Quasigroups and Related Systems **30** (2022), 47 – 62
https://doi.org/10.56415/qrs.v30.04

Simplicial polygroups and the generalized Moore complexes

Bijan Davvaz and *Murat Alp*

Abstract. A simplicial group is a simplicial object in the category of groups. A very nice application of simplicial group which is simplicial polygroup is given in this paper. Using polygroups instead of groups, we already had very good results from the well known properties due to Loday. Loday proved that a crossed module, a cat^1-group, a group object in the category of categories and a simplicial group whose Moore complex is of length one are equivalent. Using Loday's idea we present a functor from the category of groups to the category of polygroups and the simplicial groups to the simplicial polygroups. We show that there exist a functor from the category of cat^1-polygroups to the category of groups and the category of groups to the category of polygroups. We also prove that the category of simplicial groups is equivalent to the category of simplicial polygroups and the category of simplicial polygroups with generalized Moore complex with of length one is equivalent to the category of polygroups.

1. Introduction

More than last thirty years, simplicial groups play very important role to help improving homological group theory and homotopy theory. Especially simplicial group with Moore complex in grater than n provide n-types simplicial groups.

Crossed module firstly defined by J. H. C. Whitehead in [23]. Crossed module is a very powerful applications tools for simplicial groups too. Because, some applications of simplicial groups has been got from the equivalence categories idea. The good example of these applications can be found in [2]. Equivalent categories are presented by Loday [20] as a crossed module, a cat^1-group, a group object in the category of categories and a simplicial group whose Moore complex is of length one. The important application of these equivalent categories presented in this paper as well.

The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many area, such as geometry, lattices, combinatorics and color scheme. There exists a rich bibliography: publications appeared within 2013 can be found in "Polygroup Theory and Related Systems" by B. Davvaz [9]. This book contains the principal definitions endowed with examples and the basic results of the theory. Applications

2010 Mathematics Subject Classification: 20N20
Keywords: action, crossed module, polygroup, fundamental relation, simplicial group

of hypergroups appears in special subclasses like polygroups that they were studied by Comer [4], also see [1, 9, 10, 11, 15]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible in itself multigroup. A new application of crossed module which is crossed module of polygroups is presented in [3]. Also, cat1-polygroups defined by Davvaz and Alp in [12]. In this paper we show that a polygroup object in the category of categories is equivalent to a simplicial polygroup whose generalized Moore complex is of length one. To do this new application we use the idea of Mutlu and Porter in [21].

2. Simplicial groups

Simplicial groups occupy a place somewhere between homological group theory, homotopy theory, algebraic K-theory an algebraic geometry. We recall some basic concepts from [21].

A simplicial group $\mathbf{G}$ over the category of groups $\mathcal{G}$ consists of

(i) for every integer $n \geqslant 0$ an object $G_n \in \mathcal{G}$, and

(ii) for every pair of integers (i, n) with $0 \leqslant i \leqslant n$, face and degeneracy maps

$$d_i : G_n \to G_{n-1} \qquad \text{and} \qquad s_i : G_n \to G_{n+1}$$

in $\mathcal{G}$ satisfying the simplicial identities:

$$d_i d_j = d_{j-1} d_i \text{ if } i < j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ id & \text{if } i = j \text{ or } j+1, \\ s_j d_{i-1} & \text{if } i > j+1, \end{cases} \tag{1}$$

$$s_i s_j = s_j s_{i-1} \text{ if } i > j.$$

Similarly a simplicial map $f : \mathbf{G} \to \mathbf{G}'$ between two simplicial groups consists of maps

$$f_n : G_n \to G'_n \in \mathcal{G}$$

which commute with the face and degeneracy maps, i.e.

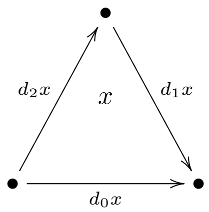
$$d_i f_n = f_{n-1} d_i \quad \text{and} \quad s_i f_n = f_{n+1} s_i \quad \text{for all } i.$$

Now, we recall a few well-known definitions and facts about simplicial groups. Suppose $\mathbf{G}$ is a simplicial group.

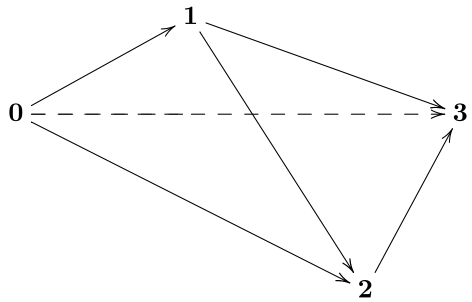
For $n = 0$, a 0-dimensional simplex is simply an element $x \in G_0$ and a 1-dimensional simplex is just, for $x \in G_1$,

$$d_1 x \bullet \xrightarrow{\ x\ } \bullet d_0 x.$$

2-dimensional simplices are just triangles: for $x \in G_2$



with the indicated faces, and 3-dimensional simplices are just tetrahedra:



The face d_ix is the face opposite the i^{th} vertex and so on.

Lemma 2.1 and Proposition 2.2 are related to semi-direct product of simplicial groups.

Lemma 2.1. [21] *Let* **G** *be a simplicial group. Then,* G_n *can be decomposed as a semi-direct product:*

$$G_n \cong kerd_0^n \ltimes s_0^{n-1}(G_{n-1}).$$

Proposition 2.2. [5] *If* **G** *is a simplicial group, then for any* $n \geqslant 0$*,*

$$\begin{aligned} G_n \quad &\cong (\ldots NG_n \ltimes s_{n-1}NG_{n-1}) \ltimes \ldots \ltimes s_{n-2} \ldots \ltimes s_1NG_1) \ltimes \\ &\quad (\ldots (s_0NG_{n-1} \ltimes s_1s_0NG_{n-2}) \ltimes \ldots \ltimes s_{n-1}s_{n-2} \ldots s_0NG_0). \end{aligned}$$

3. Crossed modules

Let G be a group and Ω be a non-empty set. A *(left) group action* is a binary operator $\tau : G \times \Omega \to \Omega$ that satisfies the following two axioms:

(1) $\tau(gh, \omega) = \tau(g, \tau(h, \omega))$, for all $g, h \in G$ and $\omega \in \Omega$,

(2) $\tau(e, \omega) = \omega$, for all $\omega \in \Omega$.

For $\omega \in \Omega$ and $g \in G$, we write ${}^{g}\omega := \tau(g, \omega)$. A *crossed module* $X = (M, N, \partial, \tau)$ consists of groups M and N together with a homomorphism $\partial : M \to N$ and a (left) action $\tau : N \times M \to M$ on M, satisfying the conditions:

(1) $\partial(\,^{g}m) = g\partial(m)g^{-1}$, for all $m \in M$ and $g \in N$,

(2) $\,^{\partial(m)}m' = mm'm^{-1}$, for all $m, m' \in M$.

The crossed module X also is denoted by $X = (\partial : M \to N)$. Let M be a group and take $G = Aut(M)$. Then, ∂ sends x to the inner automorphism $x(-)x^{-1}$. This obviously is a crossed module with the respect to the action of $Aut(M)$ on M.

A *categorical group* or *cat*^{1}*-group* is a group G together with a subgroup N and two homomorphisms $s, b : G \to N$ satisfying the following conditions:

(1) $s|_N = b|_N = id_N$,

(2) $[kers, kerb] = 1$.

This cat^1-group is denoted by $C = (G; N)$ if no confusion can arise. A morphism of cat^1-groups $C \to C'$ is a group homomorphism $f : G \to G'$ such $f(N) \subseteq N'$ and $s'f = f|_N s$, $b'f = f|_N b$.

Lemma 3.1. [20] *The following data are equivalent:*

(1) *a crossed module* $\partial : M \to N$,

(2) *a cat*^{1} *group* $C = (G; N)$,

(3) *a group object in the category of categories,*

(4) *a simplicial group whose Moore complex is of the length one.*

4. Polygroups and crossed polymodules

Let H be a non-empty set and $\star : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \star)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \star B = \bigcup_{\substack{a \in A \\ b \in B}} a \star b,\ \ A \star x = A \star \{x\} \text{ and } x \star B = \{x\} \star B.$$

A hypergroupoid $(H, \star)$ is called a *semihypergroup* if for all a, b, c of H we have $(a \star b) \star c = a \star (b \star c)$, which means that

$$\bigcup_{u \in a \star b} u \star c = \bigcup_{v \in b \star c} a \star v.$$

Let $(H, \star)$ is a semihypergroup and A be a non-empty subset of H. We say that A is a *complete part* of H if for any non-zero natural number n and for all $a_1, \ldots, a_n$ of H, the following condition holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \ \Rightarrow\ \prod_{i=1}^{n} a_i \subseteq A.$$

A hypergroupoid $(H,\star)$ is called a *quasihypergroup* if for all a of H we have $a\star H = H\star a = H$. This condition is also called the *reproduction axiom*. A hypergroupoid $(H,\star)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*. The details about algebraic hyperstructures can be found in the comprehensive reviews published by Corsini [6], Corsini and Leoreanu [7], Davvaz and Leoreanu-Fotea [13] and Vougiouklis [22]. Also, see [18, 19]. A special class of hypergroups is polygroups. We recal the following definition from [4, 9] A *polygroup* is a multi-valued system $\mathcal{M} =< P, \circ, e,^{-1} >$, with $e\in P$, $^{-1} : P \longrightarrow P$, $\circ : P\times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold for all x, y, z in P:

(1) $(x\circ y)\circ z = x\circ(y\circ z)$,

(2) $e\circ x = x\circ e = x$,

(3) $x\in y\circ z$ implies $y\in x\circ z^{-1}$ and $z\in y^{-1}\circ x$.

By using the concept of generalized permutation, in [8], Davvaz defined permutation polygroups and action of a polygroup on a set. For the definition of crossed polymodule, it is necessary the notion of polygroup action.

Definition 4.1. [8, 9] Let $\mathcal{P} =< P, \circ, e,^{-1} >$ be a polygroup and Ω be a nonempty set. A map $\alpha : P\times\Omega \to \mathcal{P}^*(\Omega)$ is called a *(left) polygroup action* on Ω if the following axioms hold:

(1) $\alpha(e,\omega) = \{\omega\} = \omega$, for all $\omega\in\Omega$,

(2) $\alpha(h,\alpha(g,\omega)) = \bigcup\limits_{x\in h\circ g}\alpha(x,\omega)$, for all $g, h\in P$ and $\omega\in\Omega$,

(3) $\bigcup\limits_{\omega\in\Omega}\alpha(g,\omega) = \Omega$, for all $g\in P$,

(4) for all $g\in P$, $x\in\alpha(g,y)\Rightarrow y\in\alpha(g^{-1},x)$.

From the second condition, we get $\bigcup\limits_{\omega_0\in\alpha(g,\omega)}\alpha(h,\omega_0) = \bigcup\limits_{x\in h\circ g}\alpha(x,\omega)$. For $\omega\in\Omega$, we write ${}^{g}\omega := \alpha(g,\omega)$. Therefore, we have

(1) ${}^{e}\omega = \omega$,

(2) ${}^{h}({}^{g}\omega) = {}^{h\circ g}\omega$, where ${}^{g}A = \bigcup\limits_{a\in A}{}^{g}a$ and ${}^{B}\omega = \bigcup\limits_{b\in B}{}^{b}\omega$, for all $A\subseteq\Omega$ and $B\subseteq P$,

(3) $\bigcup\limits_{\omega\in\Omega}{}^{g}\omega = \Omega$,

(4) for all $g\in P$, $a\in{}^{g}b\Rightarrow b\in{}^{g^{-1}}a$.

Alp and Davvaz in [3, 12] introduced and studied the concept of crossed polymodules as a generalization of crossed modules

Definition 4.2. A *crossed polymodule* $\mathcal{X} = (C, P, \partial, \alpha)$ consists of polygroups $< C, \star, e,^{-1} >$ and $< P, \circ, e,^{-1} >$ together with a strong homomorphism $\partial : C \to P$ and a (left) action $\alpha : P \times C \to \mathcal{P}^*(C)$ on C, satisfying the conditions:

(1) $\partial(\ {}^{p}c) = p \circ \partial(c) \circ p^{-1}$, for all $c \in C$ and $p \in P$,

(2) ${}^{\partial(c)}c' = c \star c' \star c^{-1}$, for all $c, c' \in C$.

Example 4.3. [12]A conjugation crossed polymodule is an inclusion of a normal subpolygroup N of P, with action given by conjugation. In particular, for any polygroup P the identity map $Id_P : P \to P$ is a crossed polymodule with the action of P on itself by conjugation. Indeed, there are two canonical ways in which a polygroup P may be regarded as a crossed polymodule: via the identity map or via the inclusion of the trivial subpolygroup.

We say the action of P on C is *productive*, if for all $c \in C$ and $p \in P$ there exist $c_1, \dots, c_n$ in C such that ${}^{p}c = c_1 \star \dots \star c_n$. The action defined in Examples 4.3 is productive.

5. Fundamental relation

Let $< P, \circ, e,^{-1} >$ be a polygroup. We define the relation β^*_P as the smallest equivalence relation on P such that the quotient P/β^*_P, the set of all equivalence classes, is a group. In this case β^*_P is called the *fundamental equivalence relation* on P and P/β^*_P is called the *fundamental group*. The product $\odot$ in P/β^*_P is defined as follows: $\beta^*_P(x) \odot \beta^*_P(y) = \beta^*_P(z)$, for all $z \in \beta^*_P(x) \circ \beta^*(y)$. This relation is introduced by Koskas [16] and studied mainly by Corsini [6], Leoreanu-Fotea [17] and Freni [14] concerning hypergroups, Vougiouklis [22] concerning H_v-groups, Davvaz concerning polygroups [9], and many others. We consider the relation β_P as follows:

$$x \ \beta_P \ y \ \Leftrightarrow \ \text{there exist } z_1, \dots z_n \text{ such that } \{x, y\} \subseteq \circ \prod_{i=1}^{n} z_i.$$

Freni in [14] proved that for hypergroups $\beta = \beta^*$. Since polygroups are certain subclass of hypergroups, we have $\beta^*_P = \beta_P$. The kernel of the *canonical map* $\varphi_P : P \longrightarrow P/\beta^*_P$ is called the *core* of P and is denoted by ω_P. Here we also denote by ω_P the unit of P/β^*_P. It is easy to prove that the following statements: $\omega_P = \beta^*_P(e)$ and $\beta^*_P(x)^{-1} = \beta^*_P(x^{-1})$, for all $x \in P$. So, the notion of a heart of a polygroup is directly connected to the fundamental relation on that polygroup.

Lemma 5.1. *ω_P is a subpolygroup of P.*

Theorem 5.2. [13] *The heart of a polygroup P is the smallest complete part subpolygroup of P*

Lemma 5.3. [3, 12] *For every $p \in P$, $p \circ p^{-1} \subseteq \omega_p$.*

Throughout the paper, we denote the binary operations of the fundamental groups P/β^*_P and C/β^*_C by $\odot$ and $\otimes$, respectively.

Theorem 5.4. [3, 12] *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crosed polymodule such that the action of P on C is productive. Then, $\mathcal{X}_{\beta^*} = (C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ is a crossed module.*

Now, we can consider another notion of the kernel of a strong homomorphism of polygroups. Let $< P, \circ, e, ^{-1} >$ and $< C, \star, e, ^{-1} >$ be two polygroups and $\partial : C \to P$ be a strong homomorphism. The *core-kernel* of ∂ is defined by

$$ker^*\partial = \{x \in C \mid \partial(x) \in \omega_P\}.$$

Lemma 5.5. [3, 12] *$ker^*\partial$ is a normal subpolygroup of C.*

Theorem 5.6. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule. Then, $ker^*\partial$ is a $P/\partial(C)$-polymodule.*

The following proposition is noted in [3]. A proof is included for completeness.

Proposition 5.7. *Let $< C, \star, e, ^{-1} >$ and $< P, \circ, e, ^{-1} >$ be two polygroups and let $\partial : C \to P$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D} : C/\beta^*_C \to P/\beta^*_P$ by setting*

$$\mathcal{D}(\beta^*_C(c)) = \beta^*_P(\partial(c)), \text{ for all } c \in C.$$

Proof. First, we prove that $\mathcal{D}$ is well defined. Suppose that $\beta^*_C(c_1) = \beta^*_C(c_2)$. Then, there exist $a_1, \ldots, a_n$ such that $\{c_1, c_2\} \subseteq \star \prod_{i=1}^{n} a_i$. So,

$$\{\partial(c_1), \partial(c_2)\} \subseteq \partial\left(\star \prod_{i=1}^{n} a_i\right) = \circ \prod_{i=1}^{n} \partial(a_i).$$

Hence, $\partial(c_1)\ \beta^*_P\ \partial(c_2)$, which implies that $\mathcal{D}\left(\beta^*_C(c_1)\right) = \mathcal{D}\left(\beta^*_C(c_2)\right)$. Now, we have

$$\begin{aligned}\mathcal{D}(\beta^*_C(c_1) \otimes \beta^*_C(c_2)) &= \mathcal{D}(\beta^*_C(c_1 \star c_2)) = \beta^*_P(\partial(c_1 \star c_2)) \\ &= \beta^*_P(\partial(c_1) \circ \partial(c_2)) = \beta^*_P(\partial(c_1)) \odot \beta^*_P(\partial(c_2)) \\ &= \mathcal{D}(\beta^*_C(c_1)) \odot \mathcal{D}(\beta^*_C(c_2)).\end{aligned}$$

□

Lemma 5.8. *Let $< P, \circ, e, ^{-1} >$ and $< C, *, e, ^{-1} >$ be two polygroups and ∂ be a strong homomorphism $C \to P$. Then, for $x \in C$, $\beta^*_C(x) \in kerD \Leftrightarrow x \in \ker^* \partial$.*

Proof. Suppose that $x \in C$. Then, we have

$$\begin{aligned}\beta^*_C(x) \in kerD &\Leftrightarrow D(\beta^*_C(x) \in kerD) = w_P \\ &\Leftrightarrow \beta^*_P(\partial(x)) = w_P \\ &\Leftrightarrow \partial(x) \in w_P \\ &\Leftrightarrow x \in ker^*\partial\end{aligned}$$

□

Definition 5.9. [12] A cat^1-polygroup $\mathcal{C} = (k; t, h : P \to C)$ consists of polygroups P and C, two strong epimorphisms $t, h : P \to C$ and an embedding $k : C \to P$ satisfying

$$\begin{array}{ll} \text{CAT-P-1}: & tk = hk = Id_C, \\ \text{CAT-P-2}: & [x, y] \subseteq w_P, \forall x \in ker^* t, \forall y \in ker^* h, \end{array}$$

where $[x, y] = \{z \mid z \in x \circ y \circ x^{-1} \circ y^{-1}\}$.

The maps t, h are called the *source* and *target*.

Lemma 5.10. [12] *Condition CAT-P-2 is equivalent to, for all $x, y \in P$,*

$$[\beta^*_P(x), \beta^*_P(y)] = w_P = 1_{P/\beta^*_P}.$$

6. Simplicial groups obtained from simplicial polygroups

In this section first we introduce the concept of simplicial polygroups. Indeed, in the theory of simplicial sets, a simplicial polygroup is a simplicial object in the category of polygroups. Then, by using the notion of fundamental relation, we make a connection between simplicial polygroups and simplicial groups.

Definition 6.1. A simplicial polygroup $\mathbf{P}$ over the category of polygroups $\mathcal{P}$ consists of

(i) for every integer $n \geqslant 0$ an object $P_n \in \mathcal{P}$, and

(ii) for every pair of integers (i, n) with $0 \leqslant i \leqslant n$, face and degeneracy maps

$$d_i : P_n \to P_{n-1} \qquad \text{and} \qquad s_i : P_n \to P_{n+1}$$

in $\mathcal{P}$ satisfying the simplicial identities:

$$d_i d_j = d_{j-1} d_i \text{ if } i < j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i \text{ if } i < j, \\ id \qquad \text{if i = j or j + 1}, \\ s_j d_{i-1} \text{ if } i > j + 1, \end{cases} \tag{2}$$

$$s_i s_j = s_j s_{i-1} \text{ if } i > j.$$

Lemma 6.2. *Every simplicial group is a simplicial polygroup.*

Proof. It is clear, since every group is a polygroup, □

Theorem 6.3. *Let $\mathbf{P}$ be a simplicial polygroup. Then, by using the fundamental relations, we obtain a simplicial group.*

Proof. For every pair of integers (i, n) with $0 \leqslant i \leqslant n$, we define

$$D_i : P_n/\beta_n^* \longrightarrow P_{n-1}/\beta_{n-1}^*$$
$$D_i\left(\beta_n^*(x)\right) = \beta_{n-1}^*(d_i(x)), \text{ for all } x \in P_n$$

and

$$S_i : P_n/\beta_n^* \longrightarrow P_{n+1}/\beta_{n+1}^*$$
$$S_i\left(\beta_n^*(x)\right) = \beta_{n+1}^*(s_i(x)), \text{ for all } x \in P_n.$$

Then, for $i < j$ we obtain

$$\begin{aligned} D_iD_j\left(\beta_n^*(x)\right) &= D_i\left(\beta_{n-1}^*(d_j(x)\right) = \beta_{n-2}^*(d_id_j(x)) \\ &= \beta_{n-2}^*(d_{j-1}d_i(x)) = D_{j-1}\left(\beta_{n-1}^*(d_i(x)\right) \\ &= D_{j-1}D_i\left(\beta_n^*(x)\right). \end{aligned}$$

So, $D_iD_j = D_{j-1}D_i$.

(1) For $i < j$, we have

$$\begin{aligned} D_iS_j\left(\beta_n^*(x)\right) &= D_i\left(\beta_{n+1}^*(s_j(x)\right) = \beta_n^*(d_is_j(x)) \\ &= \beta_n^*(s_{j-1}d_i(x)) = S_{j-1}\left(\beta_{n-1}^*(d_i(x)\right) \\ &= S_{j-1}D_i\left(\beta_n^*(x)\right). \end{aligned}$$

(2) For $i = j$ or $j + 1$, we have

$$D_iS_j\left(\beta_n^*(x)\right) = D_i\left(\beta_{n+1}^*(s_j(x)\right) = \beta_n^*(d_is_j(x)) = \beta_n^*(x)).$$

(3) For $i > j + 1$, we have

$$\begin{aligned} D_iS_j\left(\beta_n^*(x)\right) &= D_i\left(\beta_{n+1}^*(s_j(x)\right) = \beta_n^*(d_is_j(x)) \\ &= \beta_n^*(s_jd_{i-1}(x)) = S_j\left(\beta_{n-1}^*(d_{i-1}(x)\right) \\ &= S_jD_{i-1}\left(\beta_n^*(x)\right). \end{aligned}$$

Therefore, we conclude that

$$D_iS_j = \begin{cases} S_{j-1}D_i & \text{if } i < j, \\ Id & \text{if } i = j \text{ or } j+1, \\ S_jD_{i-1} & \text{if } i > j+1, \end{cases}$$

Also, for $i > j$ we obtain

$$\begin{aligned} S_iS_j\left(\beta_n^*(x)\right) &= S_i\left(\beta_{n+1}^*(s_j(x)\right) = \beta_{n+2}^*(s_is_j(x)) \\ &= \beta_{n+2}^*(s_js_{i-1}(x)) = S_j\left(\beta_{n+1}^*(s_{i-1}(x)\right) \\ &= S_jS_{i-1}\left(\beta_n^*(x)\right). \end{aligned}$$

So, $S_iS_j = S_jS_{i-1}$. This completes the proof. □

Theorem 6.4. *Let* **P** *and* **H** *be two simplicial polygroups and* $\mathbf{f}: \mathbf{P} \to \mathbf{H}$ *be a simplicial map between them consists of maps*

$$f_n : P_n \to H_n.$$

Then, $\mathbf{F}: \mathbf{P}/\beta^*_P \to \mathbf{H}/\beta^*_H$ *is a simplicial map between simplicial groups consists of maps*

$$F_n : P_n/(\beta^*_P)_n \longrightarrow H_n/(\beta^*_H)_n$$

$$F_n\left((\beta^*_P)_n(x)\right) = (\beta^*_H)_n(f(x)), \text{ for all } x \in P_n.$$

Proof. We have

$$\begin{aligned} D_i F_n\left((\beta^*_P)_n(x)\right) &= D_i\left((\beta^*_H)_n(f_n(x)\right) = (\beta^*_H)_{n-1}(d_i f_n(x)) \\ &= (\beta^*_H)_{n-1}(f_{n-1} d_i(x)) = F_{n-1}\left((\beta^*_P)_{n-1}(d_i(x)\right) \\ &= F_{n-1} D_i\left((\beta^*_P)_n(x)\right). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} S_i F_n\left((\beta^*_P)_n(x)\right) &= S_i\left((\beta^*_H)_n(f_n(x)\right) = (\beta^*_H)_{n+1}(s_i f_n(x)) \\ &= (\beta^*_H)_{n+1}(f_{n+1} s_i(x)) = F_{n+1}\left((\beta^*_P)_{n+1}(s_i(x)\right) \\ &= F_{n+1} S_i\left((\beta^*_P)_n(x)\right). \end{aligned}$$

□

Corollary 6.5. *Let* **P** *be a simplicial polygroup. Then,* $P_n/\beta^*_{P_n}$ *can be decomposed as a semi-direct product:*

$$P_n/\beta^*_{P_n} \cong ker D^n_0 \ltimes S^{n-1}_0(P_{n-1}/\beta^*_{P_{n-1}}).$$

Proof. The proof follows from Theorem 6.3 and Lemma 2.1. □

Corollary 6.6. *If* **P** *is a simplicial polygroup, then for any* $n \geqslant 0$,

$$\begin{aligned} P_n/\beta^*_{P_n} \cong (\ldots NP_n/\beta^*_{P_n} \ltimes S_{n-1}NP_{n-1}/\beta^*_{P_{n-1}}) \ltimes \ldots \ltimes S_{n-2}\ldots \ltimes S_1 NP_1/\beta^*_{P_1}) \ltimes \\ (\ldots (S_0 NP_{n-1}/\beta^*_{P_{n-1}} \ltimes S_1 S_0 NP_{n-2}/\beta^*_{P_{n-2}}) \ltimes \ldots \ltimes \\ S_{n-1}S_{n-2}\ldots S_0 NP_0/\beta^*_{P_0}). \end{aligned}$$

Proof. The proof follows from Theorem 6.3 and Proposition 2.2. □

The order of terms in this multiple semi-direct product are generated by the sequence

$$\begin{aligned} & P_1/\beta^*_{P_1} \cong (NP_1/\beta^*_{P_1} \ltimes S_0 NP_0/\beta^*_{P_0}), \\ & P_2/\beta^*_{P_2} \cong (NP_2/\beta^*_{P_2} \ltimes S_1 NP_1/\beta^*_{P_1}) \ltimes (S_0 NP_1/\beta^*_{P_1} \ltimes S_1 S_0 NP_0/\beta^*_{P_0}), \\ & P_3/\beta^*_{P_3} \cong (NP_3/\beta^*_{P_3} \ltimes S_2 NP_2/\beta^*_{P_2}) \ltimes (S_1 NP_2/\beta^*_{P_2} \ltimes S_2 S_1 NP_1/\beta^*_{P_1}) \ltimes \\ & \qquad (S_0 NP_2/\beta^*_{P_2} \ltimes S_2 S_0 NP_1/\beta^*_{P_1}) \ltimes (S_1 S_0 NP_1/\beta^*_{P_1} \ltimes S_2 S_1 S_0 NP_0/\beta^*_{P_0}), \end{aligned}$$

$$P_4/\beta^*_{P_4} \cong (NP_4/\beta^*_{P_4} \ltimes S_3NP_3/\beta^*_{P_3}) \ltimes (S_2NP_3/\beta^*_{P_3} \ltimes S_3S_2NP_2/\beta^*_{P_2}) \ltimes (S_1NP_3/\beta^*_{P_3} \ltimes S_3S_1NP_2/\beta^*_{P_2}) \ltimes (S_2S_1NP_2/\beta^*_{P_2} \ltimes S_3S_2S_1NP_1/\beta^*_{P_1}).$$

Theorem 6.7. *Let $\mathcal{C} = (k; t, h : P \to C)$ be a cat^1-polygroup. Then, by using the fundamental relation, we obtain a group object in the category of categories.*

Proof. Starting with a cat^1-polygroup $\mathcal{C} = (k; t, h : P \to C)$ we construct a small category with objects of C/β^*_C and morphisms the elements of P/β^*_P. The morphisms $\beta^*_P(p)$ and $\beta^*_P(q)$ are compatible if and only if $\beta^*_P(t(p)) = \beta^*_P(h(q))$ and we define their composition by $\beta^*_P(p)O\beta^*_P(q) = \beta^*_P(p) \odot \beta^*_P(t(p))^{-1} \odot \beta^*_P(q)$. It is easy to check that the axioms of a category are satisfied. We must show that the composition is a group homomorphism. If $\beta^*_P(p'), \beta^*_P(q')$ are two other composable morphisms, then this property holds

$$\begin{aligned}&\beta^*_P(p) \odot \beta^*_P(t(p))^{-1} \odot \beta^*_P(q) \odot \beta^*_P(p') \odot \beta^*_P(t(p'))^{-1} \odot \beta^*_P(q')\\ &= \beta^*_P(p) \odot \beta^*_P(p') \odot \beta^*_P(t(pp'))^{-1} \odot \beta^*_P(q) \odot \beta^*_P(q').\end{aligned}$$

After simplification use of the quality $\beta^*_P(t(p)) = \beta^*_P(h(q))$ proves that it is equivalent to

$$\begin{aligned}&\beta^*_P(h(q))^{-1} \odot \beta^*_P(q) \odot \beta^*_P(p') \odot \beta^*_P(t(p'))^{-1}\\ &= \beta^*_P(p') \odot \beta^*_P(t(p'))^{-1} \odot \beta^*_P(h(q))^{-1} \odot \beta^*_P(q)\end{aligned} \qquad (3)$$

Since $h(q)^{-1} \in C$, it follows that $h(h(q)^{-1}) = h(q)^{-1}$. Hence,

$$h(h(q)^{-1} \circ q) = h(h(q)^{-1}) \circ h(q) = h(q)^{-1} \circ h(q).$$

By Lemma 5.3, $h(q)^{-1} \circ h(q) \subseteq w_C$ and so $h(h(q)^{-1} \circ q) \subseteq w_C$. This implies that $h(q)^{-1} \circ q \subseteq ker^*h$. Similarly, we have $p' \circ t(p')^{-1} \subseteq ker^*t$. Therefore, we have $[p' \circ t(p')^{-1}, h(q)^{-1} \circ q] \subseteq w_P$. So by Lemma 5.10, we conclude that $[\beta^*_P(p' \circ t(p')^{-1}), \beta^*_P(h(q)^{-1} \circ q)] = 1_{P/\beta^*}$ or

$$[\beta^*_P(p') \odot \beta^*_P(t(p'))^{-1}, \beta^*_P(h(q)^{-1}) \odot \beta^*_P(q)] = 1_{P/\beta^*} \qquad (4)$$

Equation (4) is equivalent to Equation (3). In conclusion, composition in this category is a group homomorphism if and only if axiom 2 for cat^1-polygroups is valid. □

7. The generalized Moore complexes

Let **P** be a simplicial polygroup. Note that $d_i : P_n \to P_{n-1}$ and so

$$ker^*d_i = \{x \in P_n \mid d_i(x) \in \omega_{P_{n-1}}\}.$$

According to Lemma 5.5, ker^*d_i is a normal subpolygroup of P_n. Now, we define the generalized Moore complex as follows:

Definition 7.1. Let $\mathbf{P}$ be a simplicial polygroup. The *generalized Moore complex* $(\mathbf{NP}, \partial)$ of $\mathbf{P}$ is the chain complex defined by

$$NP_n = \bigcap_{i=0}^{n-1} ker^* d_i^n$$

with $\partial_n : NP_n \to NP_{n-1}$ induced from d_n^n by restriction.

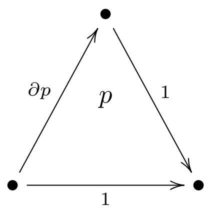
The n^{th} *homotopy polygroup* $\pi_n(\mathbf{P})$ of $\mathbf{P}$ is the n^{th} homology of the generalized Moore complex of $\mathbf{P}$, i.e.,

$$\begin{aligned}\pi_n(\mathbf{P}) &\cong H_n(\mathbf{NP}, \partial), \\ &= \bigcap_{i=0}^{n} ker^* d_i^n / d_{n+1}^{n+1}(\bigcap_{i=0}^{n} ker^* d_i^{n+1}).\end{aligned}$$

Similar to simplicial groups, the interpretation of $\mathbf{NP}$ and $\pi_n(\mathbf{P})$ is as follows:
for $n = 1,\ p \in NP_1$,

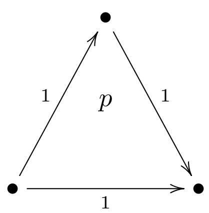
$$\partial p \bullet \xrightarrow{\ \ p\ \ } \bullet 1$$

and $p \in NP_2$ looks like



and so on.

Remark 7.2. $p \in \mathrm{NP}_2$ is in $ker^*\partial$ if it looks like



whilst it will give the trivial element of $\pi_2(\mathbf{P})$ if there is a 3-simplex x with p on its third face and all other faces identity.

This simple interpretation of the elements of $\mathbf{NP}$ and $\pi_n(\mathbf{P})$ will 'pay off' later by aiding interpretation of some of the elements in other situations.

Definition 7.3. A simplicial polygroup $\mathbf{P}$ is *augmented* by specifying a constant simplicial polygroup $\mathbf{K}(P,0)$ and a surjective polygroup strong homomorphism, $f = d_0^0 : P_0 \to P$ with $fd_0^1 = fd_1^1 : P_1 \to P$. An *augmentation* of the simplicial polygroup $\mathbf{P}$ is then a map

$$\mathbf{P} \longrightarrow \mathbf{K}(P,0),$$

or more simply $f : P_0 \longrightarrow P$. An augmented simplicial polygroup (P, f) is *acyclic* if the corresponding complex is acyclic, i.e., $H_n(\mathbf{P}) \cong 1$ for $n > 0$ and $H_0(\mathbf{P}) \cong P$.

Theorem 7.4. *The following data are equivalent:*

(1) *a polygroup object in the category of categories,*

(2) *a simplicial polygroup* $\mathbf{P}$ *whose generalized Moore complex is of length one.*

Proof. Starting from the category we obtain a simplicial set by taking the nerve. Indeed, this simplicial set is the simplicial polygroup $\mathbf{P}$, because the category is a polygroup object in the category of categories. Its generalized Moore complex is of the length one, i.e.,

$$\cdots \to \omega \to \omega \to ker^* d_1 \to P_0.$$

There is a cat^1-polygroup associated to this stuation. We put $P = P_1$ and $C = s_0(P_0)$ The structurel morphisms t and h are given by $h = d_1$ and $t = d_0$. The first condition of Definition 5.9 follows from the relations between face and degeneracy maps, i.e.,

$$d_1|_C = h\,|_C = id_C \text{ and } d_0\,|_C = t\,|_C = id_C.$$

In order to prove the second condition of Definition 5.9, we suppose that $x \in ker^* d_1$ and $y \in ker^* d_0$. Note that $d_0 : P_1 \to P_0$ and $d_1 : P_1 \to P_0$. So, we have

$$d_1(x) \in \omega_{P_0} \text{ and } d_0(y) \in \omega_{P_0}.$$

This implies that

$$\beta^*_{P_0}(d_1(x)) = \omega_{P_0} = 1_{P_0/\beta^*_{P_0}} \text{ and } \beta^*_{\mathrm{P}_0}(\mathrm{d}_0(\mathrm{y})) = \omega_{\mathrm{P}_0} = 1_{\mathrm{P}_0/\beta^*_{\mathrm{P}_0}}.$$

Now, we obtain

$$\begin{aligned}
&D_1(\beta^*_{P_2}([s_0(x), s_0(y) * s_1(y)^{-1}]))\\
&= D_1([\beta^*_{P_2}(s_0(x)), \beta^*_{P_2}(s_0(y)) \otimes \beta^*_{P_2}(s_1(y))^{-1}])\\
&= [D_1(\beta^*_{P_2}(s_0(x))), D_1(\beta^*_{P_2}(s_0(y))) \otimes D_1(\beta^*_{P_2}(s_1(y)))^{-1}]\\
&= [\beta^*_{P_1}(d_1 s_0(x)), \beta^*_{P_1}(d_1 s_0(y)) \otimes \beta^*_{P_1}(d_1 s_1(y))^{-1}]\\
&= [\beta^*_{P_1}(x), \beta^*_{P_1}(y) \otimes \beta^*_{P_1}(y)^{-1}]\\
&= [\beta^*_{P_1}(x), \omega_{P_1}]\\
&= \omega_{P_1}.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&D_2(\beta^*_{P_2}([s_0(x), s_0(y) * s_1(y)^{-1}])) \\
&= D_2([\beta^*_{P_2}(s_0(x)), \beta^*_{P_2}(s_0(y)) \otimes \beta^*_{P_2}(s_1(y))^{-1}]) \\
&= [D_2(\beta^*_{P_2}(s_0(x))), D_2(\beta^*_{P_2}(s_0(y))) \otimes D_2(\beta^*_{P_2}(s_1(y)))^{-1}] \\
&= [\beta^*_{P_1}(d_2 s_0(x)), \beta^*_{P_1}(d_2 s_0(y)) \otimes \beta^*_{P_1}(d_2 s_1(y))^{-1}] \\
&= [\beta^*_{P_1}(s_0 d_1(x)), \beta^*_{P_1}(s_0 d_1(y)) \otimes \beta^*_{P_1}(y^{-1})] \\
&= [S_0(\beta^*_{P_0}(d_1(x))), S_0(\beta^*_{P_0}(d_1(y))) \otimes \beta^*_{P_1}(y^{-1})] \\
&= [S_0(\omega_{P_0}), S_0(\omega_{P_0}) \otimes \beta^*_{P_1}(y^{-1})] \\
&= [\omega_{P_1}, \omega_{P_1} \otimes \beta^*_{P_1}(y^{-1})] \\
&= [\omega_{P_1}, \beta^*_{P_1}(y^{-1})] \\
&= \omega_{P_1}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&D_0(\beta^*_{P_2}([s_0(x), s_0(y) * s_1(y)^{-1}])) \\
&= D_0([\beta^*_{P_2}(s_0(x)), \beta^*_{P_2}(s_0(y)) \otimes \beta^*_{P_2}(s_1(y))^{-1}]) \\
&= [D_0(\beta^*_{P_2}(s_0(x))), D_0(\beta^*_{P_2}(s_0(y))) \otimes D_0(\beta^*_{P_2}(s_1(y)))^{-1}] \\
&= [\beta^*_{P_1}(d_0 s_0(x)), \beta^*_{P_1}(d_0 s_0(y)) \otimes \beta^*_{P_1}(d_0 s_1(y^{-1}))] \\
&= [\beta^*_{P_1}(x), \beta^*_{P_1}(y) \otimes \beta^*_{P_1}(s_0 d_0(y^{-1}))] \\
&= [\beta^*_{P_1}(x), \beta^*_{P_1}(y) \otimes S_0(\beta^*_{P_0}(d_0(y^{-1})))] \\
&= [\beta^*_{P_1}(x), \beta^*_{P_1}(y) \otimes S_0(\omega_{P_0})] \\
&= [\beta^*_{P_1}(x), \beta^*_{P_1}(y) \otimes \omega_{P_1}] \\
&= [\beta^*_{P_1}(x), \beta^*_{P_1}(y)].
\end{aligned}
$$

Therefore, by the above calculations we proved that

$$D_1(\beta^*_{P_2}([s_0(x), s_0(y) * s_1(y)^{-1}])) = \omega_{P_1}, \tag{5}$$

$$D_2(\beta^*_{P_2}([s_0(x), s_0(y) * s_1(y)^{-1}])) = \omega_{P_1}, \tag{6}$$

$$D_0(\beta^*_{P_2}([s_0(x), s_0(y) * s_1(y)^{-1}])) = [\beta^*_{P_1}(x), \beta^*_{P_1}(y)]. \tag{7}$$

By using Lemma 6.2, we obtain

$$[s_0(x), s_0(y) * s_1(y)^{-1}] \subseteq ker^* d_1,$$

$$[s_0(x), s_0(y) * s_1(y)^{-1}] \subseteq ker^* d_2$$

and so

$$[s_0(x), s_0(y) * s_1(y)^{-1}] \subseteq ker^* d_1 \cap ker^* d_1$$

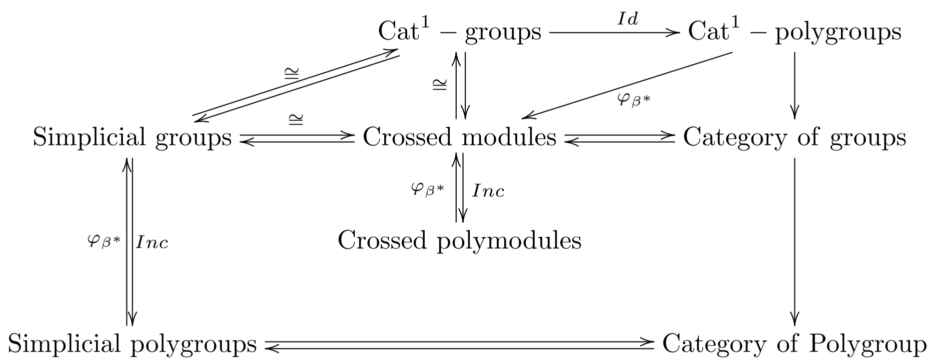
Thus,

$$\beta^*_{P_2}([s_0(x), s_0(y) * s_1(y)^{-1}]) = \omega_{P_2}$$

and its image by D_0 is $[\beta^*_{P_1}(x), \beta^*_P(y)]$. Since $kerD_1 \cap kerD_2 = 1_{P_1/\beta^*_{P_1}}$, it follows that $[\beta^*_{P_1}(x), \beta^*_P(y)] = 1_{P_1/\beta^*_{P_1}}$. By Lemma 5.10 the second condition of Definition [12] is satisfied. □

Now, according to Lemma 6.2, Theorem 6.4 and Theorem 7.4, we can expand the Corollary 3.7 of [12] as follows.

Corollary 7.5. *The following diagram shows all the results obtained and thus gives their relations.*



References

[1] **H. Aghabozorgi, B. Davvaz and M. Jafarpour**, *Solvable polygroups and derived subpolygroups*, Comm. Algebra, **41** (2013), no. 8, 3098-3107.

[2] **M. Alp**, *Enumeration of 1-truncated simplicial groups of low order*, Indian J. Pure Applied Math., **35** (2004), no. 3, 333-345.

[3] **M. Alp and B. Davvaz**, *Crossed polymodules and fundamental relations*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **77** (2015), no. 2, 129 − 140.

[4] **S.D. Comer**, *Polygroups derived from cogroups*, J. Algebra, **89** (1984), 397-405.

[5] **D. Conduché**, *Generalized crossed modules of length* 2, Pro. Luminy Confer. on Algebraic K-theory, J. Pure Appl. Algebra, **34** (1984), no. 2-3, 155-178.

[6] **P. Corsini**, *Prolegomena of Hypergroup Theory*, Second edition, Aviain editore, 1993.

[7] **P. Corsini and V. Leoreanu**, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[8] **B. Davvaz**, *On polygroups and permutation polygroups*, Math. Balkanica, (N.S.) **14** (2000), no. 1-2, 41-58.

[9] **B. Davvaz**, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[10] **B. Davvaz**, *Applications of the γ^*-relation to polygroups*, Comm. Algebra, **35** (2007), 2698-2706.

[11] **B. Davvaz**, *A survey on polygroups and their properties*, Proc. Intern. Confer. on Algebra 2010, 148 − 156, World Sci. Publ., Hackensack, NJ, 2012.

[12] **B. Davvaz and M. Alp**, *Cat^1-Polygroup and Pullback cat^1-Poly group*, Bull. Iranian Math. Soc., **40** (2014), no. 3, 721-735.

[13] **B. Davvaz and V. Leoreanu-Fotea**, *Hyperring Theory and Applications*, Internationl Academic Press, USA, 2007.

[14] **D. Freni**, *A note on the core of a hypergroup and the transitive closure β^* of β*, Riv. Mat. Pura Appl., **8** (1991), 153-156.

[15] **M. Jafarpour, H. Aghabozorgi and B. Davvaz**, *On nilpotent and solvable polygroups*, Bull. Iranian Math. Soc., **39** (2013), no. 3, 487-499.

[16] **M. Koskas**, *Groupoids, demi-groupes et hypergroupes*, J. Math. Pures Appl., **49** (1970), 155-192.

[17] **V. Leoreanu**, *The heart of some important classes of hypergroups*, Pure Math. Appl., **9** (1998), 351-360.

[18] **V. Leoreanu-Fotea, B. Davvaz, F. Feng and C. Chiper**, *Join spaces, soft join spaces and lattices*, Anal. Sti. Univ. Ovidius Constanta, Ser. Matematica, **22** (2014), no. 1, 155-167.

[19] **V. Leoreanu-Fotea, I. Rosenberg, B. Davvaz and T. Vougiouklis**, *A new class of n-ary hyperoperations*, European J. Combin., **44** (2015), 265-273.

[20] **J.-L. Loday**, *Spaces with finitely many nontrivial homotopy groups*, J. Pure Appl. Algebra, **24** (1982), no. 2, 179-202.

[21] **A. Mutlu and T. Porter**, *Applications of Peiffer pairings in the Moore complex of a simplicial group*, Theory and Appl. Categories, **4** (1998), no. 7, 148-173.

[22] **T. Vougiouklis**, *Hyperstructures and Their Representations*, Hadronic Press, Inc, 115, Palm Horber, USA, 1994.

[23] **J.H.C. Whitehead**, *Combinatorial homotopy II*, Bull. Amer. Math. Soc., **55** (1949), 453-496.

Received April 9, 2021

B. Davvaz
Department of Mathematics
Yazd University
Yazd, Iran
e-mail: davvaz@yazd.ac.ir and bdavvaz@yahoo.com

M. Alp
Department of Mathematics,
College of Engineering and Technology,
American University of the Middle East, Kuwait
e-mail: Murat.alp@aum.edu.kw