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Algebraic connections between right and middle Bol loops and their cores

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Abstract. To every right or left Bol loop corresponds a middle Bol loop. In this paper, the cores of right Bol loops (RBL) and its corresponding middle Bol loops (MBL) were studied. Their algebraic connections were considered. It was shown that the core of a RBL is elastic and right idempotent. The core of a RBL was found to be alternative (or left idempotent) if and only if its corresponding MBL is right symmetric. If a MBL is right (left) symmetric, then, the core of its corresponding RBL is a medial (semimedial). The core of a middle Bol loop has the left inverse property (automorphic inverse property, right idempotence resp.) if and only if its corresponding RBL has the super anti-automorphic inverse property (automorphic inverse property, exponent 2 resp.). If a RBL is of exponent 2, then, the core of its corresponding MBL is left idempotent. If a RBL is of exponent 2 then: the core of a MBL has the left alternative property (right alternative property) if and only if its corresponding RBL has the cross inverse property (middle symmetry). Some other similar results were derived for RBL of exponent 3.

1. Introduction

Let $(Q, \cdot)$ be a quasigroup. Then the left translation L_a and right translation R_a are bijections. For any quasigroup $(Q, \cdot)$, we have two new binary operations: right division $(/)$ and left division $(\backslash)$ and middle translation P_a defined as follows:

$$x\backslash y = yL_x^{-1} = xP_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yP_x^{-1},$$

where

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups.

Definition 1.1. A *loop* $(Q, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

(*i*) $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$ for all $x, y \in Q$,

(*ii*) $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$ for all $x, y \in Q$ and

(*iii*) $x\backslash x = y/y$ or $e \cdot x = x \cdot e = x$ for all $x, y \in Q$.

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We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x(y\backslash z)$ respectively.

In a loop $(Q,\cdot)$ with identity element e, the *left inverse element* of $x \in Q$ is the element $xJ_\lambda = x^\lambda \in Q$ such that

$$x^\lambda \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

Middle Bol loops were first studied by V. D. Belousov [5], where he gave identity

$$(x/y)(z\backslash x) = x(zy\backslash x) \tag{1}$$

characterizing loops that satisfy the universal anti-automorphic inverse property.

After this characterization, Gwaramija (see [10]) proved that a loop $(Q,\circ)$ is middle Bol if there exist a right Bol loop $(Q,\cdot)$ such that $x \circ y = (y \cdot xy^{-1})y$ for all $x, y \in Q$. If $(Q,\circ,//,\backslash\backslash)$ is a middle Bol loop and $(Q,\cdot,/,\backslash)$ is the corresponding right Bol loop, then

$$x \circ y = y^{-1}\backslash x \qquad \text{and} \qquad x \cdot y = y//x^{-1} \tag{2}$$

for every $x, y \in Q$.

Also, if $(Q,\circ,//,\backslash\backslash)$ is a middle Bol loop and $(Q,\cdot,/,\backslash)$ is the corresponding left Bol loop, then

$$x \circ y = x/y^{-1} \qquad \text{and} \qquad x \cdot y = x//y^{-1} \tag{3}$$

for every $x, y \in Q$.

Grecu [7] showed that the right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop. After then, middle Bol loops resurfaced in literature in 1994 and 1996 when Syrbu [22, 23] considered them in-relation to the universality of the elasticity law. Kuznetsov [18], while studying gyrogroups (a special class of Bol loops) established some algebraic properties of middle Bol loops and designed a method of constructing a middle Bol loop from a gyrogroup.

In 2010, Syrbu [24] studied the connections between structure and properties of middle Bol loops and of the corresponding left Bol loops. It was noted that two middle Bol loops are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic, and a general form of the autotopisms of middle Bol loops was deduced. Relations between different sets of elements, such as nucleus, left (right,middle) nuclei, the set of Moufang elements, the center, e.t.c. of a middle Bol loop and left Bol loops were established. In [8] Grecu and Syrbu proved that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic. In [25], Syrbu and Grecu established a necessary and sufficient condition for the quotient loop of a middle Bol loop and of its corresponding right Bol loop to be isomorphic. In [9], they established that the commutant (centrum) of a middle Bol loop is an AIP-subloop and gave a necessary and sufficient condition when the commutant is an invariant under the existing isostrophy between a middle Bol loop and the corresponding right Bol loop. Osoba et al. [19] investigate further the multiplication group of the middle Bol loop in relation to the left Bol loop while Jaiyéọlá [11, 13] studied second Smarandache Bol loops.

The core of a loop was originally introduced by R. H. Bruck in connection with invariants of isotopically closed Moufang loops (isotopic Moufang loops that have isomorphic core). More general definition was created by V. D Belousov [5]. The core of middle Bol loop was introduced and initially studied by Gwaramija [10]. Using Bruck's approach for the notion of core, Gwaramija introduced a definition for the core of a middle Bol loop. By using the properties of algebraic 3-nets, Gwaramija remarked that this core of a middle Bol loop is isomorphic to the core of a left Bol loop. Belousov [4] used geometrical approach, namely on evaluation of coordinates of points and lines in the corresponding Bol net to show that cores of left Bol loops, particularly cores of Moufang loops, or groups, are left distributive, left symmetric, and idempotent. In 2005, Vanžurová [26] used purely algebraic approach to clarify some of the well known results, the relationship between cores and the variety of left symmetric, left distributive, idempotent groupoids or its medial subvariety of Bol loops, Moufang loops or groups.

Adeniran et al. [1] carried out a comprehensive study on the core and some isotopic characterisation of generalised Bol loops, it was shown that the set of semi-automorphisms of the generalised Bol loops are the automorphisms of the core. Jaiyéọlá et al. [15], studied the holomorphic structure of Middle Bol loops and showed that the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle Bol loop and its automorphism group is abelian. Generalised Bol loops also are studied in [2, 3] and [17].

New algebraic identities of middle Bol loops, where necessary and sufficient conditions for a bi-variate mapping of a middle Bol loop to have RIP, LIP, RAIP, LAIP and flexible property were presented in [14] and [16].

Furtherance to earlier studies, in this work, we unveil some algebraic characterizations of right and middle Bol loops relatively from their cores. In the first section, the concept of core of middle Bol loop and the core of its corresponding right Bol loop were introduced. To every right or left Bol loop corresponds a middle Bol loop. In the second section, the cores of a right Bol loop (RBL) and its corresponding middle Bol loop (MBL) were studied.

Definition 1.2. A groupoid (quasigroup) $(Q, \cdot)$ is said to have the

1. *left inverse property* (*LIP*) if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \cdot xy = y$ holds for all $x, y \in Q$.

2. *right inverse property* (RIP) if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $yx \cdot x^\rho = y$ holds for all $x, y \in Q$.
3. *inverse property* (IP) if it has both the LIP and RIP.
4. *right alternative property* (RAP) if $y \cdot xx = yx \cdot x$ holds for all $x, y \in Q$.
5. *left alternative property* (LAP) if $xx \cdot y = x \cdot xy$ holds for all $x, y \in Q$.
6. *flexibility* or *elasticity* if $xy \cdot x = x \cdot yx$ holds for all $x, y \in Q$.
7. *cross inverse property* (CIP) if there exist the mapping $J_\lambda : x \mapsto x^\lambda$ or $J_\rho : x \mapsto x^\rho$ such that $xy \cdot x^\rho = y$ or $x \cdot yx^\rho = y$ or $x^\lambda \cdot yx = y$ or $x^\lambda y \cdot x = y$ holds for all $x, y \in Q$.

Definition 1.3. A loop $(Q, \cdot)$ is said to be

1. an *automorphic inverse property loop* (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$,
2. an *anti-automorphic inverse property loop* (AAIPL) or a *D-loop* [6] if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$,
3. a *super anti-automorphic inverse property loop* (SAAIPL) if $(x \cdot yz)^{-1} = z^{-1} \cdot \big(y^{-1}x^{-1}\big)$ for all $x, y, z \in Q$,
4. a *semi-automorphic inverse property loop* (SAIPL) if it obeys any of the identities $(xy \cdot x)^\rho = x^\rho y^\rho \cdot x^\rho$ or $(xy \cdot x)^\lambda = x^\lambda y^\lambda \cdot x^\lambda$ for all $x, y \in Q$,
5. a *power associative loo*p if $< x >$ is a subgroup for all $x \in Q$ and a *diassociative loop* if $< x, y >$ is a subgroup for all $x, y \in Q$.

Definition 1.4. A loop $(Q, \cdot)$ is called a

1. *right Bol loop* if $(xy \cdot z)y = x(yz \cdot y)$ for all $x, y \in Q$,
2. *middle Bol loop* if $(x/y)(z\backslash x) = (x/(zy))x$ or $(x/y)(z\backslash x) = x((zy)\backslash x)$ for all $x, y \in Q$.

Definition 1.5. A groupoid (quasigroup) $(Q, \cdot)$ is

1. *right symmetric* if $yx \cdot x = y$ for all $x, y \in Q$,
2. *left symmetric* if $x \cdot xy = y$ for all $x, y \in Q$,
3. *middle symmetric* if $x \cdot yx = y$ or $xy \cdot x = y$ for all $x, y \in Q$,
4. *idempotent* if $x \cdot x = x$ for all $x \in Q$,
5. *left idempotent* if $xx \cdot y = xy$ for all $x, y \in Q$,
6. *right idempotent* if $y \cdot xx = yx$ for all $x, y \in Q$,
7. *semimedial* if $xx \cdot yz = xy \cdot xz$ for all $x, y, z, \in Q$,
8. *medial* if $xy \cdot zu = xz \cdot yu$ for all $x, y, z, u \in Q$,
9. a *Steiner loop* if it is totally symmetric loop.

Theorem 1.6. (cf. [20]) *A quasigroup* $(Q, \cdot)$ *is totally symmetric if and only if it is commutative and right or left symmetric.*

Theorem 1.7. (cf. [20]) *A loop* $(Q,\cdot)$ *is totally symmetric if and only if* $(Q,\cdot)$ *is an IP loop of exponent* 2.

Corollary 1.8. (cf. [20]) *Every totally symmetric quasigroup is a commutative I.P. quasigroup.*

Definition 1.9. (cf. [5]) Let $(Q,\cdot)$ be a loop. For all $x,y\in Q$, define $x+y=xy^{-1}\cdot x$. The groupoid $(Q,+)$ is called the *core* of $(Q,\cdot)$.

2. Main results

Lemma 2.1. *A middle Bol loop* $(Q,\cdot)$ *satisfies the following identities:*

(a) $x/(x\backslash z)=x(z\backslash x)$ *or* $|P_xL_x|=2$ *or* $|L_xP_x|=2$,
(b) $x\backslash(z\backslash x)=(zx)\backslash x$,
(c) $(t/x)\backslash x=x(t\backslash x)$ *or* $P_xL_x=R_x^{-1}P_x$,
(d) $x/(x\backslash z)=x(z\backslash x)=(z/x)\backslash x$ *or* $P_xL_x=R_x^{-1}P_x$.

Proof. (a). By (1), $(Q,\cdot)$ is a middle Bol loop if and only if $(x/y)(z\backslash x)=x(zy\backslash x)$. Thus for $z=x$ we have $x\backslash(x/y)=xy\backslash x$ which implies $yP_x^{-1}L_x^{-1}=yL_xP_x$. Consequently, for all $x\in Q$, we have

$$P_x^{-1}L_x^{-1}=L_xP_x. \tag{4}$$

From (4), we get $P_x^{-1}=L_xP_xL_x$ which gives $L_x^{-1}P_x^{-1}=P_xL_x$ So, for any $z\in Q$ $zL_x^{-1}P_x^{-1}=zP_xL_x$. Hence,

$$x/(x\backslash z)=x(z\backslash x). \tag{5}$$

(b). Setting $y=x$ in (1), we have $z\backslash x=x(zx\backslash x)$ Consequently,

$$x\backslash(z\backslash x)=zx\backslash x. \tag{6}$$

(c). From the last equality, $zPL_x^{-1}=zR_xP_x$. Thus $R_x^{-1}P_xL_x^{-1}=P_x$. Hence, $R_x^{-1}P_x=P_xL_x$, i.e for any $t\in Q$, we have $(t/x)\backslash x=x(t\backslash x)$,
(d). By (a) and (b) we obtain

$$x/(x\backslash z)=x(z\backslash x)=(z/x)\backslash x. \tag{7}$$

This completes the proof. □

Lemma 2.2. *Let* $(Q,\cdot,/,\backslash)$ *be a right Bol loop with core* $(Q,+)$ *and corresponding middle Bol loop* $(Q,\circ,//,\backslash\backslash)$ *with core* $(Q,\oplus)$*. Then*

$$x+y=xy^{-1}\cdot x=x//(x\backslash\backslash y)=x\circ(y\backslash\backslash x)\ \text{and}\ x\oplus y=x^{-1}\backslash(y^{-1}\backslash x)$$

for all $x,y\in Q$.

Proof. By (2), we have

$$x + y = xy^{-1} \cdot x = x//(xy^{-1})^{-1} = x//(y^{-1}//x^{-1})^{-1} = x//(x\backslash\backslash y).$$

Applying Lemma 2.1(a), we have $x//(x\backslash\backslash y) = x \circ (y\backslash\backslash x)$.

Let x^- be the inverse of x in $(Q, \circ)$. Going by (2), $e = x^- \circ x = x^{-1}\backslash x^- \implies x^{-1} = x^-$. So, $x \oplus y = (x \circ y^-) \circ x = (x \circ y^{-1}) \circ x = \big(y^{-1}\backslash x\big) \cdot x = x^{-1}\backslash\big(y^{-1}\backslash x\big)$. □

Theorem 2.3. *The core* $(Q, +)$ *of a right Bol loop* $(Q, \cdot)$ *is elastic.*

Proof. $(Q, +)$ is flexible if and only if $(x + y) + x = x + (y + x)$. By Lemma 2.2, $(Q, +)$ is flexible if and only if $(x + y)//[(x + y)\backslash\backslash x] = x//[x\backslash\backslash(y + x)]$, i.e. if and only if

$$\big(x//(x\backslash\backslash y)\big)//[x//\big((x\backslash\backslash y)\backslash\backslash x\big)] = x//[x\backslash\backslash\big(y//(y\backslash\backslash x)\big)]. \tag{8}$$

Using (5), we have

$$(x \circ (y\backslash\backslash x))//[(x \circ (y\backslash\backslash x))\backslash\backslash x] = x//[x\backslash\backslash(y \circ (x\backslash\backslash y))]. \tag{9}$$

If $p = x \circ (y\backslash\backslash x)$ and $q = y \circ (x\backslash\backslash y)$, then from (9), it follows that $p//(p\backslash\backslash x) = x//(x\backslash\backslash q)$. Using Lemma 2.1 ($a$), we get

$$(x \circ \big(y\backslash\backslash x)\big) \circ [x\backslash\backslash(x \circ (y\backslash\backslash x))] = x \circ [(y \circ (x\backslash\backslash y))\backslash\backslash x]. \tag{10}$$

Set $a = y\backslash\backslash x$ and $b = x\backslash\backslash y$. Then from (10),

$$(x \circ a) \circ (x\backslash\backslash(x \circ a)) = x \circ ((y \circ b)\backslash\backslash x).$$

Using (1), we have

$$(x \circ a) \circ (x\backslash\backslash(x \circ a)) = (x//b) \circ (y\backslash\backslash x) \Leftrightarrow (x \circ a) \circ a = (x//b) \circ (y\backslash\backslash x)$$

Replacing a and b back into the equality, we have

$$(x \circ (y\backslash\backslash x)) \circ (y\backslash\backslash x) = (x//b) \circ (y\backslash\backslash x) \Leftrightarrow x \circ (y\backslash\backslash x) = x//(x\backslash\backslash y)$$

which is true based on Lemma 2.1(a). □

Theorem 2.4. *Let* $(Q, +)$ *be the core of a right Bol loop* $(Q, \cdot)$ *whose corresponding middle Bol loop is* $(Q, \circ)$. *Then*

(i) $(Q, \circ)$ *is right symmetric if and only if* $(Q, +)$ *is an alternative groupoid,*

(ii) $(Q, \circ)$ *is right symmetric if and only if* $(Q, +)$ *is left idempotent,*

(iii) $(Q, +)$ *is right idempotent.*

Proof. (*i*). The core $(Q,+)$ has the left alternative property: $(x+x)+y = x+(x+y)$ if and only if $(x+x)//[(x+x)\backslash\backslash y] = x//[x\backslash\backslash(x+y)]$, i.e. if and only if $(x//(x\backslash\backslash x))//[(x//(x\backslash\backslash x))\backslash\backslash y] = x//[x\backslash\backslash(x//(x\backslash\backslash y))]$. The last is equivalent to

$$x//(x\backslash\backslash y) = x//[x\backslash\backslash(x//(x\backslash\backslash y))]. \tag{11}$$

Set $x//(x\backslash\backslash y) = z$, so that (11) becomes $z = x//(x\backslash\backslash z))$ so that $z \circ (x\backslash\backslash z) = x$. Put $x\backslash\backslash z = p \Leftrightarrow x \circ p = z$, then we have $z \circ p = x$. Substituting for z gives $(x \circ p) \circ p = x$.

$(Q,+)$ has right alternative property if and only if $(y+x)+x = y+(x+x)$, or equivalently $(y+x)//((y+x)\backslash\backslash x) = y//(y\backslash\backslash(x+x))$, i.e. if and only if $(y//(y\backslash\backslash x))//[(y//y\backslash\backslash x)\backslash\backslash x] = y//[y\backslash\backslash(x//(x\backslash\backslash x))]$. This is equivalent to

$$(y//(y\backslash\backslash x))//[(y//y\backslash\backslash x)\backslash\backslash x] = y//(y\backslash\backslash x). \tag{12}$$

Let

$$q = y//(y\backslash\backslash x). \tag{13}$$

Put (13) in (12) to get $q//(q\backslash\backslash x) = q \Leftrightarrow q = q \circ (q\backslash\backslash x) \Leftrightarrow q\backslash\backslash x = e \Leftrightarrow q = x$ so that (13) now becomes

$$x = y//(y\backslash\backslash x) \Leftrightarrow x \circ (y\backslash\backslash x) = y. \tag{14}$$

For $t = y\backslash\backslash x$ (14) has the form $x \circ t = y$ which for $x = y \circ t$ gives $(y \circ t) \circ t = y$.

$(Q,+)$ satisfies the left idempotency law: $(x+x)+y = x+y$ if and only if $(x+x)//[(x+x)\backslash\backslash y] = x//(x\backslash\backslash y)$. i.e. $x//(x\backslash\backslash x)[(x//(x\backslash\backslash x))\backslash\backslash y]x//(x\backslash\backslash y)$, or $x \circ (x\backslash\backslash y) = x//(x\backslash\backslash y)$. This for $x\backslash\backslash y = t$ gives $x \circ t = x//t$ Hence $(x \circ t) \circ t = x$.

$(Q,+)$ is right idempotent since for $x, y \in Q$ $\;y+(x+x) = y//[y\backslash\backslash(x+x)] = y//[y\backslash\backslash(x//(x\backslash\backslash x))] = y//(y\backslash\backslash x) = y+x$. □

Corollary 2.5. *Let $(Q,+)$ be the core of a right Bol loop $(Q,\cdot)$ whose corresponding middle Bol loop is $(Q,\circ)$. If $(Q,+)$ is alternative or left idempotent, then $(Q,\circ)$ is Steiner loop if and only if it is commutative.*

Proof. This follows from Theorem 2.4 and Theorem 1.6. □

Theorem 2.6. *Let $(Q,\circ)$ be a middle Bol loop and let $(Q,+)$ be the core of its corresponding right Bol loop $(Q,\cdot)$. Then*

(1) *$(Q,+)$ is medial if $(Q,\circ)$ satisfies right symmetric property*

(2) *$(Q,+)$ is semimedial if $(Q,\circ)$ satisfies left symmetric property*

Proof. (1). $(Q,+)$ is medial if and only if

$$(x+y)//[(x+y)\backslash\backslash(u+v)] = (x+u)//[(x+u)\backslash\backslash(y+v)]$$

or equivalently, if and only if

$$x//(x\backslash\backslash y)//[(x//(x\backslash\backslash y))\backslash\backslash(u//(u\backslash\backslash v))] = x//(x\backslash\backslash u)//[(x//(x\backslash\backslash u))\backslash\backslash(y//(y\backslash\backslash v))].$$

Using (5), we have

$(x \circ (y\backslash\backslash x))//[(x \circ (y\backslash\backslash x))\backslash\backslash(u \circ (v\backslash\backslash u))] = (x \circ (u\backslash\backslash x))//[(x \circ (u\backslash\backslash x))\backslash\backslash(y \circ (v\backslash\backslash y))]$.

This for $y\backslash\backslash x = t$ gives

$$((y \circ t) \circ t)//[((y \circ t) \circ t)\backslash\backslash u \circ (v\backslash\backslash u)] \\ = (y \circ t) \circ \big(u\backslash\backslash(y \circ t)\big)//[((y \circ t) \circ g(u\backslash\backslash(y \circ t)))\backslash\backslash(y \circ (v\backslash\backslash y))]$$

which is equivalent to

$$y//[y\backslash\backslash u \circ (v\backslash\backslash u)] = (y \circ t) \circ (u\backslash\backslash(y \circ t))//[(y \circ t) \circ (u\backslash\backslash(y \circ t))\backslash\backslash(y \circ (v\backslash\backslash y))].$$

This, for $u\backslash\backslash(y \circ t) = s$ gives

$$y//[(y\backslash\backslash(u \circ (v\backslash\backslash u))] = ((u \circ s) \circ s)//[((u \circ s) \circ s)\backslash\backslash(y \circ (v\backslash\backslash y))],$$

which is equivalent to

$$y//((y\backslash\backslash(u \circ (v\backslash\backslash u))) = u//(u\backslash\backslash(y \circ (v\backslash\backslash y))).$$

From this, for $v\backslash\backslash u = p$ and $v\backslash\backslash y = q$, we get

$$(v \circ q)//((v \circ q)\backslash\backslash v) = (v \circ p)//((v \circ p)\backslash\backslash v).$$

Now, applying (5), we obtain $(v \circ q)\big(v\backslash\backslash(v \circ q)\big) = (v \circ p) \circ \big(v\backslash\backslash(v \circ p)\big)$, which for $r = v\backslash\backslash(v \circ q)$ and $w = v\backslash\backslash(v \circ p)$ gives $(v \circ r) \circ r = (v \circ w) \circ w$.

(2). $(Q, +)$ is semimedial since by the definition,

$$(x + x)//[(x + x)\backslash\backslash(y + z)] = (x + y)//[(x + y)\backslash\backslash(x + z)],$$

which is equivalent to

$$(x//(x\backslash\backslash x))//[(x//(x\backslash\backslash x))\backslash\backslash(y//(y\backslash\backslash z)] = (x//(x\backslash\backslash y))//[(x//\big(x\backslash\backslash y))\backslash\backslash(x//(x\backslash\backslash z))]$$

Using (7), we have

$$x//[x\backslash\backslash((z//y)\backslash\backslash y)] = ((y//x)\backslash\backslash x)//[((y//x)\backslash\backslash x)\backslash\backslash((z//x)\backslash\backslash x)]$$

or equivalently,

$$[((z//y)\backslash\backslash y)//x]\backslash\backslash x = [((z//x)\backslash\backslash x)//((y//x)\backslash\backslash x)]\backslash\backslash((y//x)\backslash\backslash x).$$

For fixed $z \in Q$, we have

$$zR_y^{-1}P_yR_x^{-1}P_x = zR_x^{-1}P_xR^{-1}_{((y//x)\backslash\backslash x)}P_{((y//x)\backslash\backslash x)}$$

which is equivalent to

$$zR_y^{-1}P_yR_x^{-1}P_x = zR_x^{-1}P_xR^{-1}_{y(R_x^{-1}P_x)}P_{(yR_x^{-1}P_x)}.$$

The last equation is true because the left symmetric property in $(Q, \circ)$ is equivalent to the fact that $R_x = P_x \Leftrightarrow yR_x = yP_x \Leftrightarrow y \circ x = y\backslash\backslash x \Leftrightarrow y \circ (y \circ x) = x$ for any $x, y \in Q$. □

Theorem 2.7. *Let $(Q, \circ)$ be a middle Bol loop with core $(Q, \oplus)$ and corresponding right Bol loop $(Q, \cdot, e)$.*

1. *$(Q, \oplus)$ has LIP if and only if $(Q, \cdot)$ is a SAAIPL.*
2. *If $(Q, \cdot)$ is of exponent 2, then $(Q, \oplus)$ has LAP if and only if $(Q, \cdot)$ has CIP.*
3. *If $(Q, \cdot)$ is of exponent 2, then $(Q, \oplus)$ has RAP if and only if $(Q, \cdot)$ has middle symmetry.*
4. *$(Q, \oplus)$ has AIP if and only if $(Q, \cdot)$ is an AIPL.*
5. *$(Q, \oplus)$ obeys right idempotent law if and only if $(Q, \cdot)$ is of exponent 2.*
6. *If $(Q, \cdot)$ is of exponent 2, then $(Q, \oplus)$ obeys left idempotent law.*
7. *If $(Q, \cdot)$ is of exponent 3, then $(Q, \oplus)$ has RAP if and only if there exists a map $t : Q \times Q \to Q$ such that $(Q, \cdot)$ satisfies the condition $x^2 \cdot t(x,y)^2 y^2 = t(x,y)$ for all $x, y \in Q$.*
8. *If $(Q, \cdot)$ is of exponent 3, then $(Q, \oplus)$ has LAP if and only if $(Q, \cdot)$ satisfies $z^2 \cdot x^2 y = y^2 \cdot x^2 z$ for all $x, y, z \in Q$.*
9. *If $(Q, \cdot)$ is of exponent 3, then $(Q, \oplus)$ has left idempotent law if and only if $(Q, \cdot)$ satisfies $y^2 \cdot x^2 y = x$ for all $x, y \in Q$.*
10. *If $(Q, \cdot)$ is of exponent 2, then $(Q, \oplus)$ is idempotent.*
11. *If $(Q, \cdot)$ is of exponent 3, then $(Q, \oplus, e)$ is of exponent 2.*
12. *If $(Q, \cdot)$ is of exponent 3, then $y \oplus (x \oplus x) = y \cdot y$ and $(x \oplus x) \oplus y = y$ for all $x, y \in Q$.*

Proof. 1. If $(Q, \oplus)$ has the left inverse property, then $x \backslash [(x^{-1} \backslash (y^{-1} \backslash x))^{-1} \backslash x^{-1}] = y$, or equivalently $x \cdot y = (x^{-1} \backslash (y^{-1} \backslash x))^{-1} \backslash x^{-1}$.

Thus, for $t = (x^{-1} \backslash (y^{-1} \backslash x))^{-1} \Leftrightarrow y^{-1} \cdot (x^{-1} \cdot t^{-1}) = x$ we have, $x \cdot y = t \backslash x^{-1}$, or equivalently $t \cdot (x \cdot y) = x^{-1}$. So, $(t \cdot (x \cdot y))^{-1} = y^{-1} \cdot (x^{-1} \cdot t^{-1})$, which means that $(Q, \cdot)$ is a SAAIPL.

2. If $(Q, \oplus)$ has LAP, then $x \oplus (x \oplus y) = (x \oplus x) \oplus y$. This is equivalent to $(x \oplus x)^{-1} \backslash [y^{-1} \backslash (x \oplus x)] = x^{-1} \backslash [(x \oplus y)^{-1} \backslash x]$, which can be rewritten in the form

$$[x^{-1} \backslash (x^{-1} \backslash x)]^{-1} \backslash [y^{-1} \backslash \big(x^{-1} \backslash (x^{-1} \backslash x)\big)] = x^{-1} \backslash [\big(x^{-1} \backslash (y^{-1} \backslash x)\big)^{-1} \backslash x]. \tag{15}$$

This is equivalent to

$$[x^{-1} \backslash (x^{-1} \backslash x)]^{-1} \backslash [y^{-1} \backslash \big(x^{-1} \backslash (x^{-1} \backslash x)\big)] = x^{-1} \backslash [\big(x^{-1} \backslash (y^{-1} \backslash x)\big)^{-1} \backslash x],$$

i.e. to $x \backslash (y^{-1} \backslash x) = x \backslash [\big(x \backslash (y^{-1} \backslash x)\big)^{-1} \backslash x]$.

This for $t = x \backslash (y^{-1} \backslash x)$ has the form $t = x \backslash (t^{-1} \backslash x)$, which can be written in the form $x \cdot t = t^{-1} \backslash x$, i.e. $\Leftrightarrow t^{-1} \cdot (x \cdot t) = x$,

3. $(Q, \oplus)$ has RAP if and only if $y \oplus (x \oplus x) = (y \oplus x) \oplus x$, i.e. if and only if

$$y^{-1} \backslash [(x^{-1} \backslash (x^{-1} \backslash x))^{-1} \backslash y] = (y^{-1} \backslash (x^{-1} \backslash y))^{-1} \backslash [x^{-1} \backslash (y^{-1} \backslash (x^{-1} \backslash y))]. \tag{16}$$

This is equivalent to $y\backslash[(x\backslash(x\backslash x))^{-1}\backslash y] = (y\backslash(x\backslash y))^{-1}\backslash[x\backslash(y\backslash(x\backslash y))]$ i.e. to

$$y\backslash(x\backslash y) = (y\backslash(x\backslash y))^{-1}\backslash[x\backslash(y\backslash(x\backslash y))].$$

This is equivalent to $e = x\backslash(y\backslash(x\backslash y))$, and consequently to $x = y\backslash(x\backslash y)$ which means that $(Q, \cdot)$ has the middle symmetry property

4. $(Q, \oplus)$ has the AIP if and only if $(x^{-1}\backslash(y^{-1}\backslash x))^{-1} = x\backslash(y\backslash x^{-1})$.

Let $t = (x^{-1}\backslash(y^{-1}\backslash x))^{-1}$. i.e. $y^{-1} \cdot (x^{-1} \cdot t^{-1}) = x$. Then $t = x\backslash(y\backslash x^{-1})$, or equivalently $y \cdot (x \cdot t) = x^{-1}$. Thus, $y^{-1} \cdot (x^{-1} \cdot t^{-1}) = (y \cdot (x \cdot t))^{-1}$, i.e $(Q, \cdot)$ is an AIPL.

5. $(Q, \oplus)$ is right idempotent law if and only if $y^{-1}\backslash\big((x \oplus x)^{-1}\backslash y\big) = y^{-1}\backslash(x^{-1}\backslash y)$, i.e. if and only if $y^{-1}\backslash\big(\big(x^{-1}\backslash(x^{-1}\backslash x)\big)^{-1}\backslash y\big) = y^{-1}\backslash(x^{-1}\backslash y)$. The last is equivalent to $\big(x^{-1}\backslash(x^{-1}\backslash x)\big)^{-1}\backslash y = x^{-1}\backslash y$ and consequently to $x^2 = e$.

6. Let $(Q, \cdot)$ be of exponent 2, then $(Q, \oplus)$ obeys the left idempotent law because

$$\begin{aligned}(x \oplus x) \oplus y &= (x \oplus x)^{-1}\backslash\big(y^{-1}\backslash(x \oplus x)\big)\\ &= \big(x^{-1}\backslash(x^{-1}\backslash x)\big)^{-1}\backslash\big(y^{-1}\backslash\big(x^{-1}\backslash(x^{-1}\backslash x)\big)\big)\\ &= (x^{-1}\backslash e)^{-1}\backslash\big(y^{-1}\backslash(x^{-1}\backslash e)\big) = x^{-1}\backslash(y^{-1}\backslash x)\\ &= x \oplus y.\end{aligned}$$

7. From (16) assuming that $(Q, \cdot)$ is of exponent 3, we have that $(Q, \oplus)$ has the RAP if and only if $y^{-1} = (y^{-1}\backslash(x^{-1}\backslash y))^{-1}\backslash[x^{-1}\backslash(y^{-1}\backslash(x^{-1}\backslash y))]$, i.e. if and only if $(y^{-1}\backslash(x^{-1}\backslash y))^{-1}y^{-1} = x^{-1}\backslash(y^{-1}\backslash(x^{-1}\backslash y))$.

Let $t_{(x,y)} = y^{-1}\backslash(x^{-1}\backslash y)$, then $t^{-1}_{(x,y)}y^{-1} = x^{-1}\backslash t_{(x,y)} \Leftrightarrow x^{-1} \cdot t_{(x,y)}{}^{-1}y^{-1} = t_{(x,y)} \Leftrightarrow x^2 \cdot t(x,y)^2 y^2 = t(x,y)$.

8. From (15) assuming that $(Q, \cdot)$ is of exponent 3, we have that $(Q, \oplus)$ has the LAP if and only if $y^{-1}\backslash e = x^{-1}\backslash\big(\big(x^{-1}\backslash(y^{-1}\backslash x)\big)^{-1}\backslash x\big)$, equivalently, if and only if $x^{-1}y = \big(x^{-1}\backslash(y^{-1}\backslash x)\big)^{-1}\backslash x$.

Let $t = x^{-1}\backslash(y^{-1}\backslash x)$, i.e. $y^{-1} \cdot x^{-1}t = x$. Then $t^{-1} \cdot x^{-1}y = y^{-1} \cdot x^{-1}t$, so $t^2 \cdot x^2 y = y^2 \cdot x^2 t$.

9. Assume that $(Q, \cdot)$ is of exponent 3. Then, $(Q, \oplus)$ obeys the left idempotent law if and only if $x^{-1}\backslash(y^{-1}\backslash x) = \big(x^{-1}\backslash(x^{-1}\backslash x)\big)^{-1}\backslash\big(y^{-1}\backslash\big(x^{-1}\backslash(x^{-1}\backslash x)\big)\big)$, or equivalently $x^{-1}\backslash(y^{-1}\backslash x) = y$, i.e. $y^2 \cdot x^2 y = x$.

The proofs for $10, 11$ and 12 are easy. □

Corollary 2.8. *Let $(Q, \oplus)$ be a core of middle Bol loop $(Q, \circ)$ and $(Q, \cdot)$ be the corresponding right Bol loop. If $(Q, \oplus)$ obey right idempotent law, then $(Q, \cdot)$ is a Steiner loop if and only if it satisfies the IP.*

Proof. This follows from Theorem 2.7(5) and Theorem 1.7 □

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