrm{Rad}(F)$.

Proof. We first show that if K is a subfilter of F with $K \subseteq \mathrm{Rad}(F)$, then $K \cap H \subseteq \mathrm{Rad}(H)$. By assumption, there exists a subfilter G of F such that $F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{Rad}(H)$. If $H = \mathrm{Rad}(H)$, then we are done. So we may assume that $H \neq \mathrm{Rad}(H)$. Assume to the contrary, let $H \cap K \nsubseteq \mathrm{Rad}(H)$. Then there is a maximal subfilter P of H such that $K \cap H \nsubseteq P$ and $G \cap H \subseteq P$. Then $H = T(P \cup T(\{\{x\}))$ for some $x \in K \cap H \setminus P$. It follows that $F = T(G \cup T(P \cup T(\{x\})) = T(T(\{x\}) \cup T(G \cup P))$. Since $x \in K \subseteq \mathrm{Rad}(F)$, we get $T(\{x\}) \ll F$ by Proposition 1.2 and this implies that $F = T(G \cup P)$. By modularity law, $H = H \cap T(G \cup P) = T(P \cup (G \cap H)) = T(P) = P$ which is impossible. Thus $K \cap H \subseteq \mathrm{Rad}(H)$. If $K = \mathrm{Rad}(F)$, then $H \cap \mathrm{Rad}(F) \subseteq \mathrm{Rad}(H)$, as required. □

Lemma 2.6. *Let G, H be subfilters of a filter F of L. If H is a Rad-supplement of G in F and K is a maximal subfilter of H, then $T(K \cup G)$ is a maximal subfilter of F.*

Proof. By assumption, $H = T(K \cup T(\{x\}))$ for all $x \in H \setminus K$. We claim that $F \neq T(G \cup K)$. Suppose $F = T(G \cup K)$ and look for a contradiction. Then we can write $x = (g \wedge k) \vee x = (x \vee g) \wedge (x \vee k)$ for some $g \in G$ and $k \in K$. It follows that $x \vee g \in G \cap H \subseteq \mathrm{Rad}(H) \subseteq K$; hence $x \in K$ which is impossible. Thus $F \neq T(G \cup K)$. Let $T(G \cup K) \subsetneqq H' \subseteq F$ for some subfilter H' of F. There is an element $y \in H'$ with $y \notin T(G \cup K)$ (so $y \notin G$ and $y \notin K$). As $y \in H' \subseteq T(G \cup H) = F$, $y = (y \vee a) \wedge (y \vee b)$ for some $a \in G$ and $b \in H$. Since $y \vee a \in G$, we get $y \vee b \notin K$ (otherwise $y \in T(G \cup K)$). It follows that $H = T(K \cup T(\{y \vee b\})) \subseteq H'$. Now $G \subseteq H'$ gives $H' = F$. Thus $T(K \cup G)$ is a maximal subfilter of F. □

Theorem 2.7. *Assume that G, H are subfilters of a filter F of L and let H be a Rad-supplement of G in F. Then the following hold:*

(1) *If H is a non-radical filter, then G contained in a maximal subfilter of F.*

(2) *If G is a maximal subfilter of F, then $\mathrm{Rad}(H) = G \cap H$ is a unique maximal subfilter of H.*

Proof. (1). By hypothesis, H contains a maximal subfilter K. Then $T(K \cup G)$ is a maximal subfilter of F by Lemma 2.6 with $G \subseteq T(G \cup K)$, as required.

(2). Since $T(G \cup H) = F$ and G is a maximal subfilter of F, then $H \nsubseteq G$; so $G \cap H \neq H$. Let K be a subfilter of H such that $G \cap H \subsetneqq K \subseteq H$. Then there is an element $x \in K \subseteq H$ with $x \notin G$. Now $G \subsetneqq T(T(\{x\}) \cup G) \subseteq F$ gives $F = T(T(\{x\}) \cup G)$. By modular law, we conclude that

$$H = H \cap T(T(\{x\}) \cup G) = T(T(\{x\}) \cup (G \cap H)) \subseteq K;$$

so $H = K$. Thus $G \cap H$ is a maximal subfilter of H which implies that $\mathrm{Rad}(H) \subseteq G \cap H$. As H is a Rad-supplement of G, $G \cap H \subseteq \mathrm{Rad}(H)$. Hence $\mathrm{Rad}(H) = G \cap H$. □

Proposition 2.8. *Let F be a filter of L. Then the following hold:*

(1) *If F is a reduced and Rad-supplemented filter, then* $\mathrm{Rad}(F) \ll F$.

(2) *Let F be a reduced filter such that every Rad-supplement subfilter of F is Rad-supplemented. Then F is supplemented.*

Proof. (1). By Proposition 2.7, every proper subfilter is contained in a maximal subfilter of F. If $F = T(\mathrm{Rad}(F) \cup U)$ with $U \neq F$, then there is a maximal subfilter P of F such that $U \subseteq P$, and so $F \subseteq T(\mathrm{Rad}(F) \cup P) = T(P) = P$ which is a contradiction. Thus $\mathrm{Rad}(F) \ll F$.

(2). By hypothesis, F is Rad-supplemented. Let G be a subfilter of F. So there is a subfilter H of F such that $F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{Rad}(H)$. Since H is Rad-supplemented and F is reduced (so H is reduced), $\mathrm{Rad}(H) \ll H$ by (1); hence $G \cap H \ll H$. Therefore F is supplemented. □

A lattice L is called *semisimple*, if for each proper filter F of L, there exists a filter G of L such that $L = T(F \cup G)$ and $F \cap G = \{1\}$). In this case, we say that F is a *direct summand* of L, and we write $L = F \oplus G$. A filter F of L is called a *semisimple* filter, if every subfilter of F is a direct summand. A *simple* filter), is a filter that has no filters besides the $\{1\}$ and itself.

Proposition 2.9. *Let F be a Rad-supplemented filter of L. If H is a subfilter of F with $H \cap \mathrm{Rad}(F) = \{1\}$, then H is semisimple. In particular, if $\mathrm{Rad}(F) = \{1\}$, then F is semisimple.*

Proof. Let H' be any subfilter of H. By assumption, there is a subfilter K of F with $F = T(H' \cup K)$ and $H' \cap K \subseteq \mathrm{Rad}(K)$. By modular law, $H = H \cap T(H' \cup K) = T(H' \cup (H \cap K))$. As $(H \cap K) \cap H' = K \cap H' \subseteq H \cap \mathrm{Rad}(K) \subseteq H \cap \mathrm{Rad}(F) = \{1\}$, we get $(H \cap K) \cap H' = \{1\}$ and $H = T(H' \cup (H \cap K))$. Thus H is semisimple. The in particular statement is clear. □

Let G be a subfilter of a filter F of L. If subfilter H of F is maximal with respect to $G \cap H = \{1\}$, then we say that H is a complement of G. Using the maximal principle we readily see that if G is a subfilter of F, then the set of those subfilters of F whose intersection with G is $\{1\}$ contains a maximal element H. Thus every subfilter G of F has a complement.

Theorem 2.10. *Let F be a Rad-supplemented filter of L. Then there exist a semisimple subfilter K and a subfilter G with $\mathrm{Rad}(G) \trianglelefteq G$ such that $F = K \oplus G$.*

Proof. Let K be a complement of $\mathrm{Rad}(F)$ in F. We first show that $T(K \cup \mathrm{Rad}(F)) \trianglelefteq F$. If $\{1\} \neq H \subseteq F$ and $T(K \cup \mathrm{Rad}(F)) \cap H = \{1\}$, then we prove that $\mathrm{Rad}(F) \cap T(H \cup K) = \{1\}$. Let $x \in \mathrm{Rad}(F) \cap T(H \cup K)$. Then $x \in \mathrm{Rad}(F)$ and $x = (a \wedge b) \vee x = (x \vee a) \wedge (x \vee b)$ for some $a \in K$ and $b \in H$. As $a \vee x \in \mathrm{Rad}(F) \cap K = \{1\}$, we get $x = b \vee x \in H$. Thus $x \in H \cap T(K \cup \mathrm{Rad}(F)) = \{1\}$, contrary to the maximality of K. Thus $K \cap \mathrm{Rad}(F) = \{1\}$ and $T(K \cup \mathrm{Rad}(F)) \trianglelefteq F$. Since F is a Rad-supplemented filter, there is a subfilter G

of F such that $F = T(K \cup G)$ and $K \cap G \subseteq \text{Rad}(G)$. Since $K \cap G = K \cap (K \cap G) \subseteq K \cap \text{Rad}(G) \subseteq K \cap \text{Rad}(F) = \{1\}$; hence $F = K \oplus G$. By Proposition 2.9, K is semisimple. Since $\text{Rad}(F) = T(\text{Rad}(G) \cup \text{Rad}(K)) = T(\text{Rad}(G) \cup \{1\}) = \text{Rad}(G)$, $T(K \cup \text{Rad}(G)) \trianglelefteq F = T(K \cup G)$. Suppose, $\text{Rad}(G)$ is not essential in G; so $\text{Rad}(G) \cap A = \{1\}$ for some subfilter $A \neq \{1\}$ of G. Let $y \in T(K \cup \text{Rad}(G)) \cap A$. Then $y \in A$ and $y = (g_1 \wedge g_2) \vee y = (y \vee g_1) \wedge (y \vee g_2)$ for some $g_1 \in K$ and $g_2 \in \text{Rad}(G)$. Then $y \vee g_2 \in A \cap \text{Rad}(G) = \{1\}$; hence $y = g_1 \vee y \in K$. Therefore $y \in G \cap K = \{1\}$. Thus $T(K \cup \text{Rad}(G)) \cap A = \{1\}$ which is impossible. Thus $\text{Rad}(G) \trianglelefteq G$, as required. □

Proposition 2.11. *Assume that F_1 and G are subfilters of a filter F of L and let F_1 be a Rad-supplemented filter. If $T(F_1 \cup G)$ has a Rad-supplement in F, then the same is true for G.*

Proof. Let X be a Rad-supplement of $T(F_1 \cup G)$ in F; so $T(X \cup T(F_1 \cup G)) = F$ and $X \cap T(F_1 \cup G) \subseteq \text{Rad}(X)$. For $D = T(X \cup G) \cap F_1 \subseteq T(X \cup G)$, since F_1 is a Rad-supplemented filter, there exists a subfilter Y of F_1 such that $T(Y \cup D) = F_1$ and $D \cap Y = T(X \cup G) \cap Y \subseteq \text{Rad}(Y)$. By Lemma 2.1, we have $F = T(X \cup T(F_1 \cup G)) = T(F_1 \cup T(X \cup G)) = T(T(Y \cup D) \cup T(X \cup G)) \subseteq$

$$T(Y \cup T(D \cup T(G \cup X))) = T(Y \cup T(X \cup G)) = T(G \cup T(X \cup Y)) \subseteq F;$$

hence $F = T(G \cup T(X \cup Y)) = T(Y \cup T(X \cup G))$ and $T(X \cup G) \cap Y \subseteq \text{Rad}(Y)$, that is, Y is a Rad-supplement of $T(G \cup X)$ in F. Now we show that $T(X \cup Y)$ is a Rad-supplement of G in F. It is enough to show that $T(X \cup Y) \cap G \subseteq \text{Rad}(T(X \cup Y))$. Since $T(G \cup Y) \subseteq T(F_1 \cup G)$, $T(G \cup Y) \cap X \subseteq X \cap T(F_1 \cup G) \subseteq \text{Rad}(X)$. To simplify our notation let $T(X \cup G) \cap Y = A$ and $T(G \cup Y) \cap X = B$. If $z \in T(X \cup Y) \cap G$, then $z = z \vee (a \wedge b) \vee z = ((z \vee a) \wedge (z \vee b)) \vee z$ for some $a \in X$ (so $z \vee a \in B$) and $b \in Y$ (so $z \vee b \in A$) which implies that $z \in T(A \cup B)$. Hence $T(X \cup Y) \cap G \subseteq T(A \cup B)$. Therefore $T(X \cup Y) \cap G \subseteq T(A \cup B) \subseteq T(\text{Rad}(X) \cup \text{Rad}(Y)) \subseteq \text{Rad}(T(X \cup Y))$, as needed. □

Theorem 2.12. *Let F_1 and F_2 be Rad-supplemented filters of L. If $F = T(F_1 \cup F_2)$, then F is a Rad-supplemented filter.*

Proof. Let G be a subfilter of F (so $T(F_2 \cup G \cup F_1) = F$). Let H be a Rad-supplement of $A = T(F_2 \cup G) \cap F_1 \subseteq T(F_2 \cup G)$ in F_1; so $T(H \cup A) = F_1$ and $A \cap H = T(G \cup F_2) \cap H \subseteq \text{Rad}(H)$. Clearly, $T(A \cup F_2 \cup G) \subseteq T(F_2 \cup G)$. By Lemma 2.1, $F = T(F_2 \cup G \cup F_1) = T(F_2 \cup G \cup T(H \cup A)) \subseteq$

$$T(H \cup T(F_2 \cup G \cup A)) \subseteq T(H \cup T(F_2 \cup G)) \subseteq F;$$

hence $F = T(H \cup T(F_2 \cup G))$ which implies that H is a Rad-supplement of $T(F_2 \cup G)$ in F. Now the assertion follows from Proposition 2.11. □

Corollary 2.13. *If $F_1, \cdots, F_n$ are Rad-supplemented filters of L, then $T(U_{i=1}^n F_i)$ is a Rad-supplemented filter.*

Theorem 2.14. *The following statements are equivalent for a filter F of L.*

(1) *Every subfilter of F is Rad-supplement in F.*
(2) *Every subfilter of F is supplement in F.*
(3) *Every subfilter of F is a direct summand of F.*
(4) *F is semisimple.*

Proof. (4) $\Rightarrow$ (3). is clear. To see (3) $\Rightarrow$ (2), let X be a subfilter of F. Then there is a subfilter X' of F such that $X \cap X' = \{1\} \ll X$ and $F = T(X \cup X')$ by (3). It follows that X is a supplement of X' in F. (2) $\Rightarrow$ (1) is clear since each supplement is a Rad-supplement.

(1) $\Rightarrow$ (3). Let H be a subfilter of F. Then there is a subfilter G of F such that $F = T(G \cup H)$ and $G \cap H \subseteq \text{Rad}(H)$. Let $x \in G \cap H$. Then by Proposition 2.4, $T(\{x\}) \ll H$ and it follows that $T(\{x\}) \ll F$; hence $T(\{x\}) \subseteq \text{Rad}(F)$. By (1), $T(\{x\})$ is a Rad-supplement in F. Now Theorem 2.5 gives $\text{Rad}(T(\{x\})) = T(\{x\}) \cap \text{Rad}(F) = T(\{x\})$ which implies that $x = 1$ since every finitely generated filter contains a maximal subfilter. Therefore $G \cap H = \{1\}$, and so H is a direct summand of F. □

For the remainder of this section we collect some basic properties concerning weakly Rad-supplemented filters and amply Rad-supplemented filters.

Definition 2.15. Let G, H be subfilters of a filter F of L. If $F = T(G \cup H)$ and $G \cap H \subseteq \text{Rad}(F)$, then H is called a *weak Rad-supplement* of G in F. If every subfilter of F has a weak Rad-supplement in F, then F is called a *weakly Rad-supplemented filter.*

Proposition 2.16. *Let F be a weakly Rad-supplemented filter of L. Then every supplement filter of F is a weakly Rad-supplemented filter.*

Proof. Let K be a subfilter of F. If G is a subfilter of K, since F is a weakly Rad-supplemented, there is a subfilter H of F such that $F = T(G \cup H)$ and $G \cap H \subseteq \text{Rad}(F)$. Hence $K = K \cap T(G \cup H) = T(G \cup (K \cap H))$ and $G \cap (K \cap H) = G \cap H = K \cap (G \cap H) \subseteq K \cap \text{Rad}(F) = \text{Rad}(K)$. Thus K is a weakly Rad-supplemented filter. □

Proposition 2.17. *Assume that F_1 and G are subfilters of a filter F of L and let F_1 be a weakly Rad-supplemented filter. If $T(F_1 \cup G)$ has a weak Rad-supplement in F, then the same is true for G.*

Proof. By hypothesis, there is a subfilter N of F such that $T(N \cup T(F_1 \cup G)) = F$ and $N \cap T(F_1 \cup G) \subseteq \text{Rad}(F)$. Since F_1 is a Rad-supplemented filter, there exists a subfilter H of F such that $T(H \cup (F_1 \cap T(N \cup G))) = F_1$ and $H \cap T(N \cup G) \subseteq \text{Rad}(F_1)$. By an argument like that in Proposition 2.11, $F = T(G \cup T(N \cup H))$. Set $A = N \cap T(F_1 \cup G)$ and $B = H \cap T(G \cup N)$. We can easily show that $G \cap T(N \cup H) \subseteq T(A \cup B) \subseteq T(\text{Rad}(F) \cup \text{Rad}(F_1)) \subseteq \text{Rad}(F)$. This completes the proof. □

Theorem 2.18. *Let F_1 and F_2 be weakly Rad-supplemented filters of L. If $F = T(F_1 \cup F_2)$, then F is a weakly Rad-supplemented filter.*

Proof. Let G be a subfilter of F (so $T(F_2 \cup G \cup F_1) = F$). Let H be a Rad-supplement of $T(F_2 \cup G) \cap F_1$ in F_1. By an argument like that in Theorem 2.12, H is a Rad-supplement of $T(F_2 \cup G)$. Now the assertion follows from Proposition 2.17. □

Corollary 2.19. *If $F_1, \ldots, F_n$ are weakly Rad-supplemented filters of L, then $T(U_{i=1}^{n} F_i)$ is a weakly Rad-supplemented filter.*

Definition 2.20. A subfilter G of a filter F of L has ample Rad-supplement in F if, whenever $F = T(G \cup H)$, G has a Rad-supplement H' with $H' \subseteq H$. If every subfilter of F has an ample Rad-supplement in F, then F is called an amply Rad-supplemented filter.

Proposition 2.21. *Let F be a filter of L. If every subfilter of F is a Rad-supplemented filter, then F is an amply Rad-supplemented filter.*

Proof. Let G and H be subfilters of F such that $F = T(G \cup H)$. By assumption, There exists a subfilter H' of H such that $H = T(H' \cup (H \cap G))$ and $(G \cap H) \cap H' = H' \cap G \subseteq \text{Rad}(H')$. Then $H = T(H' \cup (H \cap G)) \subseteq T(H' \cup G)$ gives $F = T(G \cup H) \subseteq T(G \cup T(H' \cup G)) = T(H' \cup G) \subseteq F$; hence $F = T(H' \cup G)$, as required. □

Corollary 2.22. *The following statements are equivalent for a lattice L.*
(1) *Every filter is amply Rad-supplemented.*
(2) *Every filter is Rad-supplemented.*

Theorem 2.23. *Assume that F_1 and F_2 are subfilters of a filter F of L and let $F = T(F_1 \cup F_2)$. If F_1 and F_2 have ample Rad-supplements in F, then $F_1 \cap F_2$ also has ample Rad-supplements in F.*

Proof. Let V be a subfilter of F such that $F = T(V \cup (F_1 \cap F_2))$. Suppose now that $F_1 \cap F_2 = C$ and $F_1 \cap V = D$. Then by Lemma 2.1, $F_1 \cap F_2 \subseteq F_1$ gives $F_1 = F_1 \cap T(V \cup (F_1 \cap F_2)) = T((F_1 \cap F_2) \cup (F_1 \cap V)) = T(C \cup D)$ which implies that $F = T(F_1 \cup F_2) = T(T(C \cup D) \cup F_2) = T(D \cup T(C \cup F_2)) = T(D \cup F_2) = T(F_2 \cup (F_1 \cap V))$. Similarly, $F = T(F_1 \cup (F_2 \cap V))$. Since F_1, F_2 have ample Rad-supplements in F, there exist $V_2' \subseteq V \cap F_2$ and $V_1' \subseteq F_1 \cap V$ such that $F = T(F_1 \cup V_2')$ and $F_1 \cap V_2' \subseteq \text{Rad}(V_2')$, and $F = T(F_2 \cup V_1')$ and $F_2 \cap V_1' \subseteq \text{Rad}(V_1')$. We show that $T(V_1' \cup V_2')$ is a ample Rad-supplements of $F_1 \cap F_2$ in F. Since $V_1' \cup V_2' \subseteq V$, $T(V_1' \cup V_2') \subseteq V$. Moreover, $F_1 = F_1 \cap T(V_1' \cup F_2) = T(V_1' \cup (F_1 \cap F_2))$ and $F_2 = T(V_2' \cup (F_1 \cap F_2))$. Therefore $F = T(F_1 \cup V_2') = T(V_2' \cup T(V_1' \cup (F_1 \cap F_2))) \subseteq T(T(V_1' \cup V_2') \cup (F_1 \cap F_2)) \subseteq F$; so $F = T(T(V_1' \cup V_2') \cup (F_1 \cap F_2))$. Moreover, $T(V_1' \cup V_2') \cap (F_1 \cap F_2) = (T(V_1' \cup V_2') \cap F_1) \cap F_2 = (T(V_1' \cup (V_2' \cap F_1)) \cap F_2 = T((V_2' \cap F_1) \cup (V_1' \cap F_2)) \subseteq T(\text{Rad}(V_1') \cup \text{Rad}(V_2')) \subseteq \text{Rad}(T(V_1' \cup V_2')$. This completes the proof. □

Theorem 2.24. *Let F be an amply Rad-supplemented filter of L. Then the following hold:*

(1) *Every supplement of a subfilter of F is an amply Rad-supplemented filter.*
(2) *Every direct summand of F is an amply Rad-supplemented filter.*

Proof. (1). Assume that K is a supplement in F and let U be a subfilter of K. Suppose that $K = T(U \cup V)$ for some subfilter V of K. Since K is a supplement in F, there is a subfilter A of F with $F = T(K \cup A)$. It follows that $F = T(A \cup T(U \cup V)) = T(V \cup T(A \cup U))$. By assumption, V contains a Rad-supplement H of $T(A \cup U)$ in F; so $F = T(H \cup T(A \cup U)) = T(A \cup T(H \cup U))$ and $H \cap T(A \cup U) \subseteq \text{Rad}(H)$. Note that $T(U \cup H) \subseteq K$. Since K is a supplement of A in F and $F = T(A \cup T(H \cup U))$, we have $K = T(H \cup U)$. Now $U \cap H \subseteq H \cap T(A \cup U) \subseteq \text{Rad}(H)$ gives H is a Rad-supplement of U in K. Hence K is amply Rad-supplemented.

(2). Let G be a direct summand of an amply Rad-supplemented filter F. Then there is a subfilter H of F such that $F = T(G \cup H)$ and $G \cap H = \{1\}$. Suppose that $G = T(C \cup D)$ for some subfilters C, D of G. Then

$$F = T(T(C \cup D) \cup H) = T(D \cup T(C \cup H)).$$

By assumption, there exists a subfilter K of D with $F = T(K \cup T(C \cup H))$ and $K \cap T(C \cup H) \subseteq \text{Rad}(K)$. Therefore by modular law, $G =$

$$G \cap T(K \cup T(C \cup H)) = G \cap T(H \cup T(C \cup K)) = T(T(C \cup K) \cup (G \cap H)) =$$

$T(T(C \cup K)) = T(C \cup K)$. Since the inclusion $C \cap K \subseteq T(C \cup H) \cap K$ is clear we will prove the reverse inclusion. Let $x \in T(C \cup H) \cap K$. Then $x = (a \wedge b) \vee x = (x \vee a) \wedge (x \vee b)$ for some $a \in C$ and $b \in H$. Since $x \vee b \in H \cap K \subseteq G \cap H = \{1\}$, $x = x \vee a \in C$ since C is a filter, and so we have the equality. Now we have $G = T(C \cup K)$ and $K \cap C \subseteq \text{Rad}(K)$. This completes the proof. □

3. Rad-supplemented filters in quotient lattices

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(L, \leqslant)$, we define a relation on L, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on L, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leqslant_Q$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. It is straightforward to check that $(\frac{L}{F}, \leqslant_Q)$ is a poset. The following notation below will be kept in this section: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X = \{a \wedge F, b \wedge F\}$. By definition of $\leqslant_Q$, $(a \vee b) \wedge F$ is an upper bound for the set X. If $c \wedge F$ is any upper bound of X, then we can easily show that $(a \vee b) \wedge F \leqslant_Q c \wedge F$. Thus $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$. Thus $(\frac{L}{F}, \leqslant_Q)$ is a lattice.

Remark 3.1. Let G be a subfilter of a filter F of L.

(1) If $a \in F$, then $a \wedge F = F$. By the definition of $\leqslant_Q$, it is easy to see that $1 \wedge F = F$ is the greatest element of $\frac{L}{F}$.

(2) If $a \in F$, then $a \wedge F = b \wedge F$ (for every $b \in L$) if and only if $b \in F$. In particular, $c \wedge F = F$ if and only if $c \in F$. Moreover, if $a \in F$, then $a \wedge F = F = 1 \wedge F$.

(3) By the definition $\leqslant_Q$, we can easily show that if L is distributive, then $\frac{L}{F}$ is distributive.

(4) $\frac{F}{G} = \{a \wedge G : a \in F\}$ is a filter of $\frac{L}{G}$.

(5) If K is a filter of $\frac{L}{G}$, then $K = \frac{F}{G}$ for some filter F of L.

(6) If H is a filter of L such that $G \subseteq H$ and $\frac{F}{G} = \frac{H}{G}$, then $F = H$.

(7) If H and V are filters of L containing G, then $\frac{F}{G} \cap \frac{H}{G} = \frac{V}{G}$ if and only if $V = H \cap F$.

(8) If H is a filter of L containing G, then $\frac{T(F \cup H)}{G} = T(\frac{H}{G} \cup \frac{F}{G})$.

(9) Let H be a subfilter of F with $G \subseteq H$. H is a maximal subfilter of F if and only if $\frac{H}{G}$ is a maximal subfilter of $\frac{F}{G}$.

Lemma 3.2. *Let F be a filter of L. Then the following hold:*

(1) *If K, H are subfilters of F with $H \ll F$, then $\frac{T(H \cup K)}{K} \ll \frac{F}{K}$.*

(2) *Let K, H be subfilters of F with $K \subseteq H$. If $H \ll F$, then $\frac{H}{K} \ll \frac{F}{K}$.*

(3) *Let K, H be subfilters of F with $K \subseteq H$. If $K \ll F$ and $\frac{H}{K} \ll \frac{F}{K}$, then $H \ll F$.*

(4) *Assume that G is a subfilter of F and let H be a Rad-supplement of G in F. If $\frac{F}{G}$ is radical, then H is radical.*

Proof. (1). Assume that $A = T(H \cup K)$ and let $\frac{F}{K} = T(\frac{A}{K} \cup \frac{G}{K}) = \frac{T(A \cup G)}{K}$ for some subfilter $\frac{G}{K}$ of $\frac{F}{K}$; so $F = T(T(H \cup K) \cup G) = T(H \cup T(K \cup G)) = T(H \cup G)$. Then $H \ll F$ gives $G = F$. This completes the proof.

(2). Apply (1).

(3). Let $F = T(H \cup H')$ for some subfilter H' of F. Then by assumption,

$$T(\frac{H}{K} \cup \frac{T(H' \cup K)}{K}) = \frac{T(H \cup T(H' \cup K))}{K} =$$

$\frac{T(H' \cup H)}{K} = \frac{F}{K}$ gives $\frac{T(H' \cup K))}{K} = \frac{F}{K}$; hence $T(K \cup H') = F$. It follows that $H' = F$ since $K \ll F$, as required.

(4). Assume to the contrary, let $\mathrm{Rad}(H) \neq H$. Let K be a maximal subfilter of H. Then by Lemma 2.6, $T(G \cup K)$ is a maximal subfilter of F and so $\frac{T(K \cup G)}{G}$ is a maximal subfilter of $\frac{F}{G}$ by Remark 3.1 (9). This is a contradiction. Thus $\mathrm{Rad}(H) = H$, that is, H is radical. □

Lemma 3.3. *If F, K are filters of L with $K \subseteq F$, then*

$$\frac{T(\mathrm{Rad}(F) \cup K)}{K} \subseteq \mathrm{Rad}(\frac{F}{K}).$$

Proof. Let $x \wedge K \in \frac{T(\mathrm{Rad}(F)\cup K)}{K}$. Then there exist $a \in \mathrm{Rad}(F)$ and $k \in K$ such that $x = x \vee x = x \vee [(x \vee k) \wedge (x \vee a)]$. There are elements $a_{i_1} \in F_{i_1} \ll F, \cdots, a_{i_k} \in F_{i_k} \ll F$ such that $a = (a \vee a_{i_1}) \wedge \cdots \wedge (a \vee a_{i_k})$; so $x = x \vee [(x \vee k) \wedge (x \vee a \vee a_{i_1}) \wedge \cdots \wedge (x \vee a \vee a_{i_k})]$. By Proposition 1.2 and Lemma 3.2, $T(\frac{T(F_{i_1}\cup K)}{K} \cup \cdots \cup \frac{T(K\cup F_{i_k})}{K}) \ll \frac{F}{K}$. Now

$$[(x \vee k) \wedge (x \vee a \vee a_{i_1})] \wedge K \wedge_Q \cdots \wedge_Q [(x \vee k) \wedge (x \vee a \vee a_{i_k})] \wedge K$$

$$= [(x \vee k) \wedge (x \vee a \vee a_{i_1}) \wedge \cdots \wedge (x \vee a \vee a_{i_k})] \wedge K \leqslant_Q x \wedge K$$

gives $x \wedge K \in T(\frac{T(F_{i_1}\cup K)}{K} \cup \cdots \cup \frac{T(K\cup F_{i_k})}{K}) \subseteq \mathrm{Rad}(\frac{F}{K})$. This completes the proof. □

Proposition 3.4. *Let X, U be subfilters of a filter F of L with $X \subseteq U$. Then*

(1) *If V is a Rad-supplement of U in F, then $\frac{T(X\cup V)}{X}$ is a Rad-supplement of $\frac{U}{X}$ in $\frac{F}{X}$.*

(2) *If V is a weak Rad-supplement of U in F, then $\frac{T(X\cup V)}{X}$ is a weak Rad-supplement of $\frac{U}{X}$ in $\frac{F}{X}$.*

(3) *If V is a supplement of U in F, then $\frac{T(X\cup V)}{X}$ is a supplement of $\frac{U}{X}$ in $\frac{F}{X}$.*

Proof. (1). If $A = T(V \cup X)$, then by Lemma 2.1, $T(A \cup U) =$

$$T(U \cup T(V \cup X)) = T(V \cup T(U \cup X)) = T(U \cup V) = F.$$

Now Remark 3.1 gives $T(\frac{U}{X} \cup \frac{A}{X}) = \frac{T(U\cup A)}{X} = \frac{F}{X}$. For $X \subseteq U$, we have $U \cap T(X \cup V) = T(X \cup (U \cap V))$ by modular law, and so $\frac{U}{X} \cap \frac{T(V\cup X)}{X} = \frac{U\cap T(V\cup X)}{X} = \frac{T((U\cap V)\cup X)}{X}$ by Remark 3.1. Since V is a Rad-supplement of U in F, we have $D = U \cap V \subseteq \mathrm{Rad}(V)$. Therefore by Lemma 3.3, it is enough to show that

$$\frac{T(D \cup X)}{X} \subseteq \frac{T(X \cup \mathrm{Rad}(A))}{X} \subseteq \mathrm{Rad}(\frac{A}{X}).$$

Since $X \cup D \subseteq X \cup \mathrm{Rad}(V) \subseteq X \cup \mathrm{Rad}(A)$, we get $T(X \cup D) \subseteq T(X \cup \mathrm{Rad}(A))$, as required.

(2). Similar to the proof of part (1).

(3). Since V is a supplement of U in F, we have $D = U \cap V \ll V$. By an argument like that in (1), we obtain $T(\frac{U}{X} \cup \frac{A}{X}) = \frac{F}{X}$. By the above consideration, it is enough to show that $B = \frac{T(D\cup X)}{X} \ll \frac{A}{X}$. Let

$$T(B \cup \frac{K}{X}) = \frac{A}{X}$$

for some subfilter $\frac{K}{X}$ of $\frac{A}{X}$. Then $\frac{A}{X} = \frac{T(T(D\cup X)\cup K)}{X} = \frac{T(D\cup K)}{X}$; so $A = T(D \cup K) = T(K \cup (U \cap V))$. Since $U \cap V \ll V \subseteq T(V \cup X) = A$, we get $U \cap V \ll T(V \cup X) = \mathrm{A}$; hence $K = A$, as required. □

Theorem 3.5.

(1) If F is a Rad-supplemented filter of L, then every quotient filter of F is Rad-supplemented.

(2) If F is a weakly Rad-supplemented filter of L, then every quotient filter of F is weakly Rad-supplemented.

(3) If F is a supplemented filter of L, then every quotient filter of F is supplemented.

Proof. Clear from Proposition 3.4. □

Theorem 3.6.

(1) *If F is an amply Rad-supplemented filter of L, then every quotient filter of F is amply Rad-supplemented.*

(2) *If F is an amply supplemented filter of L, then every quotient filter of F is amply supplemented.*

Proof. (1). Let $\frac{V}{X}$ be a subfilter of $\frac{F}{X}$ such that $\frac{F}{X} = T(\frac{V}{X} \cup \frac{U}{X})$ for some subfilter $\frac{U}{X}$ of $\frac{F}{X}$; so $F = T(V \cup U)$. Since F is amply Rad-supplemented, there is a subfilter $H \subseteq U$ such that H is a Rad-supplement V in F. Then by Proposition 3.4, $\frac{T(H \cup X)}{X} \subseteq \frac{U}{X}$ is a Rad-supplement $\frac{V}{X}$ in $\frac{F}{X}$. Thus $\frac{F}{X}$ is amply Rad-supplemented.

(2). Similar to the proof of part (1). □

Theorem 3.7. *If F is a Rad-supplemented filter of L, then $\frac{F}{\mathrm{Rad}(F)}$ is a semisimple filter.*

Proof. Let G be any subfilter of F containing $\mathrm{Rad}(F)$. Then there is a Rad-supplement H of G in F; so $T(G \cup H) = F$ and $H \cap G \subseteq \mathrm{Rad}(H) \subseteq \mathrm{Rad}(F)$. Then $F = T(\mathrm{Rad}(F) \cup H \cup G) \subseteq T(G \cup T(\mathrm{Rad}(F) \cup H)) \subseteq F$ which implies that $F = T(G \cup T(\mathrm{Rad}(F) \cup H))$. Set $T(\mathrm{Rad}(F) \cup H) = A$. Thus by Remark 3.1, $\frac{F}{\mathrm{Rad}(F)} = \frac{T(G \cup A)}{\mathrm{Rad}(F)} = T(\frac{G}{\mathrm{Rad}(F)} \cup \frac{A}{\mathrm{Rad}(F)})$. It suffices to show that $\frac{G}{\mathrm{Rad}(F)} \cap \frac{A}{\mathrm{Rad}(F)} = \{\bar{1}\}$, where $\bar{1} = 1 \wedge \mathrm{Rad}(F) = \mathrm{Rad}(F)$ is the greatest element of $\frac{L}{\mathrm{Rad}(F)}$. By modular law and Remark 3.1, we have $\frac{G}{\mathrm{Rad}(F)} \cap \frac{A}{\mathrm{Rad}(F)} = \frac{G \cap A}{\mathrm{Rad}(F)} = \frac{T(\mathrm{Rad}(F) \cup (G \cap H))}{\mathrm{Rad}(F)} = \frac{T(\mathrm{Rad}(F))}{\mathrm{Rad}(F)} = \frac{\mathrm{Rad}(F)}{\mathrm{Rad}(F)} = \{\bar{1}\}$. This completes the proof. □

Corollary 3.8. *If F is a supplemented filter of L, then $\frac{F}{\mathrm{Rad}(F)}$ is a semisimple filter.*

Proof. Let G be any subfilter of F containing $\mathrm{Rad}(F)$. Then there is a supplement H of G in F; so $T(G \cup H) = F$ and $H \cap G \ll H$; hence $H \cap G \subseteq \mathrm{Rad}(H) \subseteq \mathrm{Rad}(F)$. By an argument like that in Theorem 3.7, we get $\frac{F}{\mathrm{Rad}(F)}$ is a semisimple filter. □

Theorem 3.9. *Let F be a filter of L such that* $\mathrm{Rad}(F) \ll F$. *Then F is a weakly Rad-supplemented filter if and only if $\frac{F}{\mathrm{Rad}(F)}$ is semisimple.*

Proof. Assume that F is a weakly Rad-supplemented filter and let H be any subfilter of F containing $\mathrm{Rad}(F)$. By assumption, there exists a subfilter G of F such that $F = T(H \cup G)$ and $G \cap H \subseteq \mathrm{Rad}(F)$. Thus

$$\frac{F}{\mathrm{Rad}(F)} = \frac{T(H \cup G)}{\mathrm{Rad}(F)} = T(\frac{H}{\mathrm{Rad}(F)} \cup \frac{T(G \cup \mathrm{Rad}(F))}{\mathrm{Rad}(F)})$$

and $\frac{H}{\mathrm{Rad}(F)} \cap \frac{T(G \cup \mathrm{Rad}(F)}{\mathrm{Rad}(F)} = \frac{H \cap T(G \cup \mathrm{Rad}(F))}{\mathrm{Rad}(F)} = \frac{T(\mathrm{Rad}(F) \cup (H \cap G)}{\mathrm{Rad}(F)} = \frac{\mathrm{Rad}(F)}{\mathrm{Rad}(F)} = \{\bar{1}\}$; so $\frac{H}{\mathrm{Rad}(F)}$ is a direct summand of $\frac{F}{\mathrm{Rad}(F)}$. Conversely, assume that $\frac{F}{\mathrm{Rad}(F)}$ is semisimple. For any subfilter H of F, since $\frac{F}{\mathrm{Rad}(F)}$ is semisimple, there is a subfilter G of F containing $\mathrm{Rad}(F)$ such that

$$\frac{F}{\mathrm{Rad}(F)} = \frac{T(H \cup \mathrm{Rad}(F))}{\mathrm{Rad}(F)} \oplus \frac{G}{\mathrm{Rad}(F)}.$$

Thus $F = T(G \cup T(\mathrm{Rad}(F) \cup H)) = T(\mathrm{Rad}(F) \cup T(H \cup G))$; hence $T(H \cup G) = F$ since $\mathrm{Rad}(F) \ll F$. Now $\frac{T(H \cup \mathrm{Rad}(F))}{\mathrm{Rad}(F)} \cap \frac{G}{\mathrm{Rad}(F)} = \frac{G \cap T(\mathrm{Rad}(F) \cup H)}{\mathrm{Rad}(F)} = \frac{T(\mathrm{Rad}(F) \cup (H \cap G))}{\mathrm{Rad}(F)} = \frac{\mathrm{Rad}(F)}{\mathrm{Rad}(F)}$ gives $G \cap H \subseteq \mathrm{Rad}(F)$; hence F is a weakly Rad-supplemented filter. □

Let G, H be subfilters of a filter F of L with $G \subseteq H$. We say H lies above G in F if $\frac{H}{G} \ll \frac{F}{G}$. F is called a lifting filter if every subfilter of F lies above a direct summand of F. A subfilter G of F is called coclosed in F if and only if G has no proper subfilter K such that G lies above K. We call H a weak supplement of G in F if and only if $F = T(G \cup H)$ and $H \cap G \ll F$.

Lemma 3.10. *Let F be a filter of L. If $G \subseteq H$ are subfilters of F, then H lies above G in F if and only if $T(G \cup K) = F$ holds for all subfilter K of F with $T(H \cup K) = F$.*

Proof. Suppose that H lies above G in F. If $F = T(H \cup K)$, then

$$\frac{F}{G} = \frac{T(H \cup K)}{G} = T(\frac{H}{G} \cup \frac{T(K \cup G)}{G})$$

and $\frac{H}{G} \ll \frac{F}{G}$ gives $\frac{F}{G} = \frac{T(K \cup G)}{G}$; hence $T(G \cup K) = F$. Conversely, suppose that $T(G \cup K) = F$ for all subfilter K of F with $T(H \cup K) = F$. If there exists a subfilter K of F containing G such that $T(\frac{H}{G} \cup \frac{K}{G}) = \frac{T(H \cup K)}{G} = \frac{F}{G}$, then $F = T(H \cup K)$ yields $F = T(K \cup G) = K$; so H lies above G in F. □

Proposition 3.11. *Let H be a subfilter of a filter F of L. Consider the following statements:*

(1) *H is a supplement in F;*
(2) *H is coclosed in F;*
(3) *For all subfilter K of H, $K \ll F$ implies $K \ll H$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *holds and if H is a weak supplement in F, then* (3) $\Rightarrow$ (1) *holds.*

Proof. (1) ⇒ (2). Assume that H is a supplement of $G \subseteq F$ (so $T(H \cup G) = F$). For all subfilters $K \subseteq H$ such that H lies above K, we have that $T(H \cup G) = F$ implies $T(G \cup K) = F$. By the minimality of H with respect to this property we get $K = H$. Hence H is coclosed.

(2) ⇒ (3). Let $K \ll F$ and $K \subseteq H$. Assume $H = T(K \cup X)$ for $X \subseteq H$; then for every $Y \subseteq F$ with $F = T(H \cup Y) = T(Y \cup T(K \cup X)) \subseteq T(K \cup T(X \cup Y)) \subseteq F$ we get $F = T(X \cup Y)$ since $K \ll F$; hence H lies above X. As H is coclosed, we get $H = X$ and thus $K \ll H$. Assume that H is a weak supplement of G in F; we show that (3) ⇒ (1). By assumption, $G \cap H \ll F$; so $G \cap H \ll H$ by (3). Thus H is a supplement G in F. □

Proposition 3.12. *Let F be a filter of L. If F is an amply supplemented filter, then every subfilter of F that is not small in F lies above a supplement in F.*

Proof. Let G be a subfilter of F such that it is not small in F. Let $F = T(X \cup G)$ with X a supplement of G in F; then G contains a supplement Y of X in F; hence $T(X \cup Y) = F$. Let $T(\frac{G}{Y} \cup \frac{K}{Y}) = \frac{F}{Y}$ for some subfilter $\frac{K}{Y}$ of $\frac{F}{Y}$. Then $T(K \cup G) = F$ by Remark 3.1. By Lemma 2.1, we have $K = K \cap T(X \cup Y) = T(Y \cup (K \cap X))$; so $F = T(G \cup T(Y \cup (K \cap X))) = T((K \cap X) \cup T(G \cup Y)) = T(G \cup (K \cap X))$. Since X is a supplement of G in F, we get $X \subseteq K$ which implies that $F = T(X \cup Y) \subseteq K$. Thus $\frac{G}{Y} \ll \frac{F}{Y}$, and so G lies above Y in F. □

We next give three other characterizations of lifting filters

Theorem 3.13. *Let F be a filter of L. Then the following statements are equivalent:*

(1) *F is lifting;*
(2) *For every subfilter G of F there is a decomposition $F = F_1 \oplus F_2$ such that $F_1 \subseteq G$ and $G \cap F_2 \ll F$.*
(3) *Every subfilter G of F can be written as $G = G_1 \oplus G_2$ with G_1 a direct summand of F and $G_2 \ll F$.*
(4) *F is amply supplemented and every coclosed subfilter of F is a direct summand of F.*

Proof. (1) ⇒ (2). Let G be a subfilter of F. Then G lies above a direct summand F_1 of F. Thus there is a decomposition $F = F_1 \oplus F_2$ with $\frac{G}{F_1} \ll \frac{F}{F_1}$. Suppose now that $T((G \cap F_2) \cup H) = F_2$ for some subfilter H of F_2. Then

$$F = T(F_1 \cup F_2) = T(F_1 \cup T(H \cup \cup(G \cap F_2))) = T(H \cup T(F_1 \cup (G \cap F_2)))$$

$$= T(H \cup T((F_1 \cup G) \cap (F_1 \cup F_2))) \subseteq T(H \cup T(F_1 \cup G)) = T(G \cup T(H \cup F_1)) \subseteq F;$$

so $F = T(G \cup T(H \cup F_1))$. Then by Remark 3.1, $\frac{F}{F_1} = T(\frac{G}{F_1} \cup \frac{T(H \cup F_1)}{F_1})$; so $\frac{F}{F_1} = \frac{T(H \cup F_1)}{F_1}$ which implies that $F = T(H \cup F_1)$. Since F_2 is a supplement of F_1 in F and $H \subseteq F_2$, we get $H = F_2$. Thus $G \cap F_2 \ll F_2$ and so $G \cap F_2 \ll F$ by Proposition 1.2.

(2) $\Rightarrow$ (3). For every subfilter G there is a decomposition $F = G_1 \oplus F_2$ with $G_1 \subseteq G$ and $G \cap F_2 \ll F$. It follows that $G = G \cap T(G_1 \cup F_2) = T(G_1 \cup (G \cap F_2))$. Set $G_2 = F_2 \cap G$. Therefore $G_1 \cap G_2 = \{1\}$, $G = T(G_1 \cup G_2)$ and $G_2 \ll F$.

(3) $\Rightarrow$ (4). Let $F = T(G \cup K)$ for subfilters G, K of F. We will show that K contains a supplement of G. By assumption, $K = N \oplus H$ with $H \ll F$ and N is a direct summand of F. Then $F = T(G \cup K) = T(G \cup T(N \cup H)) = T(H \cup T(G \cup N))$ gives $F = T(G \cup N)$ since $H \ll F$. By hypothesis, $G \cap N = N_1 \oplus S$ with $S \ll F$ and N_1 is a direct summand of F. It follows that $N_1 \cap H' = \{1\}$ and $F = T(N_1 \cup H')$ for some subfilter H' of F which implies that $N = N \cap T(N_1 \cup H') = T(N_1 \cup (H' \cap N))$ with $N_1 \cap (H' \cap N) = H' \cap N_1 = \{1\}$. Hence N_1 is a direct summand of N and $S \ll N$ (see proposition 3.11). Let $N = N_1 \oplus N_2$ for some subfilter N_2 of N. We claim that N_2 is a supplement of $T(N_1 \cup S)$ in N. To prove this consider a subfilter $Y \subseteq N_2$ such that $N = T(Y \cup T(N_1 \cup S)) = T(S \cup T(N_1 \cup Y))$. Then $T(N_1 \cup Y) = N$ holds as $S \ll N$. Thus $Y = N_2$ since N_2 is a supplement N_1 in N. Therefore N_2 is a supplement of $G \cap N = T(N_1 \cup S)$ in N and so $N = T(N_2 \cup T(S \cup N_1))$. So $F = T(G \cup N) = T(G \cup T(N_2 \cup (G \cap N)) = T(N_2 \cup T(G \cup (G \cap N))) = T(G \cup N_2)$ and $G \cap N_2 = (G \cap N) \cap N_2 \ll N_2$ holds; so N_2 is a supplement of G in F. Thus F is an amply supplemented filter.

Let U be a coclosed subfilter of F. Then $U = U_1 \oplus U_2$ with U_2 a direct summand of F and $U_1 \ll F$. Let K be a subfilter of F such that $T(K \cup U) = F$. So $F = T(K \cup U) = T(K \cup T(U_1 \cup U_2)) = T(U_1 \cup T(K \cup U_2))$ which implies that $T(K \cup U_2) = F$ since $U_1 \ll F$. It follows that U lies above U_2 in F by Lemma 3.10. Hence $U = U_2$ as U is coclosed.

(4) $\Rightarrow$ (1). Let G be a subfilter of F. If $G \ll F$, then $\frac{G}{K} \ll \frac{F}{K}$ for every direct summand K of F with $K \subseteq G$. So we may assume that G is not small in F. By Lemma 3.12, G lies above a supplement in F (so it is coclosed in F Proposition 3.11) and hence above a direct summand. □

Remark 3.14. Lifting filters are exactly the amply supplemented filters whose supplements are direct summands.

A filter F of L is called *Noetherian* if any non-empty set of subfilters of F has a maximal with respect to set inclusion. This definition is equivalent to ascending chain condition on subfilters of F.

Proposition 3.15. *Let F be a filter of L.*

(1) *F is Noetherian if and only if every subfilter of F is finitely generated.*

(2) $\mathrm{Rad}(F)$ *is Noetherian if and only if F satisfies ACC on small subfilters.*

Proof. (1). Let G be a subfilter of F. Assume that G is not finitely generated and look for a contradiction. Let Ω be the set of all subfilters of G which are finitely generated; so $\Omega \neq \emptyset$ since $\{1\} \in \Omega$. Since F is Noetherian, it follows from the maximal condition that Ω has a maximal element H with respect to inclusion with $H \subsetneqq G$. Let $x \in G \setminus H$; then $T(H \cup \{x\})$ is a finitely generated subfilter of G and $H \subsetneqq T(H \cup \{x\})$. Thus we have a contradiction to the maximality of H in Ω. The proof of the other implication is similar.

(2). By Proposition 1.2, since the small subfilters are subfilters of Rad(F), the necessity is clear. Conversely, assume that F satisfies ACC on small subfilters. Then F contains a maximal small subfilter H. Thus Rad(F) = H, and so Rad(F) is Noetherian by Proposition 1.2. □

Theorem 3.16. *Let F be a filter of L with ACC on small subfilters. Then F is an amply Rad-supplemented filter and every Rad-supplement is a direct summand if and only if F is a lifting filter.*

Proof. Suppose that F has the stated property; we show that F is a lifting filter. Let $F = T(G \cup H)$. Since F is an amply Rad-supplemented filter, there is a subfilter K of H such that $F = T(G \cup K)$ and $G \cap K \subseteq \mathrm{Rad}(K)$. Since F satisfies ACC on small subfilters, Rad(K) is Noetherian by Proposition 3.15, and hence Rad(K) is finitely generated by Proposition 3.15, so $\mathrm{Rad}(K) = T(A)$, where $A = \{a_1, \cdots, a_n\} \subseteq \mathrm{Rad}(K)$. By Proposition 1.2, for each i $(1 \leqslant i \leqslant n)$, $T(\{a_i\}) \ll K$; so $T(A) \subseteq T(T(\{a_1\}) \cup \cdots \cup T(\{a_n\})) \ll K$ which implies that $T(A) = \mathrm{Rad}(K) \ll K$. It follows that $G \cap K \ll K$. Therefore K is a supplement of G in F. Thus F is an amply supplemented filter. Since every supplement subfilter is a Rad-supplement filter, every supplement is a direct summand of F by assumption. Thus F is lifting.

Conversely, assume that F is lifting. Then F is an amply supplemented filter by Theorem 3.13, and hence it is an amply Rad-supplemented. Let H be a Rad-supplement in F; so there is a subfilter G of F such that $F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{Rad}(H)$. By an argument like that the above, we get H is a supplement of G. So H is a direct summand of F by assumption, as needed. □

References

[1] **E. Büyükasik, E. Mermut and S. Özdemir**, *Rad-supplemented modules*, Rend. Semin. Mat. Univ. Padova, **124** (2010), 157-177.

[2] **G. Birkhoff**, *Lattice theory*, Amer. Math. Soc., 1973.

[3] **C. Bicer, C. Nebiyev and A. Pancar**, *Generalized supplemented lattices*, Miskolc Math. Notes, **19** (2018), 141-147.

[4] **J. Clark, C. Lomp, N. Vanaja and R. Wisbauer**, *Lifting modules. Supplements and projectivity in module theory*, Frontiers Math. (Birkhäuser, Boston, 2006)

[5] **G. Calugareanu**, *Lattice Concepts of Module Theory*, Kluwer Acad. Publ., 2000.

[6] **S. Ebrahimi Atani and M. Sedghi Shanbeh Bazari**, *On 2-absorbing filters of lattices*, Discuss. Math. Gen. Algebra Appl., **36** (2016), 157-168.

[7] **S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel and M. Sedghi Shanbeh Bazari**, *A simiprime filter-based identity-summand graph of a lattice*, Le Matematiche, **73** (2018), 297-318.

[8] **S. Ebrahimi Atani and M. Chenari**, *Supplemented property in the lattices*, Serdica Math. J., **46** (2020), 73-88.

[9] **S. Ebrahimi Atani**, *G-supplemented property in the lattices*, Math. Bohemica, to appear.

[10] **A. Harmanci, D. Keskin and P.F. Smith**, *On ⊕-supplemented modules*, Acta Math. Hunger., **83** (1999), 161-169.

[11] **B. Kosar, C. Nebiyev and N. Sökmez**, *G-supplemented modules*, Ukr. Math. J., **67** (2015), 975-980.

[12] **F. Kasch and E.A. Mares**, *Eine Kennzeichnung semi-perfekter Moduln*, Nagoya Math. J., **27** (1966), 525-529.

[13] **S.H. Mohamed and B.J. Müller**, *Continuous and discrete modules*, Cambridge University Press, London, 1990.

[14] **Y. Miyashita**, *Quasi-projective modules, perfect modules, and a theorem for modular lattices*, J. Fac. Sci. Hokkaido Univ.(I), **19** (1966), 86-110.

[15] **Y. Wang and N. Ding**, *Generalized supplemented module*, Taiwanese J. Math., **10** (2006), 1589-1601.

[16] **R. Wisbauer**, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

[17] **H. Zöschinger**, *Komplementierte Moduln über Dedekindringen*, J. Algebra, **29** (1974), 42-56.

Received June 17, 2021

Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran
E-mails: ebrahimi@guilan.ac.ir, mehdikhoramdel@gmail.com, saboura_dolati@yahoo.com