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Relative (pre-)anti-flexible algebras and associated algebraic structures

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Abstract. Pre-anti-flexible family algebras are introduced and used to define and describe the notions of Ω_c-relative anti-flexible algebras, left and right pre-Lie family algebras and Ω_c-relative Lie algebras. The notion of Ω_c-relative pre-anti-flexible algebras are introduced and also used to characterize pre-anti-flexible family algebras, left and right pre-Lie family algebras and significant identities associated to these algebraic structures are provided. Finally, a generalization of the Rota-Baxter operators defined on an Ω_c-relative anti-flexible algebra is introduced and it is also proved that both Rota-Baxter operators and its generalization provide Ω_c-relative pre-anti-flexible algebras structures and related consequences are derived.

1. Introduction and preliminaries

Anti-flexible algebra, originally derived in the generalization of flexible algebras (algebras satisfy identity $(xy)x = x(yx)$) leading to the introduction of several classes of nonassociative algebras ([23]), is a vector space A equipped with bilinear product "$* : A \times A \to A$" satisfying, for any $x, y, z \in A$,

$$(x * y) * z + z * (y * x) = (z * y) * x + x * (y * z), \tag{1a}$$

equivalently

$$(x, y, z) = (z, y, x), \tag{1b}$$

where,

$$(x, y, z) = (x * y) * z - x * (y * z), \tag{2}$$

is the associator of the bilinear product $* : A \times A \to A$. Anti-flexible algebras are also known as G_4-associative algebras ([15]) and center-symmetric algebras ([19]). Simplicity and semi-simplicity of anti-flexible algebras were investigated and characterized ([24]). Besides, simple and semisimple (totally) anti-flexible algebras over splitting fields of characteristic different to 2 and 3 were studied and classified in [6, 25, 26]. Moreover, the primitive structures and prime anti-flexible

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rings were investigated in [7] and it were established that a simple nearly anti-flexible algebra of characteristic prime to 30 satisfying the identity $(x,x,x)=0$ in which its commutator gives non-nilpotent structure possesses a unity element [10].

A Rota-Baxter operator, originally introduced in [4, 27], is a linear operator $R_B : A \to A$ defined on an associative algebra $(A,\cdot)$ and satisfying, for any $x,y \in A$,

$$R_B(x)\cdot R_B(y) = R_B(R_B(x)\cdot y) + R_B(x\cdot R_B(y)). \tag{3}$$

It is well known from [2] that Rota-Baxter operator of weight zero on a given associative algebra induces a dendriform algebra (introduced by Loday in [20]) structures. More precisely, for a given linear map $R : A \to A$ on an associative algebra $(A,\cdot)$, the two following bilinear products $\prec, \succ : A\times A \to A$ given by, for any $x,y\in A$,

$$x \succ y := R(x)\cdot y, \qquad x \prec y := x\cdot R(y), \tag{4}$$

satisfy the following relations, for any $x,y,z\in A$,

$$(x\succ y)\prec z - x\succ (y\prec z) = 0, \tag{5a}$$

$$(x\succ y + x\prec y)\succ z - x\succ (y\succ z) = 0, \tag{5b}$$

$$x\prec (y\succ z + y\prec z) - (x\prec y)\prec z = 0, \tag{5c}$$

if and only if $R : A\to A$ is a Rota-Baxter operator of weight zero on A, that is, R satisfies Eq. (3). Similarly, from [8, 9], it is established that for a given a Rota-Baxter operator of weight zero defined on an anti-flexible algebra $(A,*)$, the bilinear products given by Eq. (4) satisfy, for any $x,y,z\in A$,

$$(x\succ y)\prec z - x\succ (y\prec z) = (z\succ y)\prec x - z\succ (y\prec x), \tag{6a}$$

$$(x\succ y + x\prec y)\succ z - x\succ (y\succ z) = z\prec (y\succ x + y\prec x) - (z\prec y)\prec x, \tag{6b}$$

and the algebra $(A,\prec,\succ)$ is known as pre-anti-flexible algebra. More generally, for a given linear map $G_{RB} : A\to A$ defined on an anti-flexible algebra $(A,*)$ and considering the following bilinear products given by, for any $x,y\in A$,

$$x \succ' y := G_{RB}(x) * y, \qquad x \prec' y := x * G_{RB}(y), \tag{7}$$

then $(A,\prec',\succ')$ is a pre-anti-flexible algebra if and only if, for $x,y,z\in A$,

$$\begin{aligned}&(G_{RB}(G_{RB}(x)*y + x*G_{RB}(y)) - G_{RB}(x)*G_{RB}(y))*z+\\ &z*(G_{RB}(y)*G_{RB}(x) - G_{RB}(G_{RB}(y)*x + y*G_{RB}(x))) = 0.\end{aligned} \tag{8}$$

A linear map $G_{RB} : A \to A$ given on an anti-flexible algebra $(A, *)$ satisfying Eq. (8) is known as a generalization of the Rota-Baxter operator of weight zero.

Besides, dendriform and (di-)tri-algebras were introduced and related to Rota-Baxter operators and associated consequences were derived ([11]). Moreover, it is well known (from [16]) that Koszul duality of operad governing (di-)tri-algebras is corresponding to operad governing variety of (di-)tri-dendriform algebras which are embedded to zero's weight Rota-Baxter algebra. Furthermore, it is proved that a general operadic definition for the notion of splitting algebraic structures are equivalent with some Manin products of operads which are closely related to Rota-Baxter operators ([3]). More generally, splitting algebraic operations procedure in any algebraic operad theory were uniformed and linked to the notion of Rota-Baxter operators on operads ([22]) and other results established on Rota-Baxter algebras are surveyed in [18] and the references therein.

The notion of operated semi-group are introduced to build some algebraic structures on combinatoric elements mainly the binary rooted trees. The most relevant examples are the construction of free Rota-Baxter algebras in terms of Motzkin paths and planar rooted trees ([17]) and the use of typed decorated trees theory for describing combinatorial species ([5]). Given a (non)associative $\mathbb{K}$ (field of characteristic zero) algebra A, a Rota-Baxter family operators of weight λ ($\lambda \in \mathbb{K}$) is a family of linear maps $P_\omega : A \to A$, where $\omega \in \Omega$, satisfying, for any $x, y \in A$ and for any $\alpha, \beta \in \Omega$,

$$P_\alpha(x)P_\beta(y) = P_{\alpha\beta}(xP_\beta(y)) + P_{\alpha\beta}(P_\alpha(x)y) + \lambda P_{\alpha\beta}(xy). \tag{9}$$

The theory of Rota-Baxter family operators takes its origins in renormalization theory of quantum field theory ([12, page 591]). Recently, free (non)commutative Rota-Baxter family is introduced and linked to (tri)dendriform family algebras ([28]). Moreover, it is proved that Rota-Baxter family algebras indexed by an associative semigroup amounts to an ordinary Rota-Baxter algebra structure on the tensor product with the semigroup algebra. Similar results are provided with (tri)dendriform family algebras ([29]), and more generally, the notion of Ω-dendriform structures are introduced and nonassociative structures on typed binary trees are unified and generalized ([13]). Similarly, pre-Lie family algebras and free pre-Lie family algebras, are introduced and related typed decorated trees are constructed and related generalization are also derived ([21]). In addition, a general account of family algebras over a finitely presented linear operad are given and proved that this operad together with its presentation naturally define an algebraic structure on the set of parameters ([14]).

Throughout this article, Ω is an associative semi-group and Ω_c is a commutative associative semi-group, algebras are defined over a field of characteristic zero. We end this introductory section by describing the content flowchart of this paper as follows. In section , we introduce the notion of pre-anti-flexible family algebras, establish their relations with dendriform family algebras and use them to construct Ω_c-relative anti-flexible algebras as well as Ω_c-relative Lie algebras, left and right pre-Lie family algebras and related consequences are derived. In section , the notion of Ω_c-relative pre-anti-flexible algebras is introduced and viewed as a generalization of the Ω_c-relative dendriform algebras and used to build Ω_c-relative pre-anti-flexible algebras, Ω_c-relative pre-Lie and right pre-Lie algebras, Ω_c-relative Lie algebras and other associated structures are derived. In section , we prove that a Rota-Baxter family operators of weight zero defined on an Ω_c-relative anti-flexible algebra and its generalization induce an Ω_c-relative pre-anti-flexible algebra structure. Under some assumptions on Ω_c-relative anti-flexible algebra, we prove that a Rota-Baxter family operators defined on a related Ω_c-relative Lie algebra of an Ω_c-relative anti-flexible algebra also induces an Ω_c-relative pre-anti-flexible algebra structure.

2. Pre-anti-flexible family algebras

In this section, pre-anti-flexible family algebras are introduced and related consequences are established. Associated family algebras are derived as well as its Ω_c-relative algebraic structures.

Definition 2.1. A pre-anti-flexible family algebra is a quadruple $(A, \prec_\omega, \succ_\omega, \Omega_c)$ such that A is a vector space equipped with two families of bilinear products $\prec_\alpha, \succ_\alpha$: $A \times A \to A$ with $\alpha \in \Omega_c$ and satisfying for any $x, y, z \in A$ and for any $\alpha, \beta \in \Omega_c$,

$$(x \succ_\alpha y) \prec_\beta z - x \succ_\alpha (y \prec_\beta z) = (z \succ_\beta y) \prec_\alpha x - z \succ_\beta (y \prec_\alpha x), \tag{10a}$$

$$\begin{aligned}(x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z - x \succ_\alpha (y \succ_\beta z) = (z \prec_\beta y) \prec_\alpha x \\ - z \prec_{\beta\alpha} (y \succ_\beta x + y \prec_\alpha x).\end{aligned} \tag{10b}$$

Remark 2.2. If the LHS and the RHS of each Eq. (10a) and Eq. (10b) become zero, then pre-anti-flexible family algebra is dendriform family algebra ([28, 21, 29]). Consequently, pre-anti-flexible family algebras can be considered as a generalization of dendriform family algebras.

Definition 2.3. An Ω_c-relative anti-flexible algebra is a triple $(A, \cdot_{\alpha,\beta}, \Omega_c)$ in which A is a vector space equipped with the family of bilinear products "$\cdot_{\alpha,\beta} : A \times A \to A$" with $(\alpha, \beta) \in \Omega_c^2$ and satisfying, for any $x, y, z \in A$ and any $\alpha, \beta, \gamma \in \Omega_c$,

$$(x \cdot_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} z + z \cdot_{\gamma,\beta\alpha} (y \cdot_{\beta,\alpha} x) - (z \cdot_{\gamma,\beta} y) \cdot_{\gamma\beta,\alpha} x - x \cdot_{\alpha,\beta\gamma} (y \cdot_{\beta,\gamma} z) = 0, \tag{11}$$

equivalently

$$(x, y, z)_{\alpha,\beta,\gamma} = (z, y, x)_{\gamma,\beta,\alpha}, \tag{12}$$

where, for any $x, y, z \in A$ and for any $\alpha, \beta, \gamma \in \Omega_c$,

$$(x, y, z)_{\alpha,\beta,\gamma} := (x \cdot_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} z - x \cdot_{\alpha,\beta\gamma} (y \cdot_{\beta,\gamma} z). \tag{13}$$

Theorem 2.4. *Let $(A, \prec_\omega, \succ_\omega, \Omega_c)$ be a pre-anti-flexible family algebra, the following family of bilinear products given by, for any $\alpha, \beta \in \Omega_c$ and for any $x, y \in A$,*

$$x *_{\alpha,\beta} y = x \succ_\alpha y + x \prec_\beta y \tag{14}$$

is such that $(A, *_{\alpha,\beta}, \Omega_c)$ is an Ω_c-relative anti-flexible algebra.

Proof. Let $x, y, z \in A$ and $\alpha, \beta, \gamma \in \Omega_c$. We have

$$\begin{aligned}
(x,y,z)_{\alpha,\beta,\gamma} &:= (x *_{\alpha,\beta} y) *_{\alpha\beta,\gamma} z - x *_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z) \\
&= (x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z + (x \succ_\alpha y + x \prec_\beta y) \prec_\gamma z \\
&- x \succ_\alpha (y \succ_\beta z + y \prec_\gamma z) - x \prec_{\beta\gamma} (y \succ_\beta z + y \prec_\gamma z) \\
&= \{(x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z - x \succ_\alpha (y \succ_\beta z\} \\
&+ \{(x \prec_\beta y) \prec_\gamma z - x \prec_{\beta\gamma} (y \succ_\beta z + y \prec_\gamma z)\} \\
&+ \{(x \succ_\alpha y) \prec_\gamma z - x \succ_\alpha (y \prec_\gamma z)\} \\
&= \{(z \prec_\beta y) \prec_\alpha x - z \prec_{\beta\alpha} (y \succ_\beta x + y \prec_\alpha x)\} \\
&+ \{(z \succ_\gamma y + z \prec_\beta y) \succ_{\gamma\beta} x - z \succ_\gamma (y \succ_\beta x)\} \\
&+ \{(z \succ_\gamma y) \prec_\alpha x - z \succ_\gamma (y \prec_\alpha x)\} \\
&= (z \succ_\gamma y + z \prec_\beta y) \succ_{\gamma\beta} x + (z \succ_\gamma y + z \prec_\beta y) \prec_\alpha x \\
&- z \succ_\gamma (y \succ_\beta x + y \prec_\alpha x) - z \prec_{\beta\alpha} (y \succ_\beta x + y \prec_\alpha x) \\
(x,y,z)_{\alpha,\beta,\gamma} &= (z,y,x)_{\gamma,\beta,\alpha}.
\end{aligned}$$

Therefore, the family of bilinear products given by Eq. (14) satisfies Eq. (12). Thus $(A, *_{\alpha,\beta}, \Omega_c)$ is an Ω_c-relative anti-flexible algebra where, "$*_{\alpha,\beta} : A \times A \to A$" is derived by Eq. (14). □

Theorem 2.5. *Let A be a $\mathbf{k}$ vector space and consider on $A \otimes \mathbf{k}\Omega_c$, two bilinear products given by $\prec, \succ : A \otimes \mathbf{k}\Omega_c \times A \otimes \mathbf{k}\Omega_c \to A \otimes \mathbf{k}\Omega_c$. The triple $(A \otimes \mathbf{k}\Omega_c, \prec, \succ)$ is a pre-anti-flexible algebra if and only if $(A, \prec_\omega, \succ_\omega, \Omega_c)$ is a pre-anti-flexible family algebra and for any $x, y, \in A$ and for any $\alpha, \beta \in \Omega_c$,*

$$(x \otimes \alpha) \prec (y \otimes \beta) := (x \prec_\beta y) \otimes \alpha\beta, \tag{15a}$$

$$(x \otimes \alpha) \succ (y \otimes \beta) := (x \succ_\alpha y) \otimes \alpha\beta. \tag{15b}$$

Proof. Let $x, y, z \in A$ and $\alpha, \beta, \gamma \in \Omega_c$. We have

$$\begin{aligned}
&((x \otimes \alpha) \succ (y \otimes \beta)) \prec (z \otimes \gamma) - (x \otimes \alpha) \succ ((y \otimes \beta) \prec (z \otimes \gamma)) = \\
&\qquad ((x \succ_\alpha y) \prec_\gamma z - x \succ_\alpha (y \prec_\gamma z)) \otimes \alpha\beta\gamma,
\end{aligned} \tag{16a}$$

$$\begin{aligned}
&((x \otimes \alpha) \succ (y \otimes \beta) + (x \otimes \alpha) \prec (y \otimes \beta)) \succ (z \otimes \gamma) \\
&-(x \otimes \alpha) \succ ((y \otimes \beta) \succ (z \otimes \gamma)) = ((x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z \\
&-x \succ_\alpha (y \succ_\beta z)) \otimes \alpha\beta\gamma.
\end{aligned} \tag{16b}$$

Similarly, using the commutativity of Ω_c, we have

$$((z \otimes \gamma) \succ (y \otimes \beta)) \prec (x \otimes \alpha) - (z \otimes \gamma) \succ ((y \otimes \beta) \prec (x \otimes \alpha)) =$$

$$((z \succ_\gamma y) \prec_\alpha x - z \succ_\gamma (y \prec_\alpha x)) \otimes \alpha\beta\gamma, \tag{17a}$$

$$\begin{aligned}&((z \otimes \gamma) \prec (y \otimes \beta)) \prec (x \otimes \alpha) - (z \otimes \gamma) \prec ((y \otimes \beta) \succ (x \otimes \alpha) \\ &+(y \otimes \beta) \prec (x \otimes \alpha)) = ((z \prec_\beta y) \prec_\alpha x - z \prec_{\beta\alpha} (y \succ_\beta x + y \prec_\alpha x)) \otimes \alpha\beta\gamma.\end{aligned} \tag{17b}$$

Hence, if $(A \otimes \mathbf{k}\Omega_c, \prec, \succ)$ is a pre-anti-flexible algebra i.e., the RHS of Eq. (16a) and Eq. (17a) are respectively equal to that of Eq. (16b) and (17b), then $(A, \prec_\omega, \succ_\omega, \Omega_c)$ is a pre-anti-flexible family algebra.

Conversely, if $(A, \prec_\omega, \succ_\omega, \Omega_c)$ is a pre-anti-flexible family algebra in which "$\prec_\omega, \succ_\omega$: $A \times A \to A$" are given by Eqs. (15a) and (15b), according to Eqs. (16a), (16b), (17a) and (17b), we deduce that $(A \otimes \mathbf{k}\Omega_c, \prec, \succ)$ is a pre-anti-flexible algebra. □

Definition 2.6. (cf. [21]) A left pre-Lie family algebra is a triple $(A, \triangleright_\omega, \Omega_c)$ in which A is a vector space equipped with the family of bilinear products $\triangleright_\omega : A \times A \to A$ with $\omega \in \Omega_c$, satisfying for any $x, y, z \in A$ and for any $\alpha, \beta \in \Omega_c$,

$$(x \triangleright_\alpha y) \triangleright_{\alpha\beta} z - x \triangleright_\alpha (y \triangleright_\beta z) = (y \triangleright_\beta x) \triangleright_{\beta\alpha} z - y \triangleright_\beta (x \triangleright_\alpha z). \tag{18}$$

Definition 2.7. A right pre-Lie family algebra is a triple $(A, \triangleleft_\omega, \Omega_c)$ in which A is a vector space equipped with the family of bilinear products $\triangleleft_\omega : A \times A \to A$ with $\omega \in \Omega_c$, satisfying for any $x, y, z \in A$ and for any $\alpha, \beta \in \Omega_c$,

$$x \triangleleft_{\alpha\beta} (y \triangleleft_\beta z) - (x \triangleleft_\alpha y) \triangleleft_\beta z = x \triangleleft_{\beta\alpha} (z \triangleleft_\alpha y) - (x \triangleleft_\beta z) \triangleleft_\alpha y. \tag{19}$$

Remark 2.8. For a given left pre-Lie family algebra $(A, \cdot_\alpha, \Omega_c)$, setting for any $x, y \in A$ and for any $\alpha \in \Omega_c$, $x \cdot_\alpha^{opp} y = y \cdot_\alpha x$, we have for any $x, y, z \in A$ and for any $\alpha, \beta \in \Omega_c$,

$$\begin{aligned} x \cdot_{\alpha\beta}^{opp} (y \cdot_\beta^{opp} z) - (x \cdot_\alpha^{opp} y) \cdot_\beta^{opp} z &= (z \cdot_\beta y) \cdot_{\alpha\beta} x - z \cdot_\beta (y \cdot_\alpha x) \\ &= (y \cdot_\alpha z) \cdot_{\beta\alpha} x - y \cdot_\alpha (z \cdot_\beta x) \\ &= x \cdot_{\beta\alpha}^{opp} (z \cdot_\alpha^{opp} y) - (x \cdot_\beta^{opp} z) \cdot_\alpha^{opp} y. \end{aligned}$$

Therefore, $(A, \cdot_\alpha^{opp}, \Omega_c)$ is a right pre-Lie family algebra.

Theorem 2.9. *Let $(A, \prec_\omega, \succ_\omega, \Omega_c)$ be a pre-anti-flexible family algebra, the two following families of bilinear products*

$$x \triangleright_\omega y := x \succ_\omega y - y \prec_\omega x, \ \forall \omega \in \Omega_c, \tag{20a}$$

$$x \triangleleft_\omega y := x \prec_\omega y - y \succ_\omega x, \ \forall \omega \in \Omega_c, \tag{20b}$$

for any $x, y \in A$ and for any $\alpha, \beta \in \Omega_c$, are such that $(A, \triangleright_\omega, \Omega)$ is a left pre-Lie family algebra and $(A, \triangleleft_\omega, \Omega)$ is a right pre-Lie family algebra.

Proof. Let $x, y, z \in A$ and $\alpha, \beta \in \Omega_c$. We have

$$\begin{aligned}
&(x \triangleright_\alpha y) \triangleright_{\alpha\beta} z - x \triangleright_\alpha (y \triangleright_\beta z) = (x \succ_\alpha y - y \prec_\alpha x) \triangleright_{\alpha\beta} z \\
&- x \triangleright_\alpha (y \succ_\beta z - z \prec_\beta y) = (x \succ_\alpha y - y \prec_\alpha x) \succ_{\alpha\beta} z \\
&- z \prec_{\alpha\beta} (x \succ_\alpha y - y \prec_\alpha x) - x \succ_\alpha (y \succ_\beta z - z \prec_\beta y) \\
&+ (y \succ_\beta z - z \prec_\beta y) \prec_\alpha x = \{(x \succ_\alpha y) \succ_{\alpha\beta} z - x \succ_\alpha (y \succ_\beta z) \\
&- (z \prec_\beta y) \prec_\alpha x - z \prec_{\alpha\beta} (y \prec_\alpha x)\} - (y \prec_\alpha x) \succ_{\alpha\beta} z - z \prec_{\alpha\beta} (x \succ_\alpha y) \\
&+ x \succ_\alpha (z \prec_\beta y) + (y \succ_\beta z) \prec_\alpha x \\
&= -(x \prec_\beta y) \succ_{\alpha\beta} z - z \prec_{\alpha\beta} (y \succ_\beta x) - (y \prec_\alpha x) \succ_{\alpha\beta} z \\
&- z \prec_{\alpha\beta} (x \succ_\alpha y) + x \succ_\alpha (z \prec_\beta y) + (y \succ_\beta z) \prec_\alpha x \\
&= -(x \prec_\beta y + y \prec_\alpha x) \succ_{\alpha\beta} z - z \prec_{\alpha\beta} (x \succ_\alpha y + y \succ_\beta x) \\
&+ (y \succ_\beta z) \prec_\alpha x + x \succ_\alpha (z \prec_\beta y) = (y \triangleright_\beta x) \triangleright_{\beta\alpha} z - y \triangleright_\beta (x \triangleright_\alpha z).
\end{aligned}$$

Note that the third equal sign above upwards is due to Eq. (10b) while the last equal sign one is due to Eq. (10a). Therefore, $(A, \triangleright_\omega, \Omega_c)$ is a left pre-Lie family algebra.

Similarly to the above calculations, we prove that $(A, \triangleleft_\omega, \Omega_c)$ is a right pre-Lie family algebra. □

Definition 2.10. An Ω_c-relative Lie algebra is a triple $(A, [\cdot,\cdot]_{\alpha,\beta}, \Omega_c)$ in which A is a vector space equipped with a family of bilinear products $[\cdot,\cdot]_{\alpha,\beta} : A \otimes A \to A$ with $(\alpha, \beta) \in \Omega_c$, and satisfying, for any $x, y, z \in A$, and for any $\alpha, \beta, \gamma \in \Omega_c$,

$$[x, y]_{\alpha,\beta} + [y, x]_{\beta,\alpha} = 0, \tag{21a}$$

$$[[x, y]_{\alpha,\beta}, z]_{\alpha\beta,\gamma} + [[y, z]_{\beta,\gamma}, x]_{\beta\gamma,\alpha} + [[z, x]_{\gamma,\alpha}, y]_{\gamma\alpha,\beta} = 0. \tag{21b}$$

Theorem 2.11. *Let $(A, \prec_\omega, \succ_\omega, \Omega_c)$ be a pre-anti-flexible family algebra. The following family of bilinear products given by, for any $x, y \in A$ and for any $\alpha, \beta \in \Omega$,*

$$[x, y]_{\alpha,\beta} = x *_{\alpha,\beta} y - y *_{\beta,\alpha} x = (x \succ_\alpha y + x \prec_\beta y) - (y \succ_\beta x + y \prec_\alpha x), \tag{22}$$

is such that $(A, [\cdot,\cdot]_{\alpha,\beta}, \Omega_c)$ is an Ω_c-relative Lie algebra.

Proof. For any $x, y, z \in A$ and for any $\alpha, \beta, \gamma \in \Omega_c$, we have

- Skew symmetric,

  $$[x, y]_{\alpha,\beta} + [y, x]_{\beta,\alpha} = x *_{\alpha,\beta} y - y *_{\beta,\alpha} x + y *_{\beta,\alpha} x - x *_{\alpha,\beta} y = 0.$$

  Thus, Eq. (21a) is satisfied.

- Family of Jacobi identity,

  $$\begin{aligned}
  &[[x, y]_{\alpha,\beta}, z]_{\alpha\beta,\gamma} + [[y, z]_{\beta,\gamma}, x]_{\beta\gamma,\alpha} + [[z, x]_{\gamma,\alpha}, y]_{\gamma\alpha,\beta} \\
  &= (x *_{\alpha,\beta} y) *_{\alpha\beta,\gamma} z - z *_{\gamma,\alpha\beta} (x *_{\alpha,\beta} y)
  \end{aligned}$$

$$
\begin{aligned}
&- (y *_{\beta,\alpha} x) *_{\alpha\beta,\gamma} z + z *_{\gamma,\alpha\beta} (y *_{\beta,\alpha} x) \\
&+ (y *_{\beta,\gamma} z) *_{\beta\gamma,\alpha} x - x *_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z) \\
&- (z *_{\gamma,\beta} y) *_{\beta\gamma,\alpha} x + x *_{\alpha,\beta\gamma} (z *_{\gamma,\beta} y) \\
&+ (z *_{\gamma,\alpha} x) *_{\gamma\alpha,\beta} y - y *_{\beta,\gamma\alpha} (z *_{\gamma,\alpha} x) \\
&- (x *_{\alpha,\gamma} z) *_{\gamma\alpha,\beta} y + y *_{\beta,\gamma\alpha} (x *_{\alpha,\gamma} z) \\
&= (x,y,z)_{\alpha,\beta,\gamma} + (y,z,x)_{\beta,\gamma,\alpha} + (z,x,y)_{\gamma,\alpha,\beta} \\
&- (z,y,x)_{\gamma,\beta,\alpha} - (x,z,y)_{\alpha,\gamma,\beta} - (y,x,z)_{\beta,\alpha,\gamma} = 0
\end{aligned}
$$

due to Theorem 2.4 and Eq. (12). Hence, Eq. (21b) is satisfied.

Therefore, $(A, [\cdot,\cdot]_{\alpha,\beta}, \Omega_c)$ is an Ω_c-relative Lie algebra. □

Proposition 2.12. *Let $(A, \prec_\omega, \succ_\omega, \Omega_c)$ be a pre-anti-flexible family algebra. The following family of bilinear products given on A by, for any $\alpha, \beta \in \Omega_c$ and for any $x, y \in A$,*

$$[x,y]_{\alpha,\beta} := x \rhd_\alpha y - y \rhd_\beta x, \tag{23}$$

where, "$\rhd$" is defined by Eq. (20a), turns A into an Ω_c-relative Lie algebra which is the same as that given in Theorem 2.11.

Proof. Let $x, y \in A$ and $\alpha, \beta \in \Omega_c$. We have

$$
\begin{aligned}
[x,y]_{\alpha,\beta} &:= x \rhd_\alpha y - y \rhd_\beta x = x \succ_\alpha y - y \prec_\alpha x - y \succ_\beta x + x \prec_\beta y \\
&= (x \succ_\alpha y + x \prec_\beta y) - (y \succ_\beta x + y \prec_\alpha x) = x *_{\alpha,\beta} y - y *_{\beta,\alpha} x
\end{aligned}
$$

which is the commutator given by Eq. (22). □

Theorem 2.13. *Let $(A, \prec_\omega, \succ_\omega, \Omega_c)$ be a pre-anti-flexible family algebra. The following family of bilinear products given by, for any $x, y \in A$ and for any $\alpha, \beta \in \Omega_c$,*

$$x \circ_{\alpha,\beta} y = x *_{\alpha,\beta} y + y *_{\beta,\alpha} x, \tag{24}$$

in which "$*_{\alpha,\beta} : A \times A \to A$" is given by Eq. (14) is such that, for any $x, y, z \in A$ and for any $\alpha, \beta, \gamma \in \Omega_c$,

$$(x,y,z)_{\circ_{\alpha,\beta,\gamma}} = [y, [x,z]_{\alpha,\gamma}]_{\beta,\alpha\gamma}, \tag{25}$$

where,

$$(x,y,z)_{\circ_{\alpha,\beta,\gamma}} = (x \circ_{\alpha,\beta} y) \circ_{\alpha\beta,\gamma} z - x \circ_{\alpha,\beta\gamma} (y \circ_{\beta,\gamma} z), \tag{26}$$

and "$[\cdot,\cdot]_{\alpha,\beta}$" is given by Eq. (22).

Proof. Let $x, y, z \in A$, and for all $\alpha, \beta, \gamma \in \Omega_c$, we have

$$
\begin{aligned}
(x,y,z)_{\circ_{\alpha,\beta,\gamma}} &= (x \circ_{\alpha,\beta} y) \circ_{\alpha\beta,\gamma} z - x \circ_{\alpha,\beta\gamma} (y \circ_{\beta,\gamma} z) \\
&= (x *_{\alpha,\beta} y + y *_{\beta,\alpha} x) *_{\alpha\beta,\gamma} z + z *_{\gamma,\alpha\beta} (x *_{\alpha,\beta} y + y *_{\beta,\alpha} x)
\end{aligned}
$$

$$
\begin{aligned}
&- x *_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z + z *_{\gamma,\beta} y) - (y *_{\beta,\gamma} z + z *_{\gamma,\beta} y) *_{\beta\gamma,\alpha} x \\
&= \{(x *_{\alpha,\beta} y) *_{\alpha\beta,\gamma} z - x *_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z)\} \\
&- \{(z *_{\gamma,\beta} y) *_{\beta\gamma,\alpha} x - z *_{\gamma,\alpha\beta} (y *_{\beta,\alpha} x)\} \\
&+ (y *_{\beta,\alpha} x) *_{\alpha\beta,\gamma} z + z *_{\gamma,\alpha\beta} (x *_{\alpha,\beta} y) \\
&- x *_{\alpha,\beta\gamma} (z *_{\gamma,\beta} y) - (y *_{\beta,\gamma} z) *_{\beta\gamma,\alpha} x \\
&= (y *_{\beta,\alpha} x) *_{\alpha\beta,\gamma} z + z *_{\gamma,\alpha\beta} (x *_{\alpha,\beta} y) \\
&- x *_{\alpha,\beta\gamma} (z *_{\gamma,\beta} y) - (y *_{\beta,\gamma} z) *_{\beta\gamma,\alpha} x \\
&= y *_{\beta,\alpha\gamma} (x *_{\alpha,\gamma} z) + (z *_{\gamma,\alpha} x) *_{\gamma\alpha,\beta} y \\
&- y *_{\beta,\gamma\alpha} (z *_{\gamma,\alpha} x) - (x *_{\alpha,\gamma} z) *_{\alpha\gamma,\beta} y = [y, [x,z]_{\alpha,\gamma}]_{\beta,\alpha\gamma}.
\end{aligned}
$$

Note that the three last equals sign upwards are due to Eq. (11). □

Proposition 2.14. *Let $(A, \cdot_{\alpha,\beta}, \Omega_c)$ be an Ω_c-relative anti-flexible algebra. We have for any $x, y, z \in A$ and for any $\alpha, \beta, \gamma \in \Omega_c$,*

$$(x, y, z)_{\circ_{\alpha,\beta,\gamma}} + (z, x, y)_{\circ_{\gamma,\alpha,\beta}} + (y, z, x)_{\circ_{\beta,\gamma,\alpha}} = 0, \tag{27}$$

where, for any $x, y, z \in A$ and for any $\alpha, \beta, \gamma \in \Omega_c$, $(x, y, z)_{\circ_{\alpha,\beta,\gamma}}$ is given by Eq. (26) and $x \circ_{\alpha,\beta} y = x \cdot_{\alpha,\beta} y + y \cdot_{\beta,\alpha} x$.

Proof. According to Eq. (26), Theorem 2.11 and Theorem 2.13, Eq. (27) holds. □

3. Associated Ω_c-relative algebras

In this section, Ω_c-relative pre-anti-flexible algebras structures are introduced and associated Ω_c-relative algebras structures are derived. Moreover, Ω_c-relative pre-anti-flexible algebras are viewed as a generalization of pre-anti-flexible family algebras and associated consequences are deduced.

Definition 3.1. An Ω_c-relative pre-anti-flexible algebra is a quadruple $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$ in which A is a vector space equipped with two families of bilinear products $\prec_{\alpha,\beta}; \succ_{\alpha,\beta}$: $A \times A \to A$ for $(\alpha, \beta) \in \Omega_c^2$ and satisfying for any $x, y, z \in A$ and for any $\alpha, \beta, \gamma \in \Omega_c$,

$$
\begin{aligned}
(x \succ_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} z - x \succ_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z) = (z \succ_{\gamma,\beta} y) \prec_{\gamma\beta,\alpha} x \\
- z \succ_{\gamma,\beta\alpha} (y \prec_{\beta,\alpha} x),
\end{aligned} \tag{28a}
$$

$$
\begin{aligned}
(x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} z - x \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z) = \\
(z \prec_{\gamma,\beta} y) \prec_{\gamma\beta,\alpha} x - z \prec_{\gamma,\beta\alpha} (y \prec_{\beta,\alpha} x + y \succ_{\beta,\alpha} x).
\end{aligned} \tag{28b}
$$

Remark 3.2. If the LSH and RHS of Eq. (28a) and Eq. (28b) vanish, then the Ω_c-relative pre-anti-flexible algebra $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$ is called Ω_c-relative dendriform algebra ([1]). Hence, Ω_c-relative pre-anti-flexible algebras are a generalization of Ω_c-relative dendriform algebras.

Similarly to Theorem 2.5, we have

Theorem 3.3. *Let A be a $\mathbf{k}$ vector space and consider the bilinear products given on $A\otimes \mathbf{k}\Omega_c$ by $\prec,\succ$: $A\otimes \mathbf{k}\Omega_c\times A\otimes \mathbf{k}\Omega_c\to A\otimes \mathbf{k}\Omega_c$. The triple $(A\otimes \mathbf{k}\Omega_c,\prec,\succ)$ is a pre-anti-flexible algebra if and only if the quadruple $(A,\prec_{\alpha,\beta},\succ_{\alpha,\beta},\Omega_c)$ is an Ω_c-relative pre-anti-flexible algebra where, for any $x,y\in A$ and for any $\alpha,\beta\in\Omega_c$,*

$$(x\otimes\alpha)\prec(y\otimes\beta):=(x\prec_{\alpha,\beta}y)\otimes\alpha\beta, \tag{29a}$$

$$(x\otimes\alpha)\succ(y\otimes\beta):=(x\succ_{\alpha,\beta}y)\otimes\alpha\beta. \tag{29b}$$

Proposition 3.4. *Let $(A,\prec_{\alpha,\beta},\succ_{\alpha,\beta},\Omega_c)$ be an Ω_c-relative pre-anti-flexible algebra. The following family of bilinear products given by, for any $x,y\in A$ and for any $\alpha,\beta\in\Omega_c$,*

$$x\circledast_{\alpha,\beta}y=x\prec_{\alpha,\beta}y+x\succ_{\alpha,\beta}y, \tag{30}$$

turns A into an Ω_c-relative anti-flexible algebra.

Proof. Let $x,y,z\in A$ and $\alpha,\beta,\gamma\in\Omega_c$. We have

$$\begin{aligned}(x,y,z)_{\circledast_{\alpha,\beta,\gamma}} &= (x\succ_{\alpha,\beta}y+x\prec_{\alpha,\beta}y)\succ_{\alpha\beta,\gamma}z+(x\succ_{\alpha,\beta}y+x\prec_{\alpha,\beta}y)\prec_{\alpha\beta,\gamma}z\\
&\quad-x\succ_{\alpha,\beta\gamma}(y\succ_{\beta,\gamma}z+y\prec_{\beta,\gamma}z)-x\prec_{\alpha,\beta\gamma}(y\succ_{\beta,\gamma}z+y\prec_{\beta,\gamma}z)\\
&=\{(x\succ_{\alpha,\beta}y+x\prec_{\alpha,\beta}y)\succ_{\alpha\beta,\gamma}z-x\succ_{\alpha,\beta\gamma}(y\succ_{\beta,\gamma}z)\}\\
&\quad-\{x\prec_{\alpha,\beta\gamma}(y\succ_{\beta,\gamma}z+y\prec_{\beta,\gamma}z)-(x\prec_{\alpha,\beta}y)\prec_{\alpha\beta,\gamma}z\}\\
&\quad+\{(x\succ_{\alpha,\beta}y)\prec_{\alpha\beta,\gamma}z-x\succ_{\alpha,\beta\gamma}(y\prec_{\beta,\gamma}z)\}\\
&=\{(z\prec_{\gamma,\beta}y)\prec_{\gamma\beta,\alpha}x-z\prec_{\gamma,\beta\alpha}(y\prec_{\beta,\alpha}x+y\succ_{\beta,\alpha}x)\}\\
&\quad+\{(z\prec_{\gamma,\beta}y+z\succ_{\gamma,\beta}y)\succ_{\gamma\beta,\alpha}x-z\succ_{\gamma,\beta\gamma}(y\succ_{\beta,\alpha}x)\}\\
&\quad+\{(z\succ_{\gamma,\beta}y)\prec_{\gamma\beta,\alpha}x-z\succ_{\gamma,\beta\alpha}(y\prec_{\beta,\alpha}x)\}\\
&=(z\circledast_{\gamma,\beta}y)\circledast_{\gamma\beta,\alpha}x-z\circledast_{\gamma,\beta\alpha}(y\circledast_{\beta,\alpha}x)=(z,y,x)_{\circledast_{\gamma,\beta,\alpha}},\end{aligned}$$

the third equal sign upwards above is due to Eq. (28a) and Eq. (28b).

Therefore, $(A,\circledast_{\omega_1,\omega_2},\Omega_c)$ is an Ω_c-relative anti-flexible algebra. □

Definition 3.5. An Ω_c-relative pre-Lie (left-symmetric) algebra is a vector space A equipped with a family of bilinear products $*_{\alpha,\beta}:A\otimes A\to A$ with $(\alpha,\beta)\in\Omega_c^2$, such that for any $x,y,z\in A$ and for any $\alpha,\beta,\gamma\in\Omega_c$,

$$(x,y,z)_{\alpha,\beta,\gamma}=(y,x,z)_{\beta,\alpha,\gamma}, \tag{31}$$

or equivalently

$$(x*_{\alpha,\beta}y)*_{\alpha\beta,\gamma}z-x*_{\alpha,\beta\gamma}(y*_{\beta,\gamma}z)-(y*_{\beta,\alpha}x)*_{\beta\alpha,\gamma}z+y*_{\beta,\alpha\gamma}(x*_{\alpha,\gamma}z)=0. \tag{32}$$

Definition 3.6. An Ω_c-relative right-symmetric algebra is a vector space A equipped with a family of bilinear products $*_{\alpha,\beta}:A\otimes A\to A$ for $(\alpha,\beta)\in\Omega_c^2$ such that for any $x,y,z\in A$ and for any $\alpha,\beta,\gamma\in\Omega_c$,

$$(x,y,z)_{\alpha,\beta,\gamma}=(x,z,y)_{\alpha,\gamma,\beta}, \tag{33}$$

or equivalently

$$(x *_{\alpha,\beta} y) *_{\alpha\beta,\gamma} z - x *_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z) - (x *_{\alpha,\gamma} z) *_{\alpha\gamma,\beta} y + x *_{\alpha,\gamma\beta} (z *_{\gamma,\beta} y) = 0. \quad (34)$$

Theorem 3.7. *Let* $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$ *be an* Ω_c*-relative pre-anti-flexible algebra, defining for all* $\alpha, \beta \in \Omega$ *and for any* $x, y \in A$,

$$x \blacktriangleright_{\alpha,\beta} y = x \succ_{\alpha,\beta} y - y \prec_{\beta,\alpha} x, \quad (35a)$$

$$x \blacktriangleleft_{\alpha,\beta} y = x \prec_{\alpha,\beta} y - y \succ_{\beta,\alpha} x, \quad (35b)$$

then $(A, \blacktriangleright_{\alpha,\beta}, \Omega_c)$ *is an* Ω_c*-relative pre-Lie algebra and* $(A, \blacktriangleleft_{\alpha,\beta}, \Omega_c)$ *is an* Ω_c*-relative right symmetric algebra.*

Proof. Let $x, y, z \in A$ and $\alpha, \beta, \gamma \in \Omega_c$. We have

$$\begin{aligned}
(x,y,z)_{\blacktriangleright_{\alpha,\beta,\gamma}} &= (x \blacktriangleright_{\alpha,\beta} y) \blacktriangleright_{\alpha\beta,\gamma} z - x \blacktriangleright_{\alpha,\beta\gamma} (y \blacktriangleright_{\beta,\gamma} z)\\
&= (x \succ_{\alpha,\beta} y - y \prec_{\beta,\alpha} x) \succ_{\alpha\beta,\gamma} z\\
&- z \prec_{\gamma,\alpha\beta} (x \succ_{\alpha,\beta} y - y \prec_{\beta,\alpha} x)\\
&- x \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z - z \prec_{\gamma,\beta} y)\\
&+ (y \succ_{\beta,\gamma} z - z \prec_{\gamma,\beta} y) \prec_{\beta\gamma,\alpha} x\\
&= \{(x \succ_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} z - x \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z)\}\\
&- \{(z \prec_{\gamma,\beta} y) \prec_{\beta\gamma,\alpha} x - z \prec_{\gamma,\beta\alpha} (y \prec_{\beta,\alpha} x)\}\\
&+ \{(y \succ_{\beta,\gamma} z) \prec_{\beta\gamma,\alpha} x + x \succ_{\alpha,\beta\gamma} (z \prec_{\gamma,\beta} y)\}\\
&- \{(y \prec_{\beta,\alpha} x) \succ_{\alpha\beta,\gamma} z + z \prec_{\gamma,\alpha\beta} (x \succ_{\alpha,\beta} y)\}\\
&= \{(x \succ_{\alpha,\gamma} z) \prec_{\alpha\gamma,\beta} y + y \succ_{\beta,\alpha\gamma} (z \prec_{\gamma,\alpha} x)\}\\
&- \{(x \succ_{\alpha,\beta} y + y \prec_{\beta,\alpha} x) \succ_{\alpha\beta,\gamma} z\}\\
&- \{z \prec_{\gamma,\alpha\beta} (x \succ_{\alpha,\beta} y + y \succ_{\beta,\alpha} x)\} = (y,x,z)_{\blacktriangleright_{\beta,\alpha,\gamma}},
\end{aligned}$$

the second equal sign upwards in the above successive relations is due to Eq. (28a) and Eq. (28b) while the last one is due to Eq. (28a).

Therefore, $(A, \blacktriangleright_{\alpha,\beta}, \Omega_c)$ is an Ω_c-relative pre-Lie algebra.
Similarly, we prove that $(A, \blacktriangleleft_{\alpha,\beta}, \Omega_c)$ is an Ω_c-relative right symmetric algebra. □

Moreover, we have

Theorem 3.8. *Let* $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$ *be an* Ω_c*-relative pre-anti-flexible algebra. There is an* Ω_c*-relative Lie algebra structure on* A *given by for any* $x, y \in A$ *and for any* $\alpha, \beta \in \Omega_c$,

$$[x,y]_{\alpha,\beta} := (x \succ_{\alpha,\beta} y + x \prec_{\alpha,\beta} y) - (y \succ_{\beta,\alpha} x + y \prec_{\beta,\alpha} x). \quad (36)$$

Proof. Let $x, y, z \in A$ and $\alpha, \beta, \gamma \in \Omega_c$. In view of Eq. (36) and Eq. (30) we have

$$[x,y]_{\alpha,\beta} + [y,x]_{\beta,\alpha} = x \circledast_{\alpha,\beta} y - y \circledast_{\beta,\alpha} x + y \circledast_{\beta,\alpha} x - x \circledast_{\alpha,\beta} y = 0.$$

In addition, we have

$$\begin{aligned}&[[x,y]_{\alpha,\beta},z]_{\alpha\beta,\gamma}+[[y,z]_{\beta,\gamma},x]_{\beta\gamma,\alpha}+[[z,x]_{\gamma,\alpha},y]_{\gamma\alpha,\beta}\\&=(x,y,z)_{\circledast_{\alpha,\beta,\gamma}}+(y,z,x)_{\circledast_{\beta,\gamma,\alpha}}+(z,x,y)_{\circledast_{\gamma,\alpha,\beta}}\\&-(z,y,x)_{\circledast_{\gamma,\beta,\alpha}}-(x,z,y)_{\circledast_{\alpha,\gamma,\beta}}-(y,x,z)_{\circledast_{\beta,\alpha,\gamma}}.\end{aligned}$$

According to Proposition 3.4, we deduce the Ω_c-relative Jacobi identity. Therefore, A contains an Ω_c-relative Lie algebra structure. □

Theorem 3.9. *Let $(A,\prec_{\alpha,\beta},\succ_{\alpha,\beta},\Omega_c)$ be an Ω_c-relative pre-anti-flexible algebra such that its related Ω_c-relative anti-flexible algebra derived in Proposition 3.4 is $(A,\circledast_{\omega_1,\omega_2},\Omega_c)$. The family of bilinear products given by, for any $x,y\in A$ and for any $\alpha,\beta\in\Omega_c$,*

$$x\circledcirc_{\alpha,\beta}y=x\circledast_{\alpha,\beta}y+y\circledast_{\beta,\alpha}x, \tag{37}$$

satisfies the following relation, for any $x,y,z\in A$ and for any $\alpha,\beta,\gamma\in\Omega_c$,

$$(x,y,z)_{\circledcirc_{\alpha,\beta,\gamma}}=[y,[x,z]_{\alpha,\gamma}]_{\beta,\alpha\gamma}, \tag{38}$$

where $(x,y,z)_{\circledcirc_{\alpha,\beta,\gamma}}=(x\circledcirc_{\alpha,\beta}y)\circledcirc_{\alpha\beta,\gamma}z-x\circledcirc_{\alpha,\beta\gamma}(y\circledcirc_{\beta,\gamma}z)$ and $[x,y]_{\alpha,\beta}=x\circledast_{\alpha,\beta}y-y\circledast_{\beta,\alpha}x$.

Proof. Let $x,y,z\in A$ and $\alpha,\beta,\gamma\in\Omega_c$. We have

$$\begin{aligned}(x,y,z)_{\circledcirc_{\alpha,\beta,\gamma}}&=(x\circledcirc_{\alpha,\beta}y)\circledcirc_{\alpha\beta,\gamma}z-x\circledcirc_{\alpha,\beta\gamma}(y\circledcirc_{\beta,\gamma}z)\\&=(x\circledast_{\alpha,\beta}y)\circledast_{\alpha\beta,\gamma}z+(y\circledast_{\beta,\alpha}x)\circledast_{\beta\alpha,\gamma}z\\&+z\circledast_{\gamma,\alpha\beta}(x\circledast_{\alpha,\beta}y)+z\circledast_{\gamma,\alpha\beta}(y\circledast_{\beta,\alpha}x)\\&-x\circledast_{\alpha,\beta\gamma}(y\circledast_{\beta,\gamma}z)-x\circledast_{\alpha,\beta\gamma}(z\circledast_{\gamma,\beta}y)\\&-(y\circledast_{\beta,\gamma}z)\circledast_{\beta\gamma,\alpha}x-(z\circledast_{\gamma,\beta}y)\circledast_{\beta\gamma,\alpha}x\\&=(y\circledast_{\beta,\alpha}x)\circledast_{\beta\alpha,\gamma}z+z\circledast_{\gamma,\alpha\beta}(x\circledast_{\alpha,\beta}y)\\&-x\circledast_{\alpha,\beta\gamma}(z\circledast_{\gamma,\beta}y)-(y\circledast_{\beta,\gamma}z)\circledast_{\beta\gamma,\alpha}x\\&=y\circledast_{\beta,\alpha\gamma}(x\circledast_{\alpha,\gamma}z)+(z\circledast_{\gamma,\alpha}x)\circledast_{\gamma\alpha,\beta}y\\&-(x\circledast_{\alpha,\gamma}z)\circledast_{\alpha\gamma,\beta}y-y\circledast_{\beta,\gamma\alpha}(z\circledast_{\gamma,\alpha}x)=[y,[x,z]_{\alpha,\gamma}]_{\beta,\alpha\gamma}.\end{aligned}$$

The second and third equal sign upward are due to Proposition 3.4. □

Proposition 3.10. *Let $(A,\prec_{\alpha,\beta},\succ_{\alpha,\beta},\Omega_c)$ be an Ω_c-relative pre-anti-flexible algebra. Consider the algebra $(A,\circledcirc_{\alpha,\beta},\Omega_c)$ derived above. We have for any $x,y,z\in A$ and for any $\alpha,\beta,\gamma\in\Omega_c$,*

$$(x,y,z)_{\circledcirc_{\alpha,\beta,\gamma}}+(z,x,y)_{\circledcirc_{\gamma,\alpha,\beta}}+(y,z,x)_{\circledcirc_{\beta,\gamma,\alpha}}=0. \tag{39}$$

Proof. According to Theorem 3.8 and Proposition 3.4 and Theorem 3.9, the above equation is satisfied. □

Proposition 3.11. *Let* $(A, \prec_{\omega_1,\omega_2}, \succ_{\omega_1,\omega_2}, \Omega_c)$ *be an* Ω_c*-relative pre-anti-flexible algebra. Suppose for any* $\omega_1, \omega_2 \in \Omega_c$ *the family of bilinear operations* $\prec_{\omega_1,\omega_2}$: $A \otimes A \to A$ *is independent of* ω_1 *and* $\succ_{\omega_1,\omega_2}$: $A \otimes A \to A$ *is independent of* ω_2*. Then A possesses:*

(1) *a pre-anti-flexible family algebra structure which is* $(A, \prec_{\omega_2}, \succ_{\omega_1}, \Omega_c)$*. Conversely, if the quadruple* $(A, \prec_\omega, \succ_\omega, \Omega_c)$ *is a pre-anti-flexible family algebra, then it can be regarded as an* Ω_c*-relative pre-anti-flexible algebra* $(A, \prec_{\omega_1,\omega_2}, \succ_{\omega_1,\omega_2}, \Omega_c)$ *in which for any* $\omega_1, \omega_2 \in \Omega_c$ "$\prec_{\omega_1,\omega_2}$" *is independent of* ω_1 *and* "$\succ''_{\omega_1,\omega_2}$ *is independent of* ω_2.

(2) *a left pre-Lie family algebra structure and conversely, if A possesses a left pre-Lie family algebra structure, then it can be regarded as an* Ω_c*-relative pre-Lie algebra structure* (*given in Theorem* 3.7) *in which for any* $\omega_1, \omega_2 \in \Omega_c$, "$\prec_{\omega_1,\omega_2}$" *is independent of* ω_1 *and* "$\succ_{\omega_1,\omega_2}$" *is independent of* ω_2.

(3) *similarly to* (3.11), *a right pre-Lie family algebra structure and conversely, if A possesses a right pre-Lie family algebra structure* (*given in Theorem* 3.7), *then it can be viewed as an* Ω_c*-relative right-symmetric algebra in which, for any* $\omega_1, \omega_2 \in \Omega_c$, "$\prec_{\omega_1,\omega_2}$" *is independent of* ω_1 *and* "$\succ_{\omega_1,\omega_2}$" *is independent of* ω_2.

Proof. Under divers assumptions, Eq. (10a) is expressed by Eq. (28a) and Eq. (10b) is translated by Eq. (28b). □

4. Rota-Baxter operators

This section deals with the use of the Rota-Baxter operators defined on Ω_c-relative anti-flexible and Lie algebras to build Ω_c-relative pre-anti-flexible algebras. It is proved that a Rota-Baxter operator define the Ω_c-relative Lie algebra derived from a given Ω_c-relative anti-flexible algebra induces an Ω_c-relative pre-anti-flexible algebra.

Definition 4.1. Let $(A, \cdot_{\alpha,\beta}, \Omega_c)$ be an Ω_c-relative anti-flexible algebra. A Rota-Baxter operator on A is a family of linear operators $R_{B_\alpha} : A \to A$, with $\alpha \in \Omega_c$, satisfying for any $x, y \in A$ and any $\alpha, \beta \in \Omega_c$,

$$R_{B_\alpha}(x) \cdot_{\alpha,\beta} R_{B_\beta}(y) = R_{B_{\alpha\beta}}(R_{B_\alpha}(x) \cdot_{\alpha,\beta} y + x \cdot_{\alpha,\beta} R_{B_\beta}(y)). \tag{40}$$

Definition 4.2. Let $(A, \cdot_{\alpha,\beta}, \Omega_c)$ be an Ω_c-relative anti-flexible algebra. A generalized Rota-Baxter operator on A is a family of linear operators $G_{RB_\alpha} : A \to A$, with $\alpha \in \Omega_c$, satisfying for any $\alpha, \beta, \gamma \in \Omega_c$ and any $x, y, z \in A$,

$$\begin{aligned} 0 = &(G_{RB_{\alpha\beta}}(G_{RB_\alpha}(x) \cdot_{\alpha,\beta} y + x \cdot_{\alpha,\beta} G_{RB_\beta}(y)) - G_{RB_\alpha}(x) \cdot_{\alpha,\beta} G_{RB_\beta}(y)) \cdot_{\alpha\beta,\gamma} z \\ &+ z \cdot_{\gamma,\beta\alpha} (G_{RB_\beta}(y) \cdot_{\beta,\alpha} G_{RB_\alpha}(x) - G_{RB_{\beta\alpha}}(G_{RB_\beta}(y) \cdot_{\beta,\alpha} x + y \cdot_{\beta,\alpha} G_{RB_\alpha}(x))). \end{aligned} \tag{41}$$

Proposition 4.3. *Let $(A, \cdot_{\alpha,\beta}, \Omega_c)$ be an Ω_c-relative anti-flexible algebra and $G_{RB_\alpha} : A \to A$ a generalized Rota-Baxter operator. Defining for any $x, y \in A$ and any $\alpha, \beta \in \Omega_c$,*

$$x \prec_{\alpha,\beta} y := x \cdot_{\alpha,\beta} G_{RB_\beta}(y), \quad x \succ_{\alpha,\beta} y := G_{RB_\alpha}(x) \cdot_{\alpha,\beta} y, \tag{42}$$

then, $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$ is turns to an Ω_c-relative pre-anti-flexible algebra. The converse is true.

Proof. Let $x, y, z \in A$ and $\alpha, \beta, \gamma \in \Omega_c$. In view of Eq. (42) we have

$$(x \succ_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} z - x \succ_{\alpha,\beta\gamma} (y \prec_{\alpha,\beta} z) = (G_{RB_\alpha}(x), y, G_{RB_\gamma}(z))_{\alpha,\beta,\gamma},$$

$$(z \succ_{\gamma,\beta} y) \prec_{\gamma\beta,\alpha} x - z \succ_{\gamma,\beta\alpha} (y \prec_{\beta,\alpha} x) = (G_{RB_\gamma}(z), y, G_{RB_\alpha}(x))_{\gamma,\beta,\alpha},$$

$$\begin{aligned}(x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} z - x \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z) = (z, G_{RB_\beta}(y), G_{RB_\alpha}(x))_{\gamma,\beta,\alpha} \\ +(G_{RB_{\alpha\beta}}(G_{RB_\alpha}(x) \cdot_{\alpha,\beta} y + x \cdot_{\alpha,\beta} G_{RB_\beta}(y)) - (G_{RB_\alpha}(x) \cdot_{\alpha,\beta} G_{RB_\beta}(y))) \cdot_{\alpha\beta,\gamma} z,\end{aligned}$$

$$\begin{aligned}(z \prec_{\gamma,\beta} y) \prec_{\gamma\beta,\alpha} x - z \prec_{\gamma,\beta\alpha} (y \succ_{\beta,\alpha} x + y \prec_{\beta,\alpha} x) = (G_{RB_\alpha}(x), G_{RB_\beta}(y), z)_{\alpha,\beta,\gamma} \\ +z \cdot_{\gamma,\beta\alpha} (G_{RB_\beta}(y) \cdot_{\beta,\alpha} G_{RB_\alpha}(x) - G_{RB_{\beta\alpha}}(G_{RB_\beta}(y) \cdot_{\beta,\alpha} x + y \cdot_{\beta,\alpha} G_{RB_\alpha}(x))).\end{aligned}$$

Therefore, Eq.(42) turns A into an Ω_c-relative pre-anti-flexible algebra if and only if G_{RB_α} satisfy Eq. (41). □

Corollary 4.4. *Any Rota-Baxter operator on an Ω_c-relative anti-flexible algebra induces an Ω_c-relative pre-anti-flexible algebra.*

In the sequel of this section, we consider the Ω_c-relative anti-flexible algebra $(A, \cdot_{\omega_1,\omega_2}, \Omega_c)$ in which for any $\alpha \in \Omega_c$, the linear map $\varphi_\alpha : A \to A$ is is such that the elements

$$\varphi_\alpha(x) \cdot_{\alpha,\beta} \varphi_\beta(y) - \varphi_{\alpha\beta}(x \cdot_{\alpha,\beta} \varphi_\beta(y) + \varphi_\alpha(x) \cdot_{\alpha,\beta} y), \ \ \forall x, y \in A, \ \forall \alpha, \beta \in \Omega_c, \tag{43}$$

satisfy the following, for any $\alpha, \beta, \gamma \in \Omega_c$ and for any $x, y, z \in A$,

$$\begin{aligned} z \cdot_{\gamma,\beta\alpha} (\varphi_\alpha(x) \cdot_{\alpha,\beta} \varphi_\beta(y) - \varphi_{\alpha\beta}(x \cdot_{\alpha,\beta} \varphi_\beta(y) + \varphi_\alpha(x) \cdot_{\alpha,\beta} y)) = \\ (\varphi_{\alpha\beta}(x \cdot_{\alpha,\beta} \varphi_\beta(y) + \varphi_\alpha(x) \cdot_{\alpha,\beta} y) - \varphi_\alpha(x) \cdot_{\alpha,\beta} \varphi_\beta(y)) \cdot_{\alpha\beta,\gamma} z. \end{aligned} \tag{44}$$

Definition 4.5. By a Rota-Baxter operator on an Ω_c-relative Lie algebra $(A, [\cdot,\cdot]_{\alpha,\beta}, \Omega_c)$ we mean a family of linear operators $R_{B_\alpha} : A \to A$ with $\alpha \in \Omega_c$, satisfying for any $x, y \in A$, and for any $\alpha, \beta \in \Omega_c$,

$$[R_{B_\alpha}(x), R_{B_\beta}(y)]_{\alpha,\beta} = R_{B_{\alpha\beta}}([x, R_{B_\beta}(y)]_{\alpha,\beta} + [R_{B_\alpha}(x), y]_{\alpha,\beta}). \tag{45}$$

Proposition 4.6. *Let* $(A, \cdot_{\alpha,\beta}, \Omega_c)$ *be an* Ω_c*-relative anti-flexible algebra equipped with a family of linear maps* $\varphi_{\alpha} : A \to A$*, with* $\alpha \in \Omega_c$*, in which the elements as the form given in Eq.* (43) *satisfy Eq.* (44)*. The family linear products given by, for any* $x, y \in A$ *and any* $\alpha, \beta \in \Omega_c$*,*

$$x \prec_{\alpha,\beta} y = x \cdot_{\alpha,\beta} \varphi_{\beta}(y); \quad x \succ_{\alpha,\beta} y = \varphi_{\alpha}(x) \cdot_{\alpha,\beta} y \tag{46}$$

defines an Ω_c*-relative pre-anti-flexible structures on* A *if and only if the family of linear maps "*φ_α*" is a Rota-Baxter operator on the* Ω_c*-relative Lie algebra* $(A, [\cdot,\cdot]_{\alpha,\beta}, \Omega_c)$*, where for any* $x, y \in A$ *and for any* $\alpha, \beta \in \Omega_c$*,* $[x, y]_{\alpha,\beta} = x \cdot_{\beta,\alpha} y - y \cdot_{\beta,\alpha} x$*.*

Proof. According to Proposition 4.3, the family of linear maps φ_α satisfy Eq. (41) and Eq. (44) if and only if any $x, y, z \in A$ and any $\alpha, \beta, \gamma \in \Omega_c$,

$$z \cdot_{\gamma,\alpha\beta} \left([\varphi_{\alpha}(x), \varphi_{\beta}(y)]_{\alpha,\beta} - \varphi_{\alpha\beta}([x, \varphi_{\beta}(y)]_{\alpha,\beta} + [\varphi_{\alpha}(x), y]_{\alpha,\beta})\right) = 0. \tag{47}$$

Therefore, φ_α is a Rota-Baxter operator on the Ω_c-relative Lie algebra $(A, [\cdot,\cdot]_{\alpha,\beta}, \Omega_c)$, where for any $x, y \in A$ and for any $\alpha, \beta \in \Omega_c$, $[x, y]_{\alpha,\beta} = x \cdot_{\beta,\alpha} y - y \cdot_{\beta,\alpha} x$. □

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